Elasticity of Fluid-Infiltrated Porous Solids (Poroelasticity)

J. R. Rice (November 1998; revised list of references, August 2001 and April 2004)

For use in Engineering Science 265, *Advanced environmental geomechanics*.

Comments entered in the format ***[[..instructions..]]*** are suggested exercises.

Introduction: This handout formulates the equations describing coupled processes of elastic deformation and pore fluid diffusion in fluid-infiltrated elastic solids. We first consider poroelastic constitutive response -- i.e., the dependence of strain and fluid content on stress and pore pressure -- and the Darcy law for pore fluid transport. The governing field equations are then formulated, using considerations of stress equilibrium and mass conservation. We consider the general case when neither the solid nor the fluid phase is incompressible. (Both phases were considered separately incompressible for the elementary theory of one-dimensional consolidation developed in the lectures.)

A short list of references is given at the end. A recent introduction to the field is provided by Guéguen et al. [2004] and that article, as well as Rice and Cleary [1976], Coussy [1995], Terzaghi et al. [1996], and Wang [2000], can help you track back into the earlier literature. The subject was created by Karl Terzaghi in 1923 for describing the one-dimensional consolidation of clay soils. Its modern development and three-dimensional generalization is due principally to Maurice Anthony Biot, in a series of works starting in the early 1940s [Biot, 1941]; those included consideration of effects of dynamic loading and stress waves [Biot, 1956], and of nonlinear elasticity [Biot, 1973].

The presentation here is restricted to linear elastic solids undergoing quasistatic deformations, and is based on the treatment in Rice and Cleary [1976]. That replaced the new elastic constants introduced by Biot by more familiar constants (Poisson ratio, bulk modulus) evaluated in both the *drained* and *undrained* states. It also showed how the formulation could be developed without writing separate equations of motion for the two constituents, as is done in an alternative approach to the subject that is sometimes described as *mixture theory* [Bowen, 1982; Coussy, 1995].

Notations: Like for most discussions in elasticity theory, it is convenient to use a concise notation in which spatial coordinates x,y,z are replaced by x_1, x_2, x_3, respectively. Displacements (e.g., of the solid phase) are denoted u_1, u_2, u_3, the notation σ_{ij} is used for stresses and ε_{ij} for strains. In terms of a common elementary notation for stress and strain, the correspondence is

$$\sigma_{11} = \sigma_x,\ \sigma_{22} = \sigma_y,\ \sigma_{33} = \sigma_z;\ \sigma_{12} = \tau_{xy},\ \sigma_{23} = \tau_{yz},\ \sigma_{31} = \tau_{zx};$$

$$\varepsilon_{11} = \varepsilon_x,\ \varepsilon_{22} = \varepsilon_y,\ \varepsilon_{33} = \varepsilon_z;\ \varepsilon_{12} = \frac{\gamma_{xy}}{2},\ \varepsilon_{23} = \frac{\gamma_{yz}}{2},\ \varepsilon_{31} = \frac{\gamma_{zx}}{2}.$$

Stress: Stress can be defined with reference to an infinitesimal cube with faces pointing in the coordinate directions: σ_{ij} is the force in the x_j direction, per unit area, acting on a face of the cube whose normal points in the x_i direction. Normal stresses like σ_{11} $(= \sigma_x)$ will therefore be positive if corresponding to tension; that is opposite to the convention used in most of rock and soil mechanics, and in most the other problems of this course that mention stress. For balance of torque acting on all such infinitesimal elements of material, it is necessary that shear stresses be equal on adjoining faces, which is concisely expressed by requiring that $\sigma_{ij} = \sigma_{ji}$ for all i and j. Also, for force equilibrium, it is necessary that ***[[please derive]]***

$$\sum_{i=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_i} + f_j = 0 \quad \text{for all } j,$$

where f_j is the body force per unit volume. This f_j is typically just the weight force, so that $f_j = -\gamma \partial z_{elev} / \partial x_j$ (where z_{elev} is vertical elevation above some datum, so that $\partial z_{elev} / \partial x_j$ are the components of a vertical unit vector).

