Engineering Sciences 247: Fracture Mechanics J. R. Rice, 1991

Homework Problems

1) Assuming that the stress field near a crack tip in a linear elastic solid is singular in the form $\sigma_{ij} = r^{\lambda}\, \Sigma_{ij}(\theta)$, it was shown in the lectures that $\lambda = -1/2$ and that, for Mode I (tensile) loading of an isotropic solid, the form of the singular near-tip stress state is

$$\frac{(\sigma_{rr} + \sigma_{\theta\theta})}{2} = \frac{(\sigma_{11} + \sigma_{22})}{2} = \frac{K_I}{\sqrt{2\pi r}} \cos(\theta/2) ,$$

$$\frac{(\sigma_{\theta\theta} - \sigma_{rr})}{2} + i\, \sigma_{r\theta} = e^{2i\theta} \left[\frac{(\sigma_{22} - \sigma_{11})}{2} + i\, \sigma_{12} \right] = \frac{i\, K_I\, e^{i\theta/2}}{2\sqrt{2\pi r}} \sin(\theta) .$$

Also, $\sigma_{33} = \nu\, (\sigma_{11} + \sigma_{22})$ for plane strain and general 3D cracks, and $\sigma_{33} = 0$ for the 2D plane stress model. Here K_I is undetermined by the near-tip analysis and is called the Mode I *stress intensity factor*.

Show that the corresponding displacement field is

$$u_r + i\, u_\theta = e^{-i\theta} (u_1 + i\, u_2) = \frac{K_I}{2\mu} \sqrt{\frac{r}{2\pi}}\, (\kappa - \cos\theta)\, e^{-i\theta/2}$$

where $\kappa = 3 - 4\nu$ for plane strain or for the general 3D crack, and where $\kappa = (3-\nu)/(1+\nu)$ for the 2D plane stress model.

2) Derive the form of the singular near-tip stress field for Mode II (in-plane shear) conditions in an isotropic linear elastic solid, beginning with the terms

$$U = r^{\lambda + 2}\, [\, C \sin\lambda\theta + D \sin(\lambda + 2)\theta\,]$$

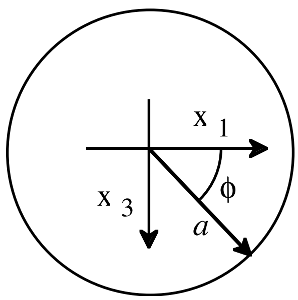
in the Airy stress function which correspond to an anti-symmetric dependence on θ. The undetermined constant should be re-defined so that (after you show $\lambda = -1/2$) the shear stress on $\theta = 0$ is written as $(\sigma_{12})_{\theta = 0} = K_{II} / \sqrt{2\pi r}$. Check your results by verifying that your expressions for the $\sigma_{\alpha\beta}$ coincide with those given in one of books or articles in the course bibliography, and state which source you used for this.

3) A circular crack lies on the region $x_1^2 + x_3^2 \le a^2$ of the plane $x_2 = 0$ in an unbounded

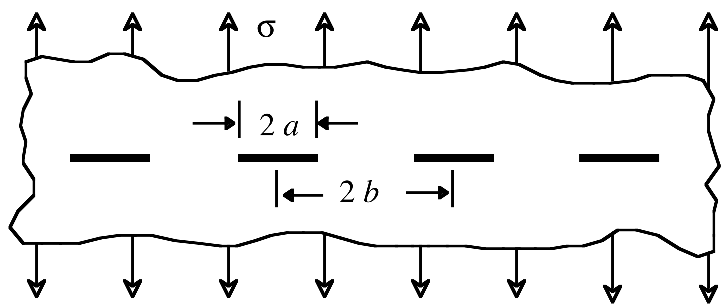
isotropic elastic solid. When the solid is subjected to the remote stress field $\sigma_{12}^{\infty} = \sigma_{21}^{\infty} = \tau$, $\sigma_{22}^{\infty} = \sigma$, with all other $\sigma_{ij}^{\infty} = 0$, and when the crack surfaces are traction-free, the displacement discontinuity between the upper (+) and lower (–) crack surfaces is

$$u_1^+ - u_1^- = \frac{8(1-\nu)\tau}{\pi(2-\nu)\mu}\sqrt{a^2 - x_1^2 - x_3^2}, \quad u_2^+ - u_2^- = \frac{4(1-\nu)\sigma}{\pi\mu}\sqrt{a^2 - x_1^2 - x_3^2},$$

$u_3^+ - u_3^- = 0$. Show that the stress intensity factors at the point along the crack front where $x_3 / x_1 = \tan\phi$ (see figure) are $K_I = 2\sigma\sqrt{a/\pi}$, $K_{II} = [4\tau/(2-\nu)]\sqrt{a/\pi}\cos\phi$, $K_{III} = -[4\tau(1-\nu)/(2-\nu)]\sqrt{a/\pi}\sin\phi$.



for problems 3 and 4



for problem 5

4) A 3D elastic solid is under loadings that may be characterized by a generalized force Q; the work-conjugate displacement to Q is q. When a crack in introduced on a surface S, lying on $x_2 = 0$ in the solid, there is a displacement Δq and the change ΔU_{el} in elastic strain energy is given by

$$\Delta U_{el} = Q\,\Delta q - \frac{1}{2}\int_S \sigma_{2j}^{\text{before}} (u_j^+ - u_j^-)^{\text{after}}\, dS$$

when the solid is modeled as linear, where $\sigma_{2j}^{\text{before}}$ is the stress acting on S before crack introduction and $(u_j^+ - u_j^-)^{\text{after}}$ is the displacement discontinuity there afterwards. The energy change could also be calculated by growing the crack incrementally from zero size up to its final size S. Let $\Gamma_{c.f.}$ denote the crack front at some stage during this growth process, let s denote arc-length along $\Gamma_{c.f.}$, and let G(s) be the energy release rate at position s (G will be proportional to Q^2). Then during an infinitesimal growth increment during which the crack locally advances by

$\delta a(s)$ normal to itself, the increment δU_{el} of elastic strain energy is

