ES 241 *Notes on convected coordinates* ξ^1, ξ^2, ξ^3 (JRR, 4/97 and 2/03)

• *Base vectors:* Reference configuration: $\mathbf{G}_i = \dfrac{\partial \mathbf{X}}{\partial \xi^i}$. Current configuration: $\mathbf{g}_i = \dfrac{\partial \mathbf{x}}{\partial \xi^i}$.

Relation between them: Since $d\mathbf{x} = \mathbf{F}\cdot d\mathbf{X}$, $\mathbf{g}_i = \mathbf{F}\cdot\mathbf{G}_i$.

• *Reciprocal base vectors:* $\mathbf{g}^i$ and $\mathbf{G}^i$ satisfy $\mathbf{g}^i\cdot\mathbf{g}_j = \mathbf{G}^i\cdot\mathbf{G}_j = \delta^i_j = 0$ *if* $i \neq j$, 1 *if* $i = j$.

Since $\mathbf{g}^i\cdot(\mathbf{F}\cdot\mathbf{G}_j) = \mathbf{G}^i\cdot\mathbf{G}_j$, they are related by $\mathbf{g}^i\cdot\mathbf{F} = \mathbf{G}^i$.

• *Relations between stress and strain measures and rates:*

• $\mathbf{E}^G = (1/2)(\mathbf{F}^T\cdot\mathbf{F} - \mathbf{I})$ and $\mathbf{D} = \mathrm{sym}(\dot{\mathbf{F}}\cdot\mathbf{F}^{-1})$ $\Rightarrow \mathbf{D} = \mathbf{F}^{-1T}\cdot\dot{\mathbf{E}}^G\cdot\mathbf{F}^{-1}$.

Thus, if we write $\mathbf{E}^G = \eta_{ij}\ \mathbf{G}^i\mathbf{G}^j$, where $\eta_{ij} \equiv \mathbf{G}_i\cdot\mathbf{E}^G\cdot\mathbf{G}_j = (1/2)(\mathbf{g}_i\cdot\mathbf{g}_j - \mathbf{G}_i\cdot\mathbf{G}_j)$ $= (1/2)(g_{ij} - G_{ij})$ [$g_{ij} = \mathbf{g}_i\cdot\mathbf{g}_j$, $G_{ij} = \mathbf{G}_i\cdot\mathbf{G}_j$ are metric tensors] , then $\mathbf{D} = \dot{\eta}_{ij}\ \mathbf{g}^i\mathbf{g}^j$.

• $\boldsymbol{\tau} = \mathbf{F}\cdot\mathbf{S}^{PK2}\cdot\mathbf{F}^T$ $\Rightarrow$ If $\boldsymbol{\tau} = \tau^{ij}\ \mathbf{g}_i\mathbf{g}_j$, then $\mathbf{S}^{PK2} = \tau^{ij}\mathbf{G}_i\mathbf{G}_j$.

• $\dot{\boldsymbol{\tau}}^* = \dot{\boldsymbol{\tau}} + \boldsymbol{\tau}\cdot\boldsymbol{\Omega} - \boldsymbol{\Omega}\cdot\boldsymbol{\tau}$ and $\boldsymbol{\Omega} = \mathrm{antisym}(\dot{\mathbf{F}}\cdot\mathbf{F}^{-1}) \Rightarrow \dot{\boldsymbol{\tau}}^* = \boldsymbol{\tau}\cdot\mathbf{D} + \mathbf{D}\cdot\boldsymbol{\tau} + \mathbf{F}\cdot\dot{\mathbf{S}}^{PK2}\cdot\mathbf{F}^T$.

Thus $\dot{\boldsymbol{\tau}}^* = \boldsymbol{\tau}\cdot\mathbf{D} + \mathbf{D}\cdot\boldsymbol{\tau} + \dot{\tau}^{ij}\ \mathbf{g}_i\mathbf{g}_j$, so that $\dot{\tau}^{ij} = \mathbf{g}^i\cdot(\dot{\boldsymbol{\tau}}^* - \boldsymbol{\tau}\cdot\mathbf{D} - \mathbf{D}\cdot\boldsymbol{\tau})\cdot\mathbf{g}^j$.