Strain: Strains can be defined most simply in the case of extremely small deformations, in which case the coordinates x_1, x_2, x_3 of material points are virtually the same before and after deformation. Extensional strains like ε_{11} $(= \varepsilon_x)$ are simple changes in length per unit length for an infinitesimal line element aligned with the x_1 direction. Shears strains like ε_{12} are defined so that $2\varepsilon_{12}$ $(= \gamma_{xy})$ is the reduction from $\pi/2$ of the angle between a pair of infinitesimal line elements pointing, respectively, in the x_1 and x_2 directions. The strains can be defined equivalently, for very small deformations as we consider here, in terms of derivatives of the displacement components of form

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad \text{for all } i,j.$$

Note that this makes $\varepsilon_{ij} = \varepsilon_{ji}$. ***[[Convince yourself that this expression for strain is consistent with characterization of strain in terms of "changes in length per unit length" and "reduction from $\pi/2$ of the angle" as above.]]***

Ordinary elastic stress-strain relations: For circumstances of ordinary elasticity for which there are no pore pressure effects, the stress-strain relations (for an isotropic linear material) are

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\delta_{ij}(\sigma_{11} + \sigma_{22} + \sigma_{33})$$

where δ_{ij} is the *Kronecker delta*, defined so that $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ otherwise. These embody the accepted understanding of *Young's tensile modulus E* and *Poisson ratio* ν ***[[please show that]]***, and also the result ***[[show that too]]*** that the elastic *shear modulus,* or *rigidity*, *G* is given by

$$G = \frac{E}{2(1+\nu)}.$$

The bulk modulus *K* is defined by writing

$$\frac{\Delta V}{V} = \frac{(\sigma_{11} + \sigma_{22} + \sigma_{33})}{3K}$$

where $\Delta V / V \equiv \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$ is the *fractional change in volume* ***[[explain why]]***. One may readily show ***[[please do]]*** that the above stress-strain relations lead to

$$K = \frac{E}{3(1-2\nu)} = \frac{2(1+\nu)G}{3(1-2\nu)}.$$

Finally, re-express the stress-strain relations above as

$$\varepsilon_{ij} = \frac{1}{2G}\sigma_{ij} + \left(\frac{1}{9K} - \frac{1}{6G}\right)\delta_{ij}(\sigma_{11} + \sigma_{22} + \sigma_{33}),$$

and obtain their inverse

$$\sigma_{ij} = 2G\varepsilon_{ij} + \left(K - \frac{2G}{3}\right)\delta_{ij}(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}).$$

[[Carry out the details of these last two steps.]]

New elastic parameters when pore pressure is present: When pore pressure *p* is present, and we regard the material as isotropic and linear elastic, the only possible generalization of this last expression is

$$\sigma_{ij} = 2G\varepsilon_{ij} + \left(K - \frac{2G}{3}\right)\delta_{ij}(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) - \alpha\delta_{ij}p.$$

Here α is a new elastic constant for a porous material, with the property that if pore pressure is increased by Δp, and all the normal stresses are decreased (that is, increased in compression) by $\alpha\Delta p$, then there is no change in strain.

Now the bulk modulus K should be understood as the bulk modulus under *drained* conditions. Drained conditions correspond to deformation at fixed p, with the fluid being allowed to flow in or out of the deforming element however is required to keep p constant.

The opposite limit is *undrained* deformation, in which the fluid is constrained from flowing in or out during deformation (and, in general, changes of p are induced). We shall soon want to introduce a bulk modulus defined for undrained conditions, and this will be denoted by K_u. The pair of new parameters, α and K_u, completely characterize elastic response with fluid infiltration.

We may observe that for a linear isotropic solid as considered here, shearing under undrained conditions cannot induce a pressure change in the pore fluid ***[[why is that?]]***, and thus G is the proper shear modulus for *both* drained and undrained conditions.