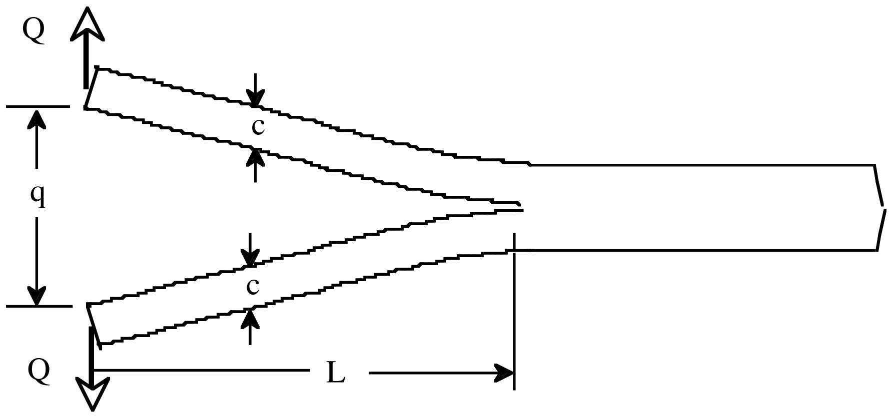
$$\delta U_{el} = Q\,\delta q - \int_{\Gamma_{c.f.}} G(s)\,\delta a(s)\;ds$$

where δq is the increment of displacement. We can recall that for an isotropic solid, $G = \{ (1-\nu)[K_I^2 + K_{II}^2] + K_{III}^2 \}/2\mu$.

Suppose now that the crack size is small compared to overall body dimensions and that it is circular in shape, like in problem 3. Also, assume that the loading is by remote shear stress $\tau\ (= \sigma_{21}^{\infty})$. Then, on the basis of Eshelby's analysis of the ellipsoidal inclusion of one solid in another, in particular, of the fact that such an inclusion sustains a spatially homogeneous strain, we may assert that the displacement discontinuity on the crack has the form $u_1^+ - u_1^- = \lambda\,\tau\,\sqrt{a^2 - x_1^2 - x_3^2}\ ,\quad u_2^+ - u_2^- = u_3^+ - u_3^- = 0$, when the crack radius is a, where λ is a constant that is independent of τ and a. The value of λ has been given in problem 3, but the idea here is to pretend that we do not know it. Using the ideas stated in the paragraph above, and recalling that the relations between K_I, K_{II}, K_{III} and the near-tip components of the crack surface displacement discontinuity should enable you to express the K's in terms of λ, solve for λ. Make sure that you reproduce the result given in problem 3.

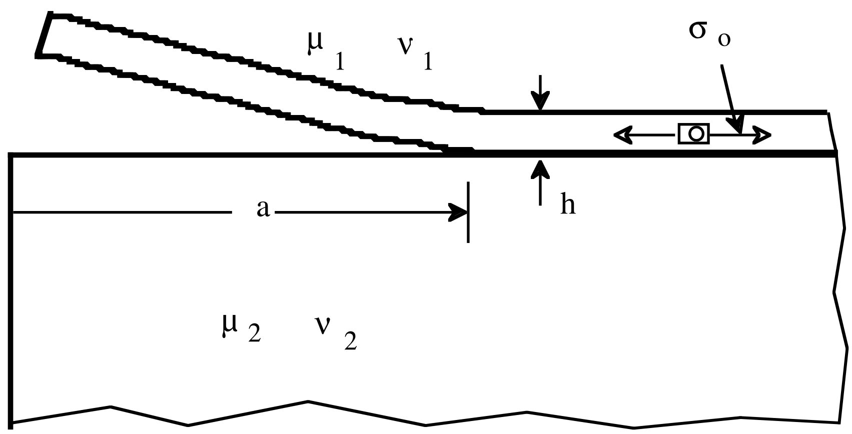
5) Given that $K_I = \sigma\sqrt{2b\ \tan(\pi a/2b)}$ for a thin sheet, of thickness h, under tension σ, that contains an infinite periodic row of collinear cracks, show that each crack opens by a volume equal to $\Delta(\text{Volume}) = (16\,\sigma\, b^2\, h / \pi E)\ ln\,[\,1/\cos(\pi a/2b)\,]$, where h = sheet thickness.

6) A crack is extended along the center line of a plate specimen, of thickness h, which may be assumed to deform in plane stress. The crack is driven by wedging forces Q. Using simple beam concepts, estimate the crack mouth opening displacement q and the stress intensity factor K_I at the tip. [Ans.: $q \approx 8\{(Q/hc)/E\}\,L^3/c^2$; $K_I \approx 2\,(Q/hc)\,L\,\sqrt{3/c}$]



for problem 6

7) An elastic layer which has thickness h and elastic properties μ_1, ν_1 is bonded at a high temperature to a massive elastic quarter-space which has properties μ_2, ν_2. Because of thermal



for problem 7

expansion mismatch, upon cooldown the layer is under a residual *biaxial* tensile stress of magnitude σ_o. This stress is uniform within the layer, at least at locations that are not too close to the corner. Suppose that the layer starts to debond in a 2D plane strain mode so that a crack of length a, where a is several time larger than h, develops along the interface. Show that the energy release rate is $G = (1 - \nu_1) \sigma_o^2 h / 4 \mu_1$. (If you get a different answer: Have you thought carefully about what is the final stress state in the debonded part of the layer?)

