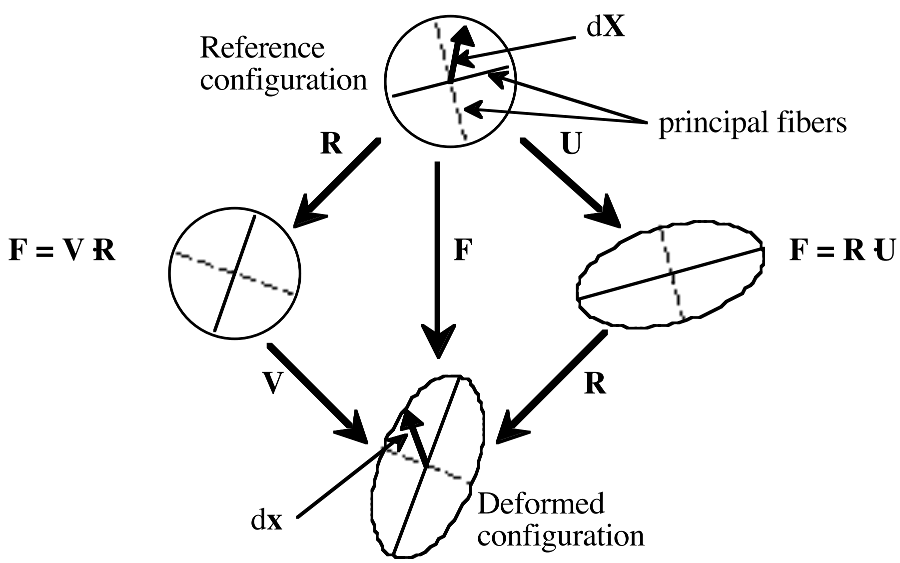
Engineering Sciences 241 Advanced Elasticity, Spring 2001 J. R. Rice

Homework Problems / Class Notes Mechanics of finite deformation (list of references at end)

Distributed Thursday 8 February.

Problems 1 to 5 due by Thursday 15 February.
Problems 6 to 10 due by Tuesday 27 February.

1. The *Polar Decomposition Theorem* for $\mathbf{F}$ (defined by $d\mathbf{x} = \mathbf{F} \cdot d\mathbf{X}$, or $\mathbf{F} = \partial\mathbf{x}/\partial\mathbf{X}$) is that $\mathbf{F} = \mathbf{V} \cdot \mathbf{R} = \mathbf{R} \cdot \mathbf{U}$, where $\mathbf{R}$ is a proper orthogonal transformation (rotation), with $\mathbf{R}^T \cdot \mathbf{R} = \mathbf{I}$ and $\det(\mathbf{R}) = 1$, and where $\mathbf{U}$ and $\mathbf{V}$ are pure deformations, i.e., $\mathbf{U} = \mathbf{U}^T$, $\mathbf{V} = \mathbf{V}^T$, and $\det(\mathbf{U}) = \det(\mathbf{V}) > 0$. The diagram below should clarify.



Develop analytical proofs for the following but, where possible, try to interpret answers geometrically in terms of the diagram:

(a) Assuming that $|\mathbf{F} \cdot \mathbf{a}| > \mathbf{0}$ for any vector $\mathbf{a} \neq \mathbf{0}$ (why is that reasonable?), show that $\mathbf{F}^T \cdot \mathbf{F}$ and $\mathbf{F} \cdot \mathbf{F}^T$ are symmetric and positive definite.

(b) Show that $\mathbf{U}^2 = \mathbf{F}^T \cdot \mathbf{F}$ and $\mathbf{V}^2 = \mathbf{F} \cdot \mathbf{F}^T$. *Note:* Sometimes $\mathbf{F}^T \cdot \mathbf{F}$ is called the *right Cauchy-Green tensor* $\mathbf{C}$, and $\mathbf{F} \cdot \mathbf{F}^T$ is called the left Cauchy-Green tensor $\mathbf{B}$.

(c) Show that the eigenvalues of $\mathbf{U}^2$ and $\mathbf{V}^2$ are identical and are positive; denote these λ_1^2, λ_2^2, λ_3^2, with positive roots λ_1, λ_2, λ_3. Explain why λ_1, λ_2, λ_3 are stretch ratios.

