Notes on elastodynamics, Green's function, and response to transformation strain and crack or fault sources

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Prepared for Harvard courses Earth and Planetary Sciences 263 (*Earthquake source processes*) and Engineering Sciences 241 (*Advanced elasticity*)

(I) A Derivation of the Elastodynamic Green's Function (Unbounded Isotropic and Homogeneous Medium)

Elastodynamic (Navier) equations:

The Navier equations of motion for a homogeneous and isotropic linear elastic solid are

$$(\lambda+\mu)\nabla(\nabla\cdot\boldsymbol{u})+\mu\nabla^2\boldsymbol{u}+\boldsymbol{f}=\rho\partial^2\boldsymbol{u}/\partial t^2\text{, or}$$

$$(\lambda+\mu)\frac{\partial}{\partial x_\alpha}\frac{\partial u_\nu}{\partial x_\nu}+\mu\frac{\partial^2}{\partial x_\nu\partial x_\nu}u_\alpha+f_\alpha=\rho\partial^2 u_\alpha/\partial t^2\ .$$

Green's function $G_{\nu\beta}(\boldsymbol{x},t)$ is the response to a concentrated impulsive force. That is, $G_{\nu\beta}(\boldsymbol{x},t)$ is the solution for $u_\nu(\boldsymbol{x},t)$ when the body force density $f_\alpha=\delta_{\alpha\beta}\delta_{\text{Dirac}}(t)\delta_{\text{Dirac}}(\boldsymbol{x})$. The solution to the Navier equations is first developed here for $f_\alpha=F_\alpha(t)\delta_{\text{Dirac}}(\boldsymbol{x})$, which represents a time-dependent concentrated force $\boldsymbol{F}(t)$ at $\boldsymbol{x}=\mathbf{0}$.

Observations:

The following observations reduce the determination of the response of a solid to a concentrated force to finding spherically symmetric solutions to a pair of scalar wave equations.

(i) ***Orthogonal operators*** (on an arbitrary vector field $\boldsymbol{v}=\boldsymbol{v}(\boldsymbol{x},t)$) may be defined by:

$$M^p\boldsymbol{v}=\nabla(\nabla\cdot\boldsymbol{v}),\ M^s\boldsymbol{v}=\nabla^2\boldsymbol{v}-\nabla(\nabla\cdot\boldsymbol{v})=-\nabla\times\nabla\times\boldsymbol{v}\ .$$

They have the following properties:

$$M^p(M^s\boldsymbol{v}) = M^s(M^p\boldsymbol{v}) = \mathbf{0}$$
$$M^p(M^p\boldsymbol{v}) = M^p(\nabla^2\boldsymbol{v})$$
$$M^s(M^s\boldsymbol{v}) = M^s(\nabla^2\boldsymbol{v})$$

(ii) ***Navier equations*** can be re-written as

$$\rho c_p^2 M^p\boldsymbol{u} + \rho c_s^2 M^s\boldsymbol{u} + \boldsymbol{f} = \rho\partial^2\boldsymbol{u}/\partial t^2 \qquad (\rho c_p^2 = \lambda + 2\mu\,,\ \rho c_s^2 = \mu\,),$$

and if we write

$$\boldsymbol{u}(\boldsymbol{x},t) = M^p\boldsymbol{A}^p(\boldsymbol{x},t) + M^s\boldsymbol{A}^s(\boldsymbol{x},t)\ ,$$

then those equations become

$$\rho M^p(c_p^2\nabla^2\boldsymbol{A}^p - \partial^2\boldsymbol{A}^p/\partial t^2) + \rho M^s(c_s^2\nabla^2\boldsymbol{A}^s - \partial^2\boldsymbol{A}^s/\partial t^2) + \boldsymbol{f} = \mathbf{0}\ .$$

(iii) ***Poisson's equation*** $\nabla^2\phi = q(\mathbf{x})$ has the general solution

$$\phi = -\frac{1}{4\pi}\int_{\text{all space}}\frac{q(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,dV'.$$

Observe, when $q(\boldsymbol{x}) = \delta_{\text{Dirac}}(\boldsymbol{x})$, $\phi = -1/4\pi r$ (where $r = |\boldsymbol{x}|$), and so $\nabla^2(-1/4\pi r) = \delta_{Dirac}(\boldsymbol{x})$.