8) Find the stress intensity factor K_I for the case of a crack with a uniform pressure loading s over a width b very near the crack tip; b is assumed here to be very much smaller than crack size or overall body dimensions. Also, solve for the opening displacement $\delta = u_y^+ - u_y^-$ as a

function of

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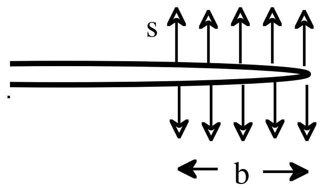
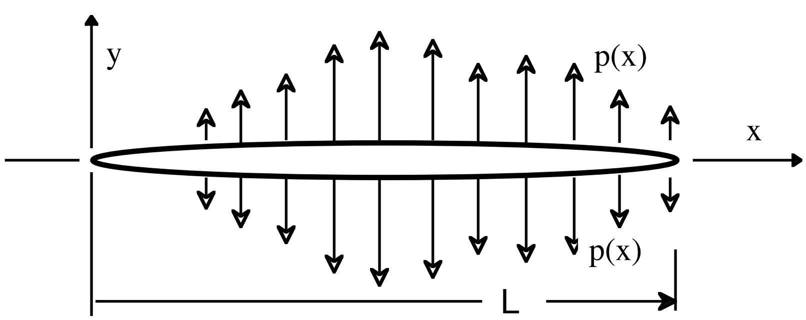


for problem 8

distance r from the crack tip, giving separate results for $0 < r < b$ and for $b < r$. Use this as an opportunity to apply weight function concepts. To check of your work, $K_I = 2\,s\sqrt{2\,b/\pi}$ and $\delta = 4(1-\nu)\,s\,b/\pi\mu$ at $r = b$, wheras $\delta \approx [\,4\,(1-\nu)\,K_I/\mu\,]\sqrt{r/2\pi}$ for $r << b$.

9) Consider the plane strain problem of a crack of length L in an unbounded elastic body, where the crack surfaces are subjected to a general distribution of opening pressure p(x). Recalling that the crack surface opening is of the form $\delta = \lambda\,\sigma\sqrt{x\,(L-x)}$ for some appropriate λ when the crack without surface loadings is subjected to remote tension σ, find the crack surface weight function $k_{Iy}(x; L)$ and show that K_I at the right end ($x = L$) and the opening displacement are:



for problem 9

$$K_I = \sqrt{\frac{2}{\pi L}}\int_0^L p(x)\sqrt{\frac{x}{L-x}}\,dx \quad ; \quad \delta(x) = \int_0^L p(x')\,M(x,\,x';L)\,dx'$$

where $$M = \frac{2\,(1-\nu)}{\pi\,\mu}\,ln\left[\frac{L\,(x+x') - 2\,x\,x' + 2\sqrt{x\,x'\,(L-x)\,(L-x')}}{L\,|x-x'|}\right]$$. Explain why this same solution applies for in-plane shear loadings involving general distributions p(x) of shear tractions on the crack faces, and also for anti-plane shear loadings when the factor $(1-\nu)$ is replaced by unity. For in-plane and anti-plane cases

K_I is replaced by K_{II} and K_{III}, respectively, and the δ corresponds to slip in the respective x and z directions.

Homework Problems (Continued)

10) This problem and the next are exercises in uses of the representation of 2D linear elastic fields in terms of analytic functions. In the isotropic case, with $z = x_1 + i\, x_2$, this representation is

$$(\sigma_{11} + \sigma_{22}) / 2 = \varphi'(z) + \bar{\varphi}'(\bar{z})$$
$$(\sigma_{22} - \sigma_{11}) / 2 + i\,\sigma_{21} = \bar{z}\,\varphi''(z) + \psi'(z)$$
$$2\,\mu\,(u_1 + i\,u_2) = \kappa\,\varphi(z) - z\,\bar{\varphi}'(\bar{z}) - \bar{\psi}(\bar{z})$$

$$\sigma_{23} + i\,\sigma_{13} = \omega'(z)$$
$$2\,i\,\mu\,u_3 = \omega(z) - \bar{\omega}(\bar{z})$$

Here $\kappa = 3 - 4\nu$ for plane strain and $\kappa = (3 - \nu) / (1 + \nu)$ for plane stress. Also, the replacement of $\psi(z)$ by $\Omega(z) - z\,\varphi'(z)$ is often convenient for crack problems.

Consider a crack of length 2a lying along $x_2 = 0, -a < x_1 < +a$ in an infinite 2D solid. The solid is subjected to zero remotely applied stresses and no tractions are applied to the crack faces; i.e., all $\sigma_{ij} = 0$ at infinity and $\sigma_{2j} = 0$ on the crack faces, i,j = 1,2,3. However, the solid is loaded by a *dislocation* process: We assume that a cut is made, starting at some location on the crack surface and extending to infinity. One side of the cut is displaced relative to the other by a uniform relative displacement vector **b** (the Burgers vector) and then, after filling in any gaps or removing any overlaps, the sides of the cut are welded back together. Thus, if C is some contour surrounding the crack, and if s denotes arc length along C, measured positive anti-clockwise, the displacement field must satisfy the condition $\int_C (\partial u_i / \partial s)\, ds = b_i$. Some notes: The stress field thus

induced in the solid is independent of the cut location (explain why!). Also, if the above integral condition on displacement is satisfied for any one contour C surrounding the crack, it is necessarily satisfied for any other contour (hence you can take C far from the crack to evaluate the integral).

Show that the boundary conditions on stress require that $\Omega'(z) - \bar{\varphi}'(z)$ and $\omega'(z) - \bar{\omega}'(z)$ be continuous across the crack and vanish at infinity, so that one must have $\Omega'(z) = \bar{\varphi}'(z)$ and

$\omega'(z) = \overline{\omega}'(z)$ for all z, and further that

$$\varphi'(x_1)^+ + \varphi'(x_1)^- = 0 \quad \text{and} \quad \omega'(x_1)^+ + \omega'(x_1)^- = 0, \quad -a < x_1 < +a .$$

Then, observe that $F(z) = (z^2 - a^2)^{-1/2}$ is a function which is analytic everywhere except along the crack, and which satisfies $F^+ + F^- = 0$ on the crack (at least when we choose the branch cut of $(z^2 - a^2)^{-1/2}$ to coincide with the crack, e.g., by restricting the phase angle of factors $z - a$ and $z + a$ to the range $-\pi <$ phase angle $< +\pi$). Thus, show that the solution is

$$\varphi'(z) = [\mu(b_2 - i b_1)/\pi(\kappa + 1)](z^2 - a^2)^{-1/2}, \quad \omega'(z) = (\mu b_3 / 2\pi)(z^2 - a^2)^{-1/2}$$

and solve for the stress intensity factors K_I, K_{II}, K_{III} at the crack tip at $x_1 = a$.