The equation above for stress in terms of strain and pore pressure may be inverted to solve for strain, leading to ***[[carry out the steps]]***

$$\varepsilon_{ij} = \frac{1}{2G}\sigma_{ij} + \left(\frac{1}{9K} - \frac{1}{6G}\right)\delta_{ij}(\sigma_{11} + \sigma_{22} + \sigma_{33}) + \frac{\alpha}{3K}\delta_{ij}p.$$

Changes of porosity and fluid mass content: To complete the description of elastic response under fluid infiltration, we also need to specify how the storage of fluid within material elements changes due to stressing and pressurization.

To this end, let us define the "*porosity*", *n,* and *fluid mass content, m,* as

$$n = \frac{V_f}{V} \ , \quad m = \frac{M_f}{V}$$

Here V_f is the volume of fluid, and M_f is the mass of fluid, contained in a lump of porous material which would occupy volume V in an unstressed and unpressurized reference state. We assume full saturation of all connected pore space, so that V_f is also the volume of void space. It is evident that $m = \rho_f n$, where ρ_f is fluid density, and thus, letting Δ denote the small changes in quantities due to the elastic deformation,

$$\Delta m = n\Delta\rho_f + \rho_f\Delta n = n\rho_f\frac{p}{K_f} + \rho_f\Delta n$$

where $\Delta\rho_f = \rho_f p / K_f$ has been used and K_f is the bulk modulus of the fluid phase. Our goal now is to find the dependence of Δm on the strains $\boldsymbol{\varepsilon}$ and *p*.

Now observe that an infinitesimal increment of work, per unit volume, to deform an element and alter the amount of fluid within it, is

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\sigma_{ij}d\varepsilon_{ij} + pdn$$

and that this must be a *perfect*, or *exact*, differential if a strain energy function is to exist (as thermodynamics requires, for elastic response). An equivalent statement ***[[explain why]]*** is that

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\sigma_{ij}d\varepsilon_{ij} - \Delta n dp$$

must be an exact differential. A consequence of that exactness is that, if we represent the description of stress and deformation in the form $\sigma_{ij} = \sigma_{ij}(\boldsymbol{\varepsilon}, p)$ and $\Delta n = \Delta n(\boldsymbol{\varepsilon}, p)$, then ***[[explain why, and explain origin of last term to follow]]***

$$\frac{\partial \Delta n(\boldsymbol{\varepsilon}, p)}{\partial \varepsilon_{ij}} = -\frac{\partial \sigma_{ij}(\boldsymbol{\varepsilon}, p)}{\partial p} = \alpha\delta_{ij}$$

Thus, if we integrate with respect to strain at fixed *p*, we find that ***[[do the steps]]***

$$\Delta n = \alpha(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + \text{ linear term in } p$$

It is most concise to combine that linear term with one already contained in the equation above for Δm, and to write the net coefficient of *p* as $\rho_f \alpha^2 / (K_u - K)$ so that

$$\Delta m = \rho_f \alpha \left(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} + \frac{\alpha}{K_u - K} p \right).$$

It is at this point that the symbol denoted K_u first enters our equations. One may show that it does indeed correspond to the bulk modulus under undrained conditions, as already announced in the previous section. This is done by observing that $\Delta m = 0$ for undrained deformation, so that then $\alpha p = -(K_u - K)(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})$. If we substitute that into the stress-strain relation at the start of the previous section ***[[please do so]]***, we find that it reduces to

$$\sigma_{ij} = 2G\varepsilon_{ij} + \left(K_u - \frac{2G}{3}\right)\delta_{ij}(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})$$

for undrained conditions, which proves ***[[explain why]]*** that K_u is the bulk modulus under those conditions.