(iv) ***Body force*** of interest to us is a point force and can be written as

$$\boldsymbol{f} = \boldsymbol{F}(t)\delta_{\text{Dirac}}(\boldsymbol{x}) = -\nabla^2\left(\frac{\boldsymbol{F}(t)}{4\pi r}\right) = -(M^p + M^s)\left(\frac{\boldsymbol{F}(t)}{4\pi r}\right)$$

Thus, Navier equations will be satisfied if:

$$c_p^2\nabla^2\boldsymbol{A}^p - \partial^2\boldsymbol{A}^p/\partial t^2 = \boldsymbol{F}(t)/4\pi\rho r\,,\ \ c_s^2\nabla^2\boldsymbol{A}^s - \partial^2\boldsymbol{A}^s/\partial t^2 = \boldsymbol{F}(t)/4\pi\rho r\,.$$

Solution for A^p and A^s, and hence for arbitrary concentrated force $\boldsymbol{F}(t)$:

Define a vector function $\boldsymbol{P}(t)$ in terms of the given $\boldsymbol{F}(t)$ by

$$\ddot{\boldsymbol{P}}(t) = \boldsymbol{F}(t)\text{, with } \boldsymbol{P}(0^-) = \dot{\boldsymbol{P}}(0^-) = \mathbf{0}.$$

Then, to solve the two wave equations in **(iv)** above, we want to solve

$$c^2\nabla^2\boldsymbol{A} - \partial^2\boldsymbol{A}/\partial t^2 = \ddot{\boldsymbol{P}}(t)/4\pi\rho r\ .$$

Look for spherically symmetric solutions, in form $\boldsymbol{A} = \boldsymbol{A}(r,t)$, and observe that $r\nabla^2\boldsymbol{A} = \dfrac{\partial^2}{\partial r^2}(r\boldsymbol{A})$ for spherical symmetry.

$$\therefore c^2\frac{\partial^2}{\partial r^2}(r\boldsymbol{A}) - \frac{\partial^2}{\partial t^2}(r\boldsymbol{A}) = \ddot{\boldsymbol{P}}(t)/4\pi\rho\ ,$$

which has the general solution

$$r\boldsymbol{A} = -\boldsymbol{P}(t)/4\pi\rho + \boldsymbol{Q}_1(t - r/c) + \boldsymbol{Q}_2(t + r/c)$$

where $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$ are arbitrary functions. The first term is a *particular* solution; the last two give the general *homogeneous* solution. We set $\boldsymbol{Q}_2 = \mathbf{0}$ (no incoming waves). In order to avoid a singularity in $\boldsymbol{A}$ as $r \to 0$, we must set $\boldsymbol{Q}_1(t) = \boldsymbol{P}(t)/4\pi\rho$.

$\therefore$ Solution is

$$\boldsymbol{A}^p = \frac{\boldsymbol{P}(t - r/c_p) - \boldsymbol{P}(t)}{4\pi\rho r},\quad \boldsymbol{A}^s = \frac{\boldsymbol{P}(t - r/c_s) - \boldsymbol{P}(t)}{4\pi\rho r}\ ,$$

so that the solution for $\boldsymbol{u} = M^p\boldsymbol{A}^p + M^s\boldsymbol{A}^s$ is

$$u_\nu(\boldsymbol{x},t) = -\delta_{\nu\beta}\nabla^2\left(\frac{P_\beta(t)}{4\pi\rho r}\right) \quad \leftarrow \text{ vanishes for } r>0$$

$$+\frac{\partial^2}{\partial x_\nu \partial x_\beta}\left(\frac{P_\beta(t-r/c_p)}{4\pi\rho r}\right)+\left(\delta_{\nu\beta}\nabla^2-\frac{\partial^2}{\partial x_\nu \partial x_\beta}\right)\left(\frac{P_\beta(t-r/c_s)}{4\pi\rho r}\right),$$

where $r=|\boldsymbol{x}|$ and $\ddot{P}_\beta(t)=F_\beta(t)$.