11) Consider again the infinite 2D solid with a crack of length 2a. Now we assume that there is a single-valued displacement field outside the crack (i.e., no Burgers vector) and that, rather, the loading is by applying tractions to the crack faces. Thus we require that all $\sigma_{ij} = 0$ at infinity but that the stresses σ_{2j} on the upper crack face, at $x_2 = 0^+$, satisfy $\sigma_{2j} = -p_j^+(x_1)$ and on the lower crack face, at $x_2 = 0^-$, satisfy $\sigma_{2j} = -p_j^-(x_1)$. The p_j are given bounded functions of x_1 and are not necessarily equal to one another, and so the possibility is allowed of a net force being exerted on the crack faces.

Show that the following must hold along the crack, $-a < x_1 < +a$:

$$\begin{aligned}
\varphi'(x_1)^+ + \overline{\Omega}'(x_1)^- &= -[p_2^+(x_1) - i\, p_1^+(x_1)], \\
\varphi'(x_1)^- + \overline{\Omega}'(x_1)^+ &= -[p_2^-(x_1) - i\, p_1^-(x_1)], \\
\omega'(x_1)^+ + \overline{\omega}'(x_1)^- &= -2\, p_3^+(x_1), \\
\omega'(x_1)^- + \overline{\omega}'(x_1)^+ &= -2\, p_3^-(x_1).
\end{aligned}$$

Now, find the equations satisfied by the functions $\varphi'(z) - \overline{\Omega}'(z)$, $\varphi'(z) + \overline{\Omega}'(z)$, $\omega'(z) - \overline{\omega}'(z)$, and $\omega'(z) + \overline{\omega}'(z)$ along the crack, and thus show that

$$\varphi'(z) - \overline{\Omega}'(z) = \frac{-1}{2\pi i}\int_{-a}^{+a} \frac{[p_2^+(t) - p_2^-(t)] - i\,[p_1^+(t) - p_1^-(t)]}{t - z}\, dt ,$$

$$\varphi'(z) + \overline{\Omega}'(z) = \frac{-(z^2 - a^2)^{-1/2}}{2\pi} \int_{-a}^{+a} \frac{\{[p_2^+(t) + p_2^-(t)] - i\,[p_1^+(t) + p_1^-(t)]\}\sqrt{a^2 - t^2}}{t - z}\,dt$$

$$- \frac{(\kappa - 1)(F_1 + i F_2)(z^2 - a^2)^{-1/2}}{2\pi(\kappa + 1)} \qquad \left(\text{where } F_j = \int_{-a}^{+a} [p_j^+(t) - p_j^-(t)]\,dt\right);$$

$$\omega'(z) = \frac{-(z^2 - a^2)^{-1/2}}{2\pi} \int_{-a}^{+a} \frac{[p_3^+(t) + p_3^-(t)]\sqrt{a^2 - t^2}}{t - z}\,dt - \frac{1}{2\pi i} \int_{-a}^{+a} \frac{[p_3^+(t) - p_3^-(t)]}{t - z}\,dt .$$

Also, find the stress intensity factors at the crack tip at $x_1 = a$.

12) Consider the 2D problem of a crack on the interface between joined dissimilar elastic solids. Solid "1" occupies $x_2 > 0$ and solid "2" occupies $x_2 < 0$; the crack coincides with the region $x_1 < 0$ on $x_2 = 0$. It was shown in the lectures that the complex stress functions near the crack tip have the form:

$$\varphi_1'(z) = \overline{K}\, z^{-1/2 - i\varepsilon}\, e^{-\pi\varepsilon} / [\, 2\sqrt{2\pi}\cosh(\pi\varepsilon)\,]$$
$$\Omega_1'(z) = K\, z^{-1/2 + i\varepsilon}\, e^{+\pi\varepsilon} / [\, 2\sqrt{2\pi}\cosh(\pi\varepsilon)\,]$$
$$\varphi_2'(z) = \overline{K}\, z^{-1/2 - i\varepsilon}\, e^{+\pi\varepsilon} / [\, 2\sqrt{2\pi}\cosh(\pi\varepsilon)\,]$$
$$\Omega_2'(z) = K\, z^{-1/2 + i\varepsilon}\, e^{-\pi\varepsilon} / [\, 2\sqrt{2\pi}\cosh(\pi\varepsilon)\,]$$

where $\varepsilon = (1/2\pi)\, ln\,[(\kappa_1/\mu_1 + 1/\mu_2)/(\kappa_2/\mu_2 + 1/\mu_1)]$ and K is the complex generalization of $K_I + i K_{II}$; e.g., $K = (\sigma_{22}^{\infty} + i\,\sigma_{21}^{\infty})(1 + 2i\varepsilon)\, L^{-i\varepsilon}\sqrt{\pi L/2}$ for the tunnel crack of length L in a remotely uniform stress field. Show that the stress along the bond line, at distance r ahead of the crack tip, and the displacement discontinuity at distance r behind the tip, have the respective asymptotic forms:

$$\sigma_{22} + i\,\sigma_{21} = K\, r^{i\varepsilon} / \sqrt{2\pi r}\,,$$

$$(u_2^+ - u_2^-) + i\,(u_1^+ - u_1^-) = (c_1 + c_2)\, K\, r^{i\varepsilon}\sqrt{r/2\pi}\, / \,[\, 2(1 + 2i\varepsilon)\cosh(\pi\varepsilon)\,]\,,$$

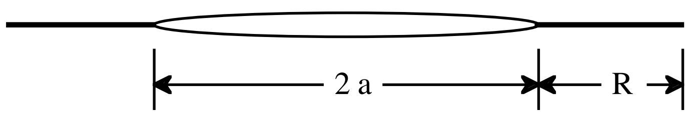
where $c_j = (\kappa_j + 1)/\mu_j$, j=1,2. Also, derive $G = (c_1 + c_2)\, K\overline{K} / [\, 16\cosh^2(\pi\varepsilon)\,]$ for the energy release rate. Letting $\tan\theta = \sigma_{21}^{\infty} / \sigma_{22}^{\infty}$, for the tunnel crack case, and assuming L = 4 mm, find the ranges of θ for which the contact zone at the tip is (a) smaller than an atomic spacing (about $3 \cdot 10^{-10}$ m), and (b) smaller than 10 microns, for the case of Ti bonded to Al_2O_3

($\varepsilon \approx 0.04$).