• *Rate form of constitutive relation:* Suppose $\dot{\boldsymbol{\tau}}^* = \mathbf{L}:\mathbf{D}$, where $\mathbf{L}$ is the 4th rank incremental modulus tensor. Then

$$
\begin{aligned}
\dot{\tau}^{ij} &= \mathbf{g}^i\cdot(\mathbf{L}:\mathbf{D} - \boldsymbol{\tau}\cdot\mathbf{D} - \mathbf{D}\cdot\boldsymbol{\tau})\cdot\mathbf{g}^j \\
&= [(\mathbf{g}^j\mathbf{g}^i):\mathbf{L}:(\mathbf{g}^l\mathbf{g}^k) - (\mathbf{g}^i\cdot\boldsymbol{\tau}\cdot\mathbf{g}^k)(\mathbf{g}^j\cdot\mathbf{g}^l) - (\mathbf{g}^i\cdot\mathbf{g}^k)(\mathbf{g}^j\cdot\boldsymbol{\tau}\cdot\mathbf{g}^l)]\dot{\eta}_{kl} \\
&= [L^{ijkl} - \tau^{ik}g^{jl} - g^{ik}\tau^{jl}]\dot{\eta}_{kl}
\end{aligned}
$$

where $g^{ij} = \mathbf{g}^i\cdot\mathbf{g}^j$ and L^{ijkl} are the components of $\mathbf{L}$ when written $\mathbf{L} = L^{ijkl}\ \mathbf{g}_i\mathbf{g}_j\mathbf{g}_k\mathbf{g}_l$.

• *Prandtl-Reuss equations:* $\dot{\boldsymbol{\sigma}}^* + \boldsymbol{\sigma}\,tr(\mathbf{D}) = \mathbf{L}^{PR}:\mathbf{D}$ where, in *cartesian* coordinates,

$$
L^{PR}_{ijkl} = \frac{E}{1+\nu}\left\{\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \frac{\nu}{1-2\nu}\delta_{ij}\delta_{kl} + \frac{3\sigma'_{ij}\sigma'_{kl}}{2\bar{\sigma}^2[1+2(1+\nu)h/3E]}\right\}.
$$

Since $\boldsymbol{\tau} = \boldsymbol{\sigma}\det(\mathbf{F})$, these relations are equivalent to $\dot{\boldsymbol{\tau}}^* = \mathbf{L}:\mathbf{D}$ with $\mathbf{L} = \det(\mathbf{F})\mathbf{L}^{PR}$. To identify L^{ijkl}, note that $\mathbf{I} = \mathbf{e}_i\mathbf{e}_i = \delta^{\text{cartesian}}_{ij}\mathbf{e}_i\mathbf{e}_j = \mathbf{g}^j\mathbf{g}_j = g^{ij}\ \mathbf{g}_i\mathbf{g}_j$ shows that $\delta^{\text{cartesian}}_{ij}$ corresponds to g^{ij} when components are relative to the base vectors $\mathbf{g}_i$.

Thus the *Prandtl-Reuss* equations are $\tau^{ij} = [L^{ijkl} - \tau^{ik}g^{jl} - g^{ik}\tau^{jl}]\dot{\eta}_{kl}$ with

$$
L^{ijkl} = \det(\mathbf{F})\frac{E}{1+\nu}\left\{\frac{1}{2}(g^{ik}g^{jl} + g^{il}g^{jk}) + \frac{\nu}{1-2\nu}g^{ij}g^{kl} + \frac{3\sigma'^{ij}\sigma'^{kl}}{2\bar{\sigma}^2[1+2(1+\nu)h/3E]}\right\}
$$

where $\sigma'^{ij} = \mathbf{g}^i\cdot[\boldsymbol{\sigma} - (1/3)\mathbf{I}(\boldsymbol{\sigma}:\mathbf{I})]\cdot\mathbf{g}^j = \sigma^{ij} - g^{ij}g_{kl}\sigma^{kl}$ and $\bar{\sigma}^2 = (3/2)g_{ik}g_{jl}\sigma'^{ij}\sigma'^{kl}$.