We are generally interested in Δm, and not specifically in Δn, but the expression for the latter can be obtained by using the equation for Δm towards the beginning of this section to obtain ***[[verify]]***

$$\Delta n = \frac{\Delta m}{\rho_f} - \frac{np}{K_f} = \alpha(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + \left(\frac{\alpha^2}{K_u - K} - \frac{n}{K_f}\right)p.$$

Expressions in terms of bulk modulus of the solid phase: There is a simple but often applicable situation in which both the new constants, α and K_u, just introduced can be determined in terms of the bulk moduli of the fluid and solid constituents.

Suppose that all pore space is fluid infiltrated, and that all the solid phase consists of material elements which respond isotropically to pure pressure stress states, with the same bulk modulus K_s. Suppose we simultaneously apply a pore pressure $p = p_o$ and macroscopic stresses amounting to compression by p_o on all faces ($\sigma_{11} = \sigma_{22} = \sigma_{33} = -p_o$). That results in a local stress state of $-p_o\delta_{ij}$ at each point of the solid phase. So each point of the solid phase undergoes the strain $-p_o\delta_{ij} / 3K_s$, which means that all linear dimensions of the material, including those characterizing void size, reduce by the (very small) fractional amount $p_o / 3K_s$, causing the macroscopic strains, and change in porosity, $\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} = -p_o / 3K_s$ and $\Delta n / n = -p_o / K_s$.

The stress-strain-pressure relation at the beginning of the second previous section must be consistent with the special state just discussed, and by substituting in it we obtain ***[[please verify]]***

$$-p_o\delta_{ij} = 2G\left(\frac{-p_o\delta_{ij}}{3K_s}\right) + \left(K - \frac{2G}{3}\right)\delta_{ij}\left(\frac{p_o}{K_s}\right) - \alpha\delta_{ij}p_o .$$

Thus one gets ***[[show]]***

$$\alpha = 1 - \frac{K}{K_s}.$$

Note that $0 \le \alpha \le 1$ and that α will be near its upper limit for soil-like materials, since then $K << K_s$. The equation for Δn at the end of the previous section must also be consistent. It reduces to ***[[please verify]]***

$$n\frac{p_o}{K_s} = \alpha\left(\frac{p_o}{K_s}\right) + \left(\frac{\alpha^2}{K_u - K} - \frac{n}{K_f}\right)p_o ,$$

from which we obtain ***[[show]]***

$$K_u = K + \frac{\alpha^2 K_s K_f}{nK_s + (\alpha - n)K_f} \; .$$

Darcy flow and conservation of fluid mass: If q_1, q_2, q_3 are the components of discharge velocity of the fluid relative to the solid, then Darcy's law (expressed in terms of the permeability measure *k*), is

$$q_i = -\frac{k}{\mu_f}\left(\frac{\partial p}{\partial x_i} + \gamma_f \frac{\partial z_{elev}}{\partial x_i}\right)$$

where μ_f is viscosity of the pore fluid and γ_f is its weight density. Conservation of fluid mass then requires ***[[derive]]*** that

$$\sum_{i=1}^{3} \frac{\partial(\rho_f q_i)}{\partial x_i} + \frac{\partial(\Delta m)}{\partial t} = 0 .$$

Equations describing perturbations: Let us now re-define p and σ_{ij} to describe perturbations away from some static initial state p^{init} and σ_{ij}^{init} that equilibrates gravitational loading, i.e., that satisfies

$$\sum_{i=1}^{3} \frac{\partial \sigma_{ij}^{init}}{\partial x_i} - \gamma \frac{\partial z_{elev}}{\partial x_j} = 0 \quad \text{and} \quad \frac{\partial p^{init}}{\partial x_i} + \gamma_f \frac{\partial z_{elev}}{\partial x_i} = 0 \; .$$

Let deformation and strain be measured from that initial state. Further, assume that all material parameters are spatially uniform.