13) Consider the Dugdale/Bilby-Cottrell-Swinden elastic-plastic model for a crack of length 2a in an infinite 2D body subjected to remote Mode II shear loading τ^∞. In this model, all nonelastic deformation near the tip is represented by sliding zones of length R, prolonging the crack, within which the local value of σ_{12} is constant at τ_y, where τ_y is the yield stress in shear. Show that the plastic zone size R and the sliding displacement discontinuity δ_{tip} at the crack tip are given by:

$$R = a \left[\sec\left(\frac{\pi \tau^\infty}{2 \tau_y}\right) - 1 \right], \quad \delta_{tip} = \frac{4 (1 - \nu) \tau_y a}{\pi \mu} \, ln \left[\sec\left(\frac{\pi \tau^\infty}{2 \tau_y}\right) \right] .$$

(Either complex variable or weight function methods could provide a good approach for analysis of this problem.) Verify that in the limit $\tau^\infty << \tau_y$, R and δ_{tip} depend on τ^∞ and a only as τ^∞ and a enter the combination $K_{II} \equiv \tau^\infty \sqrt{\pi a}$.



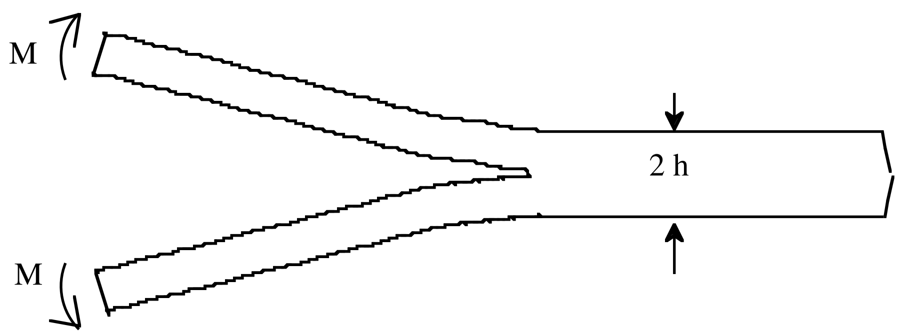
14) Consider a penny-shaped crack of radius a in an infinite 3D solid subjected to remote tension σ^∞ acting perpendicular to the plane of the crack. By remembering Eshelby's observation on homogeneous strain within an inclusion, we may infer in this case that, for a purely elastic crack, the opening displacement at distance ρ from the crack center has the form $u_2^+ - u_2^- = B \sigma^\infty \sqrt{a^2 - \rho^2}$ for some appropriate constant B. You should know how to solve for B (look back to problem 4), although that is not essential for completing the rest of the problem.

Develop a generalization of weight function theory which applies for circular cracks subjected to axi-symmetric loading. Thus, solve for the crack face weight function $k(\rho, a)$, defined such that $k(\rho, a)$ P is the K_I induced at the crack tip by a concentrated ring loading of intensity P, per unit length, acting at radius ρ on the upper crack face, and by a similar ring loading –P acting at radius ρ on the lower crack surface. [Answer is: $k(\rho, a) = 2 \rho / \sqrt{\pi a (a^2 - \rho^2)}$.]

Use the result to analyze a Dugdale/BCS model of the penny-shaped tensile crack (see previous figure for notation), showing that $R = a \left[\frac{1}{\sqrt{1 - (\sigma^\infty / \sigma_y)^2}} - 1 \right]$, and verify that this R can

likewise be expressed solely in terms of $K_I \equiv 2\,\sigma^\infty \sqrt{a/\pi}$ when $\sigma^\infty << \sigma_y$ (σ_y = tensile yield stress).

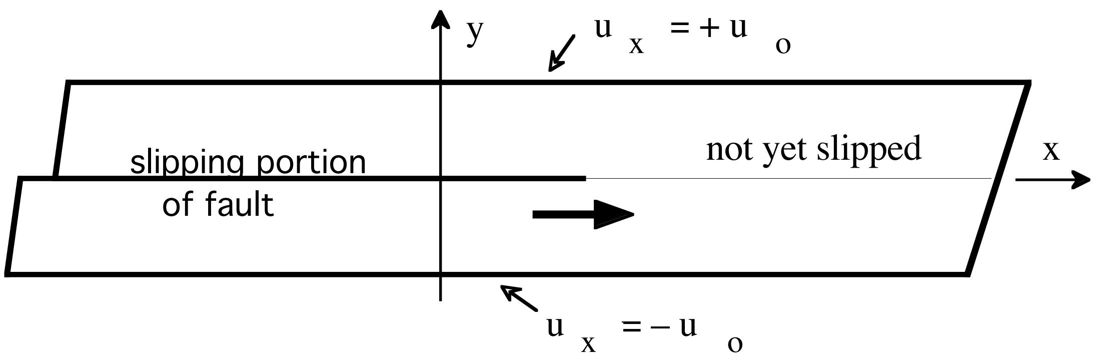
15) The two beam arms are loaded by moments M per unit thickness. Treat this as a problem in plane stress and use the J integral to evaluate G, thus showing, in the case when the material is isotropic and linear elastic, that $K_I \equiv 2\sqrt{3}\; M\, h^{-3/2}$.



16) An infinite slab of material of thickness 2h contains a fault on its centerplane (i. e., on the plane y = 0) and is loaded by forcing shear displacements in the $\pm$ x directions along the top and bottom surfaces of the slab. The boundary conditions at those surfaces are:

at $y = +h,\ u_x = +u_o,\ u_y = u_z = 0$ and at $y = -h,\ u_x = -u_o,\ u_y = u_z = 0$.