(d) Letting $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ be the unit eigenvectors of $\mathbf{U}^2$ (= $\mathbf{C}$) and $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be the unit eigenvectors of $\mathbf{V}^2$ (= $\mathbf{B}$), show that $\mathbf{U} = \sum_{i=1}^{3} \lambda_i \, \mathbf{u}_i \, \mathbf{u}_i$ and $\mathbf{V} = \sum_{i=1}^{3} \lambda_i \, \mathbf{v}_i \, \mathbf{v}_i$. Note that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ correspond to the orientations of the principal fibers in the reference configuration and $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to their orientations in the deformed configuration.

(e) Show that $\mathbf{F} = \sum_{i=1}^{3} \lambda_i \, \mathbf{v}_i \, \mathbf{u}_i$, that $\mathbf{R} = \sum_{i=1}^{3} \mathbf{v}_i \, \mathbf{u}_i$, and that $\mathbf{U} = \mathbf{R}^T \cdot \mathbf{V} \cdot \mathbf{R}$.

2. Consider the finitely strained state of simple shear, $x_1 = X_1 + \gamma X_2$, $x_2 = X_2$, $x_3 = X_3$. Try to construct *concise* and *instructive* (and easily readable!) derivations of the answers to (a), (b) and (c) which follow.

(a) Find the greatest and least values of the principal stretches $\lambda_1, \lambda_2, \lambda_3$ and of the principal values of $\mathbf{E}^M$. [Ans.: $\lambda_{\text{max,min}} = \sqrt{1+\gamma^2/4} \pm \gamma/2$, $E^M_{\text{max,min}} = \pm(\gamma/2)\sqrt{1+\gamma^2/4} + \gamma^2/4$]

(b) Assuming henceforth that $\gamma > 0$, find the orientation of the unit vector $\mathbf{N}$ which was aligned, in the undeformed configuration, with the fiber which was to undergo the greatest stretch. [Ans.: $N_2 / N_1 = \sqrt{1+\gamma^2/4} + \gamma/2$]

(c) Explain why the orientation of the unit vctor $\mathbf{n}$ which aligns, in the deformed configuration, with the fiber of greatest stretch is given by $n_2 / n_1 = N_2 / (N_1 + \gamma N_2)$, where (N_1, N_2) is $\mathbf{N}$ above, and thus solve for n_2 / n_1. [Ans.: $n_2 / n_1 = 1 / (\sqrt{1+\gamma^2/4} + \gamma/2)$, which is the inverse of N_2 / N_1 given in the last part.] *Note:* Since n_2 / n_1 here is the inverse of N_2 / N_1 in (b), then if $\mathbf{N}$ makes an angle $45° + \phi$ with the 1 direction, $\mathbf{n}$ makes an angle of $45° - \phi$ with that direction; ϕ is a small angle when γ is small.

(d) Consider the strain $\gamma = 3/2$. Show from part (a) above that $\lambda_{\text{max}} = 2$, $\lambda_{\text{min}} = 1/2$, and from the results of parts (b) and (c), solve for the unit eigenvectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ and $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. [Partial answer: If $\lambda_1 = \lambda_{\text{max}}$, then $\mathbf{u}_1 = (\mathbf{e}_1 + 2\mathbf{e}_2)/\sqrt{5}$, $\mathbf{v}_1 = (2\mathbf{e}_1 + \mathbf{e}_2)/\sqrt{5}$.]

(e) For $\gamma = 3/2$, solve for, and compare with one another, the cartesian components of the following: the infinitesimal strain tensor $\boldsymbol{\varepsilon}$ (which is, of course, irrelvant at such large deformation gradients), the Green strain (or change-of-metric strain) $\mathbf{E}^M$ based on $g(\lambda) = (\lambda^2 - 1)/2$, and the Biot strain $\mathbf{E}^B$ based on $g(\lambda) = \lambda - 1$. The latter two are members of the family of material strain tensors defined by $\mathbf{E} = \sum_{i=1}^{3} g(\lambda_i) \, \mathbf{u}_i \, \mathbf{u}_i$ with $g(1) = 0$ and $g'(1) = 1$. Check your results for $\mathbf{E}^M$ via that mode of expression with what you can calculate from $\mathbf{E}^M = (\mathbf{F}^T \cdot \mathbf{F} - \mathbf{I}) / 2$.