We begin with the equilibrium equation $\sum_{i=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_i} = 0$, and substitute into it from the stress-strain-pressure relation, $\sigma_{ij} = 2G\varepsilon_{ij} + \left(K - \frac{2G}{3}\right)\delta_{ij}(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) - \alpha\delta_{ij}p$, after expressing strains in terms of displacements by $\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$. The result ***[[please verify]]*** is the set of three partial differential equations for u_1, u_2, u_3 and p:

$$\left(K + \frac{G}{3}\right)\frac{\partial}{\partial x_j}\left(\sum_{i=1}^{3}\frac{\partial u_i}{\partial x_i}\right) + G\nabla^2 u_j - \alpha\frac{\partial p}{\partial x_j} = 0 \qquad \text{(for } j = 1,2,3\text{).}$$

Next we use the fluid mass expression $\Delta m = \rho_f \alpha\left(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} + \frac{\alpha}{K_u - K}p\right)$, again substituting from $\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$, and use Darcy's law $q_i = -\frac{k}{\mu_f}\frac{\partial p}{\partial x_i}$ to substitute into the mass conservation equation $\rho_f \sum_{i=1}^{3}\frac{\partial q_i}{\partial x_i} + \frac{\partial(\Delta m)}{\partial t} = 0$. That gives ***[[please verify]]*** the needed fourth partial differential equation for u_1, u_2, u_3 and p:

$$-\frac{k}{\mu_f}\nabla^2 p + \alpha\frac{\partial}{\partial t}\left(\sum_{i=1}^{3}\frac{\partial u_i}{\partial x_i} + \frac{\alpha}{K_u - K}p\right) = 0.$$

By doing the operation $\frac{\partial}{\partial x_j}$ on each of the first three pde's, and then summing, we notice that $\left(K + \frac{4G}{3}\right)\nabla^2\left(\sum_{i=1}^{3}\frac{\partial u_i}{\partial x_i}\right) - \alpha\nabla^2 p = 0$. It is possible to do a linear combination of this equation and the fourth pde above ***[[please work out the details]]*** to get

$$c\nabla^2\left(\sum_{i=1}^{3}\frac{\partial u_i}{\partial x_i} + \frac{\alpha}{K_u - K}p\right) = \frac{\partial}{\partial t}\left(\sum_{i=1}^{3}\frac{\partial u_i}{\partial x_i} + \frac{\alpha}{K_u - K}p\right),$$

which is a convenient alternative to that fourth equation. In this equation the diffusivity term c is ***[[derive]]***

$$c = \frac{k(K_u - K)(K + 4G/3)}{\mu_f \alpha^2 (K_u + 4G/3)} = \frac{\hat{K}(K_u - K)(K + 4G/3)}{\gamma_f \alpha^2 (K_u + 4G/3)}.$$

[The second version of the result is written in terms of hydraulic conductivity $\hat{K}$ (notation used here to avoid confusion with K for bulk modulus), where $\hat{K} = \gamma_f k / \mu_f$. Compare to $c = \hat{K} / (\gamma_f m_v)$, as derived in the lectures for one-dimensional consolidation in the case of incompressible constituents; "incompressible" means K / K_f and K / K_s

$<<$ 1, in which case $K_u >> K$ and G, so that the above expression for c reduces to $c = \hat{K} / (\gamma_f m_v)$ ***[[show that*; *as part of doing so, you will have do show that*** $K + 4G/3$ ***is indeed the modulus corresponding to*** $1/m_v$ ***in the case of one-dimensional straining]]***.]

Note that the collection of terms on which the differential operators act, in this last pde, is the same on both sides, so the pde is a pure diffusion equation. In fact, the collection of terms is directly proportional to the alteration Δm of fluid mass content. So it is the alteration in fluid mass content, and not generally the pore pressure, which satisfies the diffusion equation in this rigorously developed, coupled theory of deformation and diffusion. That is $c\nabla^2(\Delta m) = \dfrac{\partial(\Delta m)}{\partial t}$ in all cases, but this reduces to an identical pure diffusion equation for p, namely $c\nabla^2 p = \dfrac{\partial p}{\partial t}$, only in special cases (one such case is one-dimensional consolidation under a constant applied stress).