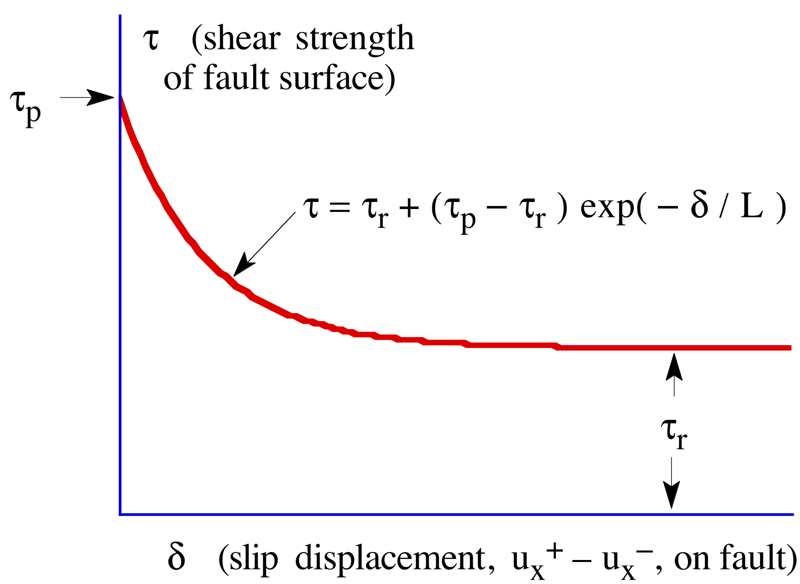
The fault surface satisfies the "slip-weakening" law illustrated in the figure on the next page; no slip occurs until τ ($\equiv \sigma_{yx}$ on the fault) reaches the "peak" strength τ_p, and then sliding occurs following the relation $\tau = \tau_r + (\tau_p - \tau_r) \exp(-\delta / L)$, where $\delta = (u_x^+ - u_x^-)_{y=0}$ is the slip and L = constant, so that the strength reduces towards the "residual" value τ_r at large slip. Assume that failure occurs by the quasistatic crack-like propagation of a slip event from the far left to the far right of the slab, in the view shown.



(a) First, assume that during this process the fault zone at the far left slips sufficiently that the strength level of the fault there is effectively reduced to τ_r , i.e., that $\delta >> L$ at $x = -\infty$, and hence that $\sigma_{yx} = \tau_r$ at $x = -\infty$. Then, using the J integral and treating this as a problem in plane strain, with linear and isotropic elastic response of the slab material outside the fault zone, show that the critical value of u_o at which the slip propagation occurs is given by

$$\frac{u_o}{h} = \frac{\tau_r}{\mu} + \sqrt{\frac{(\tau_p - \tau_r)\, L}{\mu\, h}}$$

where μ is the shear modulus.



(b) Now analyze the problem without the simplifying assumption that $\delta >> L$ at $x = -\infty$ and show that the slip δ at $x = -\infty$ is given by

$$\frac{\delta / L}{\sqrt{1 - (1 + \delta / L)\, e^{-\delta / L}}} = 2 \sqrt{\frac{(\tau_p - \tau_r) / L}{\mu / h}}$$

and express u_o / h for propagation in terms of that δ / L ratio.

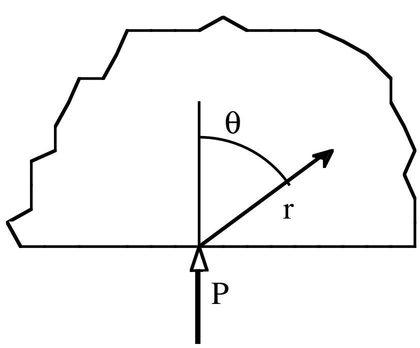
(c) The above results were obtained by assuming a crack-propagation-like mode of failure. However, if those results require a stress σ_{yx} at $x = + \infty$ which becomes equal to τ_p, then that assumed mode of failure must be incorrect. Instead, the fault would, presumably, fail simultaneously everywhere, in a dynamically unstable fashion, when σ_{yx} at $x = + \infty$ reaches τ_p. Show that the crack-propagation-like mode of failure is possible only when the "brittleness number" $B \equiv \left[(\tau_p - \tau_r) / L\right] / (\mu / h)$ satisfies $B > 0.5$.

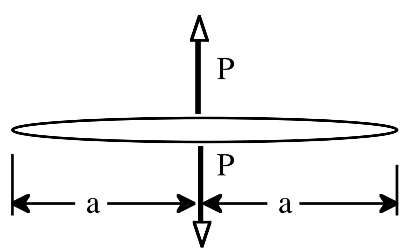
What is the smallest h allowing a crack-propagation-like failure mode for granite if $\mu = 300$ kbar, $\tau_p - \tau_r = 1$ kbar, and $L = 0.5$ mm?

17) The solution for a concentrated line force P (per unit thickness) acting on an elastic half-space, i.e., the plane strain problem, is (figure at left on next page)

$$\sigma_{rr} = -\frac{2 P \cos \theta}{\pi r}, \quad \sigma_{\theta\theta} = \sigma_{r\theta} = 0, \quad \sigma_{zz} = \nu \sigma_{rr} .$$

Using the M integral, and noting this solution, derive the K_I induced by line forces +P and –P acting along the centers of the upper and lower walls of a tunnel crack of length 2a. [Ans.: $K_I = P/\sqrt{\pi a}$.]





18) Show that the integral on a closed contour C (in 2D deformation fields)

$$L = L_3 = \int_C e_{3ij} [n_i x_j W + n_k \sigma_{ki} u_j - n_k \sigma_{km} u_{m,i} x_j] ds = 0$$

for isotropic solids with properties that are unaffected by rotation about the x_3 axis. Show this directly by transforming to an integral over the area enclosed by C, but verify also by the Noether procedure that L = 0 arises from a suitable invariant transformation.

19) A screw dislocation line of Burgers vector b lies parallel to the tip of what can be regarded as a half-plane crack in an infinite body. The dislocation line is at position $x_1 + i x_2 = z_o = \rho e^{i \phi}$ relative to the crack, where coordinates are measured from the crack tip and the crack lies on $x_2 = 0$, $x_1 < 0$. Verify that $\omega'(z) = \mu b / 2 \pi (z - z_o)$ gives the stress field of the dislocation in the absence of the crack, and then derive the solution

$$\omega'(z) = \frac{1}{2}\left[1 + (z_o / z)^{1/2}\right] \frac{\mu b}{2 \pi (z - z_o)} - \frac{1}{2}\left[1 - (\bar{z}_o / z)^{1/2}\right] \frac{\mu b}{2 \pi (z - \bar{z}_o)}$$

describing the stress field created by the dislocation in the presence of the crack. Thus show that

$$K_{III} = -\mu b \cos(\phi / 2) / \sqrt{2 \pi \rho} \quad \text{and} \quad f_r = -\mu b^2 / (4 \pi \rho)$$

are the intensity factor and radial component of the configurational "force" on the dislocation.