3. Consider an isotropic elastic material, for which we may make (among various equivalent forms) the representation of stress-strain relations $\boldsymbol{\sigma} = h_0\mathbf{I} + h_1\mathbf{B} + h_2\mathbf{B}^2$ for Cauchy stress $\boldsymbol{\sigma}$, where $\mathbf{B} = \mathbf{F}\cdot\mathbf{F}^T$ is the *left Cauchy-Green tensor* and where h_0, h_1, h_2 are functions of the invariants I_1, I_2, I_3 of $\mathbf{B}$.

(a) Show that an equivalent representation is $\boldsymbol{\sigma} = m_0\,\mathbf{I} + m_1\,\mathbf{B} + m_2\,\mathbf{B}^{-1}$, and derive expressions for the functions m_0, m_1, m_2 in terms of h_0, h_1, h_2 and I_1, I_2, I_3 . [Hint: $\mathbf{B}^{-1}$ is an isotropic function of $\mathbf{B}$, so we can write $\mathbf{B}^{-1} = n_0\,\mathbf{I} + n_1\,\mathbf{B} + n_2\,\mathbf{B}^2$, which is equivalent to $\mathbf{I} = n_0\,\mathbf{B} + n_1\,\mathbf{B}^2 + n_2\,\mathbf{B}^3$. To determine the scalars n_0, n_1, n_2 involved here, note that $\mathbf{B}$ must satisfy its own characteristic value equation, so that $\mathbf{0} = -\mathbf{B}^3 + I_1\,\mathbf{B}^2 - I_2\,\mathbf{B} + I_3\,\mathbf{I}$.]

(b) Show that the *second Piola-Kirchhoff* stress $\mathbf{S}$ (i.e., the stress which is work-conjugate to the Green strain; look ahead to problem 6) for this same elastic material is $\mathbf{S} = \sqrt{I_3}\,(m_0\,\mathbf{C}^{-1} + m_1\,\mathbf{I} + m_2\,\mathbf{C}^{-2})$ where $\mathbf{C} = \mathbf{F}^T\cdot\mathbf{F} = \mathbf{I} + 2\,\mathbf{E}^{Green}$ is called the *right Cauchy-Green tensor* and where m_0, m_1, m_2 are the same functions of I_1, I_2, I_3 as above; note that I_1, I_2, I_3 are also the invariants of $\mathbf{C}$. *Note:* By considerations like in part (a), this means that $\mathbf{S}$ has the form $\mathbf{S} = g_0\mathbf{I} + g_1\mathbf{C} + g_2\mathbf{C}^2$ where g_0, g_1, g_2 are functions of the invariants I_1, I_2, I_3 of $\mathbf{C}$ -- which are the same as the invariants I_1, I_2, I_3 of $\mathbf{B}$.

4. Consider again the simple shear of problem 2 for an isotropic elastic material.

(a) Using $\boldsymbol{\sigma} = m_0\,\mathbf{I} + m_1\,\mathbf{B} + m_2\,\mathbf{B}^{-1}$ as above, explain why m_0, m_1, m_2 are functions of γ^2 (i.e., are *even* functions of γ) and write the cartesian components σ_{11}, σ_{12}, σ_{22}, σ_{33} of $\boldsymbol{\sigma}$ in terms of γ and the functions m_0, m_1, m_2. (Note that when we retain terms beyond the linear in γ, the imposition of simple shear strain requires not only shear stress, but also normal stresses which are even in γ; the presence of these normal stresses is known as the *Poynting effect*.)

(b) Show that the results of part (a) imply that $\sigma_{11} - \sigma_{22} = \gamma\,\sigma_{12}$ must hold for any elastic material in simple shear, and develop a direct derivation of that result from consideration of principal directions in simple shear.