Some additional relations: The above formulation introduces the shear modulus G, the drained bulk modulus K, and two new parameters α and K_u, of which the latter is the undrained bulk modulus, to describe the elastic response of fluid infiltrated materials. Other parameters are sometimes used too.

For example, a measurable material property is the pressure which is induced when stresses are applied under undrained conditions. This response must have the form

$$p = -B \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3}$$

in a linear isotropic elastic material, and the new coefficient which enters is called the *Skempton coefficient*. Note that $0 \leq B \leq 1$, with the upper limit being approached for soil-like materials in which we can consider the fluid and solid constituents to be separately incompressible (in comparison to the compressibility of the material under drained conditions). From the above set of equations, B is given by ***[[please derive]]***

$$B = \frac{K_u - K}{\alpha K_u}$$

Also, it is sometimes convenient to treat G and the Poisson ratio ν as the two primary elastic properties, using the relation $K = \dfrac{2(1+\nu)G}{3(1-2\nu)}$ to replace the drained bulk modulus K with them. In that case a convenient alternative to using the undrained bulk modulus K_u is to use the Poisson ratio ν_u in undrained deformation. An expression for it is ***[[please derive]]***

$$\nu_u = \frac{\nu + \alpha B(1-2\nu)/3}{1 - \alpha B(1-2\nu)/3}$$

and, of course, it is related to K_u by $K_u = \frac{2(1+\nu_u)G}{3(1-2\nu_u)}$. Note that $\nu \le \nu_u \le 1/2$. The upper limit is approached for soil-like materials.

The Wang [2000] book is recommended to see myriad applications of the linear theory presented here.

Some References:

Biot, M. A., General theory of three-dimensional consolidation, *Journal of Applied Physics*, vol. 12, pp. 155-164, 1941

Biot, M. A., "Theory of propagation of elastic waves in a fluid-saturated porous solid, part I: low frequency range" and "part II: higher frequency range", *Journal of the Acoustical Society of America*, vol. 28, pp. 168-178 and 179-191, 1956.

Biot, M. A., Nonlinear and semilinear rheology of porous solids, *Journal of Geophysical Research*, vol.78, no.23, pp.4924-4937, 1973.

Bowen, R. M., "Compressible porous media models by use of the theory of mixtures," *International Journal of Engineering Science*, vol. 20, pp. 697-735, 1982.

Coussy, O., *Mechanics of Porous Continua*, transl. by F. Ulm, John Wiley & Son Ltd., 1995.

Guéguen, Y., and V. Palciauskas, *Introduction to the Physics of Rocks*, Princeton University Press, chapter 6, 1994.

Guéguen, Y., L. Dormieux and M. Boutéca, Fundamentals of poromechanics, in *Mechanics of Fluid Saturated Rocks,* eds. Y. Guéguen and M. Boutéca), Elsevier Academic Press (*Int'l. Geophys. Ser.*, vol. 89), chapter 1, pp. 1-54, 2004.

Rice, J. R., and M. P. Cleary, Some basic stress-diffusion solutions for fluid-saturated elastic porous media with compressible constituents, *Reviews of Geophysics and Space Physics*, vol. 14, pp. 227-241, 1976.

Terzaghi, K., Die berechnung der durchlassigkeitzifer des tones aus dem verlauf der hydrodynamischen spannungserscheinungen, *Akademie der Wissenschaften, Mathematish-naturwissenschaftliche, Klasse,* Vienna, Part IIa, vol. 132, pp. 125-138, 1923.

Terzaghi, K., R. B. Peck and G. Mesri, *Soil Mechanics in Engineering Practice*, 3rd edition, John Wiley and Sons, 1996 (1st and 2nd editions, authored by Terzaghi and Peck, were published in 1948 and 1967, respectively).

Wang, H. F., *Theory of Linear Poroelasticity with Applications to Geomechanics and Hydrogeology*, Princeton Univ. Press, 2000.