20) A thin sheet of viscoelastic material contains a tensile-loaded crack. The crack is assumed to be very long compared to a nonlinear "craze" zone which develops at its tip, so that you can assume "small scale yielding". The crazing process is to be modeled approximately using a plane stress Duddale/BCS fracture model with constant yield strength σ_y. The crazing material is assumed to fail, dropping its strength to zero, when the crack tip opening displacement reaches a critical value (say, δ_{crit}). We let K* be the critical stress intensity factor for the immediate onset of crack growth under step loading, and recall that it is related to δ_{crit} by $\delta_{crit} = J^*/\sigma_y = (K^*)^2/E\sigma_y$.

For simplicity, the viscoelastic material is to be described by the elementary Maxwell model, such that the strain ε in response to a uniaxial tensile stress (σ) history is given by

$$\frac{d\varepsilon}{dt} = \frac{1}{E}\frac{d\sigma}{dt} + \frac{\sigma}{E t_r}$$ where t_r is a material relaxation time.

(a) Show that the waiting time for onset of crack growth in response to a step loading to intensity factor K (< K*) is $t = t_r\,[(K^*/K)^2 - 1]$.

(b) Show that for steady state growth at K < K*, the crack speed V is

$$V = \frac{\pi K^4}{24\, t_r\, \sigma_y^2\, [\,(K^*)^2 - K^2\,]}$$

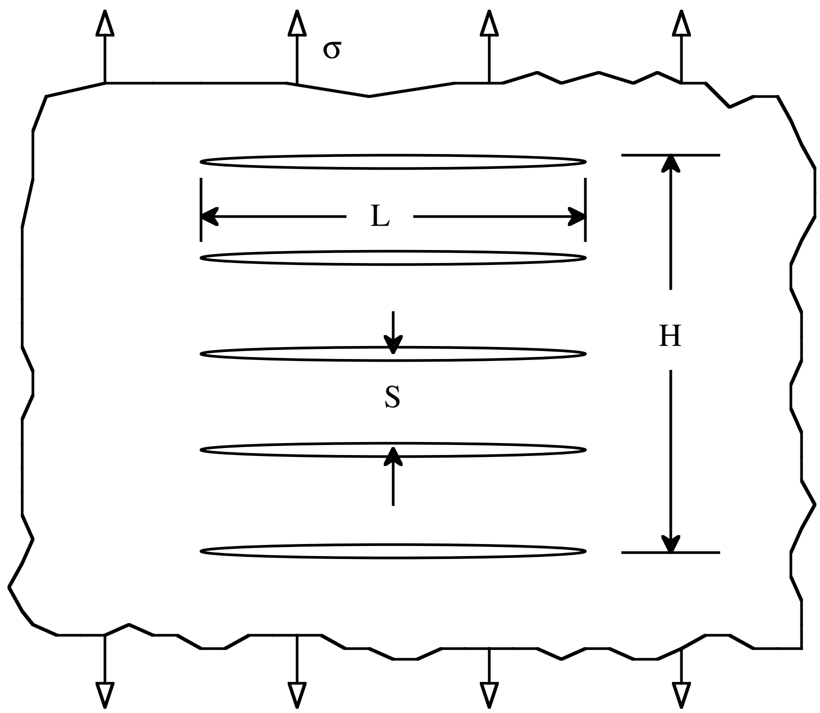
ES 247 Fracture Mechanics Final Exam 20 May 1991 J. R. Rice

(Completed exams must be returned to 207-A Pierce Hall by 10:00 AM on Tuesday 21 May.)

(Homework must reach 207-A Pierce Hall by 5:00 PM on Tuesday 21 May to be counted.)

Note: You are not to discuss the questions on this exam with classmates or others. You may use books and/or notes but, for each such use, you should give a clear reference explaining the source for what you use. Your work must be presented clearly and concisely. Print your name. Number each problem and letter each subsection.

1. (15 pts. total) Stress σ acts on a plate containing a stack of thin slots (many more than shown in the figure). The overall height of the stack is H. Each slot has length L and they are spaced by distance S.

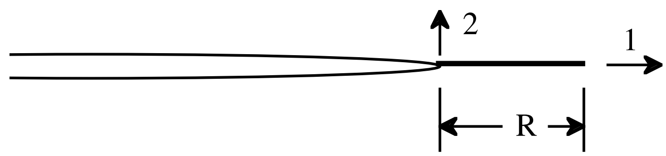


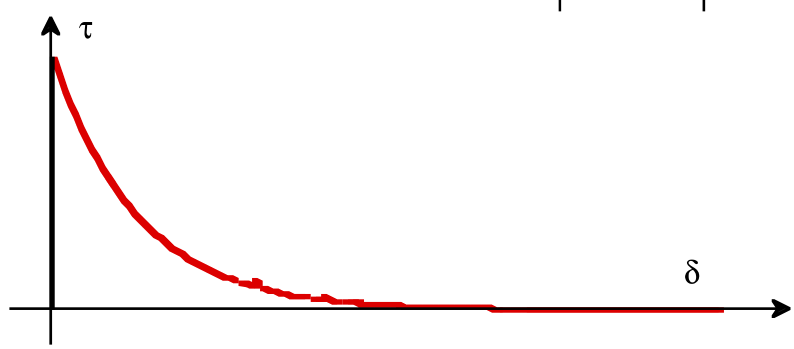
(a) (10 pts.) Assuming that H >> L and that L >> S, and treating the thin slots as if they were cracks, derive an expression for K_I that will be valid at locations not too near the ends of the stack. Be explicit about any assumptions that you make.