5. Write the Cauchy stress as $\boldsymbol{\sigma} = \sigma_{ij}\,\mathbf{e}_i\,\mathbf{e}_j = \bar{\sigma}_{ij}\,\bar{\mathbf{e}}_i\,\bar{\mathbf{e}}_j$ (summation on repeated indices) where the orthogonal unit base vectors $\bar{\mathbf{e}}_i$ ($i = 1,2,3$) happen to coincide with the fixed cartesian background base vectors $\mathbf{e}_i$ at the moment considered, but rotate relative to them with spin $\boldsymbol{\Omega}$. Here Ω_{ij} = antisym($v_{i,j}$) = (1/2) ($v_{i,j} - v_{j,i}$) and $d\bar{\mathbf{e}}_j / dt = \boldsymbol{\Omega}\cdot\bar{\mathbf{e}}_j = \boldsymbol{\Omega}\cdot\mathbf{e}_j = \Omega_{kj}\mathbf{e}_k$ at the moment considered. The *corotational (Jaumann) stress rate* $\dot{\boldsymbol{\sigma}}^*$ is then *defined* by $\dot{\boldsymbol{\sigma}}^* = \dot{\bar{\sigma}}_{ij}\;\bar{\mathbf{e}}_i\,\bar{\mathbf{e}}_j$. Show that

$$\dot{\boldsymbol{\sigma}}^{*} = \dot{\boldsymbol{\sigma}} + \boldsymbol{\sigma}\cdot\,\boldsymbol{\Omega} - \boldsymbol{\Omega}\cdot\,\boldsymbol{\sigma} \quad \text{(i.e., } \dot{\sigma}^{*}_{ij} = \dot{\sigma}_{ij} + \sigma_{ik}\,\Omega_{kj} - \Omega_{ik}\,\sigma_{kj}\text{).}$$

6. Various stress measures may be inter-related by the expression for work (per unit volume of reference state) associated with a change $\delta\mathbf{F}$ in the deformation gradient. That is,

$$\mathbf{t}\!:\!\delta\mathbf{F} = \boldsymbol{\tau}\!:\!(\delta\mathbf{F}\cdot\mathbf{F}^{-1}) = \mathbf{S}\!:\!\delta\mathbf{E}\text{ , where:}$$

- $\mathbf{t}$ = nominal stress ($\mathbf{t}^{\mathrm{T}}$ = first Piola-Kirchhoff stress),
- $\boldsymbol{\tau} = \boldsymbol{\sigma}\,\det(\mathbf{F})$ = Kirchhoff stress, where $\boldsymbol{\sigma}$ = Cauchy (or "true") stress, and
- $\mathbf{S} = \mathbf{S}^{\mathrm{T}}$ = stress conjugate to strain $\mathbf{E} = \sum_{i=1}^{3} g(\lambda_i)\,\mathbf{u}_i\,\mathbf{u}_i$ with $g(1) = 0$ and $g'(1) = 1$.

(When $\mathbf{E}$ is the strain based on change of metric, or the *Green strain*, $\mathbf{E}^{\mathrm{M}} = (1/2)\,(\mathbf{F}^{\mathrm{T}}\!\cdot\mathbf{F} - \mathbf{I})$, generated by $g(\lambda) = (\lambda^2-1)/2$, the associated $\mathbf{S}$ is called the *second Piola-Kirchhoff stress*, denoted $\mathbf{S}^{\mathrm{PK2}}$ below.) The rate of deformation tensor $\mathbf{D}$, defined by $D_{ij} = \mathrm{sym}(v_{i,j})$, so that $(\nabla\mathbf{v})^{\mathrm{T}} = \dot{\mathbf{F}}\cdot\,\mathbf{F}^{-1} = \mathbf{D} + \boldsymbol{\Omega}$, is used in what follows.

(a) Show that when the current state and reference state happen to be momentarily coincident,

$$\dot{\mathbf{t}} = \dot{\boldsymbol{\tau}}^{*} - \boldsymbol{\sigma}\cdot\,\mathbf{D} - \mathbf{D}\cdot\,\boldsymbol{\sigma} + \boldsymbol{\sigma}\cdot\,(\nabla\mathbf{v}) \text{ and that } \dot{\mathbf{S}}^{\mathrm{PK2}} = \dot{\boldsymbol{\tau}}^{*} - \boldsymbol{\sigma}\cdot\,\mathbf{D} - \mathbf{D}\cdot\,\boldsymbol{\sigma}\text{, where } \dot{\boldsymbol{\tau}}^{*} = \dot{\boldsymbol{\sigma}}^{*} + \boldsymbol{\sigma}\,\mathrm{tr}(\mathbf{D}).$$

(b) Show that any $\mathbf{E}$ has the series representation $\mathbf{E} = \mathbf{E}^{\mathrm{M}} + (m-1)\,\mathbf{E}^{\mathrm{M}}\cdot\,\mathbf{E}^{\mathrm{M}} + \ldots$ in terms of Green strain, where $m = [g''(1) + 1]/2$, and thus show that the rate of the $\mathbf{S}$ conjugate to that $\mathbf{E}$ is $\dot{\mathbf{S}} = \dot{\boldsymbol{\tau}}^{*} - m(\boldsymbol{\sigma}\cdot\mathbf{D} + \mathbf{D}\cdot\boldsymbol{\sigma})$, again when the current and reference states are momentarily coincident. (Observe also that since the logarithmic strain, generated by $g(\lambda) = \mathit{ln}\lambda$, has $m = 0$, the stress conjugate to logarithmic strain satisfies $\dot{\mathbf{S}} = \dot{\boldsymbol{\tau}}^{*}$ momentarily.)

(c) Suppose that a particular rate-independent solid satisfies the constitutive relation $\dot{\boldsymbol{\tau}}^{*} = \mathbf{L}^{\mathrm{o}}\!:\!\mathbf{D}$ where L^{o}_{ijkl} is the set of incremental moduli, necessarily symmetric under interchange of *ij* and chosen to be symmetric under interchange of *kl*. Explain why the moduli L^{o}_{ijkl} must also be symmetric under interchange of the pair *ij* with *kl* when the material is *hyper* elastic, so that a strain energy exists.

(d) The constitutive relation for this same material as in (c) may be rewritten in terms of any conjugate stress and strain measures, in the form $\dot{\mathbf{S}} = \mathbf{L}\!:\!\dot{\mathbf{E}}$. Show that in the simple situation when the current and reference configurations momentarily coincide,

$$L_{ijkl} = L^{o}_{ijkl} - \frac{m}{2}(\delta_{ik}\sigma_{jl} + \delta_{il}\sigma_{jk} + \sigma_{ik}\delta_{jl} + \sigma_{il}\delta_{jk}) .$$

(Note that symmetry under the interchange of the pair *ij* with *kl* thus applies for all, or for no, choices of conjugate strain and stress measures.)

7. Consider a hyperelastic material with strain energy density W = W(**F**) per unit volume of reference configuration. Since $\mathbf{t}:\delta\mathbf{F}$ (= $t_{ij}\ \delta F_{ji}$) = δW, it follows that the nominal stress is given by $t_{ij} = \partial W(\mathbf{F})/\partial F_{ji} = \partial W(\partial \mathbf{u}/\partial \mathbf{X})/\partial(\partial u_j/\partial X_i)$.

(a) Explain why invariance of the strain energy to a superposed rigid rotation of material elements allows us to write W = W(**E**), where **E** is any member of the family of material strain tensors discussed above, and show that when **E** is chosen as $\mathbf{E}^M$, and W is chosen to depend symmetrically on the components of $\mathbf{E}^M$, that $\mathbf{t} = [\partial W(\mathbf{E}^M)/\partial \mathbf{E}^M] \cdot \mathbf{F}^T$. [Observe that this equation automatically satisfies $\mathbf{F}\cdot\mathbf{t} = (\mathbf{F}\cdot\mathbf{t})^T$ (i.e., that $\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$, since $\mathbf{F}\cdot\mathbf{t} = \det(\mathbf{F})\,\boldsymbol{\sigma}$) which is the only consequence of the angular momentum principle not derivable on the basis of the linear momentum principle.]

(b) When a material is isotropic, it will suffice to assume that W depends only on the invariants of **E**, which is the same as assuming that it depends only on the invariants of **C** (the right Cauchy-Green tensor). These are defined by $\det(\mathbf{C}-\mu\mathbf{I}) = -\mu^3 + I_1\mu^2 - I_2\mu + I_3$ (where μ denotes λ^2) and are given as $I_1 = \mathrm{tr}(\mathbf{C})$, $I_2 = \{[\mathrm{tr}(\mathbf{C})]^2 - \mathbf{C}:\mathbf{C}\}/2$, $I_3 = \det(\mathbf{C})$. Verify with $W = W(I_1, I_2, I_3)$ that the form of stress strain relations given in part (a) leads to an expression for $\boldsymbol{\sigma}$ that is consistent with what is stated in problem 3.