(b) (5 pts.) Suppose that the slots are not cracks but, rather, have rounded tips with radius of curvature ρ_{tip}. Estimate the maximum stress at their tips.

2. (15 pts.) A solid is loaded by a field of conservative body force f_α (per unit volume), having potential $\phi = \phi(\mathbf{u})$ such that $f_\alpha = -\partial\phi(\mathbf{u})/\partial u_\alpha$. We assume that $\phi(\mathbf{u})$ has no explicit dependence on $\mathbf{x}$. (E.g., for a gravity field with the x_2 axis vertical, $\phi(\mathbf{u}) = \rho g u_2$.) Derive the analog of the path-independent crack tip integral J for 2D solids loaded by such body force fields.

3. (10 pts.) Consider a long mode II shear crack in plane strain with a slip-weakening zone along the prolongation of the crack at its tip. For $\delta > 0$ the relation between stress and displacement in the slip-weakening zone is $\tau\,(=\sigma_{21}) = \tau_o \exp(-\delta/c)\ ;\ \delta = u_1^+ - u_1^-$. Here τ_o and c are constants. Suppose that the length R of the slip weakening zone is very much smaller than crack length and other overall dimensions of the cracked body (so that the "small scale yielding" concept is applicable), that the crack surfaces are traction-free, that the body is isotropic, and that it is loaded such that, for the classical singular crack model, there would be a stress intensity factor K_{II} at the tip. Write out the relation between the crack sliding displacement, δ_{tip}, at the tip and K_{II}.





4. (25 pts. total) A 2D isotropic solid in plane strain is loaded with two sets of forces, whose amplitudes are characterized by the parameters Q_1 and Q_2, respectively. Assume that either of these force sets, acting in isolation, produces only mode I conditions at the tip of a crack of length L within the body. Regarding Q_1 and Q_2 as generalized forces, we can let q_1 and q_2 be the corresponding generalized displacements, defined such that $Q_1\,dq_1 + Q_2\,dq_2$ is an increment of work (per unit thickness) of the applied forces. Assuming that the solid is linear we may obviously write

$$K_I = k_1(L)\,Q_1 + k_2(L)\,Q_2$$

$$q_1 = C_{11}(L)\,Q_1 + C_{12}(L)\,Q_2\ ,\quad q_2 = C_{21}(L)\,Q_1 + C_{22}(L)\,Q_2$$

(a) (10 pts.) Derive an expression for $dC_{11}(L)/dL$ in terms of $k_1(L)$ and discuss experimental and computational applications of this result.

(b) (5 pts.) Derive an expression for $dC_{21}(L)/dL$ in terms of $k_1(L)$ and $k_2(L)$.

(c) (10 pts.) Recalling that the opening gap between the surfaces of a tunnel crack of length L under uniform pressure p_o on its surfaces is $[\,2\,(1-\nu)\,p_o/\mu\,]\ \sqrt{x_1\,(L-x_1)}$ (the crack lies on $0 < x_1 < L$), use the result of part (b) to obtain the expression for K_I at the crack tip at $x_1 = L$ due to a pair of line forces of magnitude F (per unit thickness) which wedge the crack walls open at $x_1 = b$.

5. (15 pts. total) A crack grows dynamically along the x_1 axis in an isotropic solid under anti-plane shear loading. The crack speed V(t) is non-uniform.

(a) (5 pts.) Starting from basic physical principles, and explaining all assumptions, write out the form of a differential equation satisfied by $u_3(x, y, t)$ which suffices to let one deduce the form of the singular field at the tip; here $y = x_2$ and $x = x_1 - V(t)\, dt$, such that x = 0 at the moving crack tip.

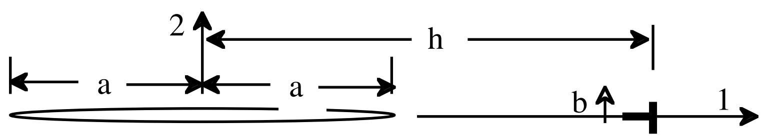
(b) (5 pts.) Show that the singularity of the stress field is of $r^{-1/2}$ type.

(c) (5 pts.) Defining $K_{III} = \text{limit}[\, (2\pi r)^{1/2} (\sigma_{23})_{\theta=0}\,]$ as $r \to 0$, express $\delta_3 \equiv u_3^+ - u_3^-$ very near the crack tip as a function of r, K_{III} and V.

6. (20 pts. total) Recall that the representation of 2D elastic fields in terms of analytic functions is

$$(\sigma_{11} + \sigma_{22})/2 = \varphi'(z) + \overline{\varphi}'(\bar{z})$$
$$(\sigma_{22} - \sigma_{11})/2 + i\,\sigma_{21} = \bar{z}\,\varphi''(z) + \psi'(z)$$
$$2\mu\,(u_1 + i\,u_2) = \kappa\,\varphi(z) - z\,\overline{\varphi}'(\bar{z}) - \overline{\psi}(\bar{z})$$

where $\kappa = 3 - 4\nu$ for plane strain and $\kappa = (3-\nu)/(1+\nu)$ for plane stress. Also, the replacement of $\psi(z)$ by $\Omega(z) - z\,\varphi'(z)$ is often convenient.



(a) (5 pts.) Show that $\varphi'(z) = \Omega'(z) = [\,\mu b / \pi\,(\kappa + 1)\,(z - h)\,]$ solves the problem of a dislocation of Burgers vector b at the position shown when there is no crack present.

(b) (10 pts.) Solve for $\varphi'(z)$ for the problem shown of a dislocation next to a tunnel crack, when there is a net dislocation within the crack such that $\delta_2 \equiv u_2^+ - u_2^- = b$ at $x_1 = a$, whereas $\delta_2 = 0$ at $x_1 = -a$. (We can think of this as the problem of beginning with a crack in a dislocation-free and stress-free solid, then cutting the solid along the x_1 axis between $x_1 = a$ and $x_1 = h$, inserting a layer of material of thickness b in the cut, and welding the walls of the layer to the solid on each side.)

(c) (5 pts.) Solve for K_I at the crack tip at $x_1 = a$.