8. A hyperelastic solid is homogeneous in its reference configuration. Let S^o denote a closed surface in that configuration, with unit outer normal **N**.

(a) Assuming that the solid is free of body force and that S^o encloses no singularities, show that any deformation field **u** = **u**(**X**) sustained by the solid satisfies the Eshelby conservation integrals

$$\int_{S^o} \left[N_i\, W(\partial \mathbf{u}/\partial \mathbf{X}) - N_j \frac{\partial W(\partial \mathbf{u}/\partial \mathbf{X})}{\partial(\partial u_k/\partial X_j)} \partial u_k/\partial X_i \right] dS^o = 0$$

9. The incremental form of the principle of virtual work, appropriate to the quasistatic rate of deformation problem, is

$$\int_{V^o} \dot{t}_{ij}\, \delta(\partial \dot{u}_j / \partial X_i)\, dV^o = \int_{V^o} \dot{f}_i^o\, \delta \dot{u}_i\, dV^o + \int_{S^o} \dot{T}_i^o\, \delta \dot{u}_i\, dS^o.$$

(a) Letting the reference configuration in which this is written be momentarily coincident with the current state, so that V^o coincides with V and S^o with S, show that the left side of this equation has the possible rearrangements

$$\int_V [\, \dot{\tau}^*_{ij}\, \delta D_{ji} - \frac{1}{2} \sigma_{ij}\, \delta(2D_{ik} D_{kj} - v_{k,i} v_{k,j})\,]\, dV \quad \text{and} \quad \int_V [\, \dot{S}_{ij}\, \delta D_{ji} + \frac{1}{2} \sigma_{ij}\, \delta(v_{k,i} v_{k,j})\,]\, dV$$

where $\mathbf{S} = \mathbf{S}^{PK2}$ in the latter form, and where $D_{ij} = \mathrm{sym}(v_{i,j})$.

(b) Suppose that the constitutive relation is given in the form $\dot{\tau}^*_{ij} = L^o_{ijkl} D_{kl}$. Discuss the formulation of a finite element procedure for the rate problem, assuming that $\{\Delta\}$ denotes nodal displacements and that the velocity field in the current configuration is interpolated by $\{v\} = [B(x)]\{\dot{\Delta}\}$. In getting equations $[K]\{\dot{\Delta}\} = \{\dot{F}\}$, you should identify three types of contribution to the tangent stiffness [K], one involving $\mathbf{L}^o$ that is formulated just as in conventional "small strain" theory, another involving the current stress $\boldsymbol{\sigma}$, and yet another stemming from the fact that, often, the nominal stress loading rate $\dot{\mathbf{T}}^o$ is not fully specified on the surface but depends on the deformation itself (consider a pressure loading).

10. A (hyper)elastic solid is under a stress distribution $\boldsymbol{\sigma}$ in its reference configuration, and we consider a field of small displacements **u** from that configuration, induced by additional nominal loadings $\mathbf{T}^o$ over a part S_T of its surface, the rest of the surface being fixed against further displacement (i.e.,the loadings $\mathbf{T}^o$ are additional to the $\mathbf{N}\cdot\boldsymbol{\sigma}$ necessary to create the initial stress state $\boldsymbol{\sigma}$). Let **L** be the modulus tensor, in terms of $\mathbf{S}^{PK2}$, in the linearized relation $\mathbf{S} = \boldsymbol{\sigma} + \mathbf{L}:\mathbf{E}$.

(a) Show that the displacement field within the body makes stationary the potential energy

$$\Pi = \frac{1}{2}\int_{V^o} [\, L_{ijkl}\, u_{i,j}\, u_{k,l} + \sigma_{ij}\, (u_{k,i}\, u_{k,j})\,]\, dV^o - \int_{S^o_T} T^o_i u_i\, dS^o$$

where here the energy is approximated to quadratic order only.

(b) In the case of a critically loaded Euler column, the variational problem $\delta\Pi = 0$ evidently has a solution when $\mathbf{T}^o = \mathbf{0}$. Discuss the buckling problem from that standpoint and, using approximations appropriate for a thin column, calculate the buckling stress. (See the Prager book if you get stuck on this.)

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