By carrying out the differentiations indicated, one may show further that

$$u_\nu(\boldsymbol{x},t)=\frac{\gamma_\nu\gamma_\beta}{4\pi\rho c_p^2 r}F_\beta(t-r/c_p)+\frac{\delta_{\nu\beta}-\gamma_\nu\gamma_\beta}{4\pi\rho c_s^2 r}F_\beta(t-r/c_s)+\frac{3\gamma_\nu\gamma_\beta-\delta_{\nu\beta}}{4\pi\rho r^3}\int_{r/c_p}^{r/c_s}\tau F_\beta(t-\tau)\,d\tau$$

where here the unit vector $\boldsymbol{\gamma}=\boldsymbol{x}/r$.

Green's function:

$$u_\nu = G_{\nu\beta}(\boldsymbol{x},t) \quad \text{when} \quad F_\alpha(t)=\delta_{\alpha\beta}\delta_{\text{Dirac}}(t)\,,$$

describing an impulsive point force in direction β. The corresponding $P_\alpha(t)=\delta_{\alpha\beta}R(t)$, where

$$R(t) = \text{unit ramp function} = \begin{cases} t & \text{for } t>0 \\ 0 & \text{for } t<0 \end{cases}, \quad \text{and } \ddot{R}(t)=\delta_{\text{Dirac}}(t)\,.$$

$$\therefore\; G_{\nu\beta}(\boldsymbol{x},t)=-\delta_{\nu\beta}\nabla^2\left(\frac{R(t)}{4\pi\rho r}\right)+\frac{\partial^2}{\partial x_\nu\partial x_\beta}\left(\frac{R(t-r/c_p)}{4\pi\rho r}\right)+\left(\delta_{\nu\beta}\nabla^2-\frac{\partial^2}{\partial x_\nu\partial x_\beta}\right)\left(\frac{R(t-r/c_s)}{4\pi\rho r}\right).$$

Static limit of solution:

Let the force $\boldsymbol{F}(t)$ considered above be $\boldsymbol{F}(t)=\boldsymbol{F}$, a constant, for $t>t_F$ and observe that because $\ddot{\boldsymbol{P}}(t)=\boldsymbol{F}(t)$, $\boldsymbol{P}(t)=C_1+C_2t+\boldsymbol{F}t^2/2$ for $t>t_F$ (the constants C_1, C_2 will make no contribution to $G_{\nu\beta}(\boldsymbol{x},t)$ so do not matter). Thus

$$\boldsymbol{P}(t-r/c)-\boldsymbol{P}(t)=-C_2r/c-\boldsymbol{F}rt/c+\boldsymbol{F}r^2/2c^2 \quad \text{whenever } t-r/c>t_F\,.$$

Since $\boldsymbol{P}(t-r/c)-\boldsymbol{P}(t)$ is divided by *r* before taking the spatial derivatives above, only the last term in $\boldsymbol{P}(t-r/c)-\boldsymbol{P}(t)$ contributes to the expression for displacements. Thus the static field (which is established just behind the *s* wave front which emanates from the point of force application at the moment $\boldsymbol{F}(t)$ becomes constant at $\boldsymbol{F}$), is

$$u_\nu(\boldsymbol{x}) = \frac{\partial^2}{\partial x_\nu \partial x_\beta}\left(\frac{F_\beta r}{8\pi\rho c_p^2}\right) + \left(\delta_{\nu\beta}\nabla^2 - \frac{\partial^2}{\partial x_\nu \partial x_\beta}\right)\left(\frac{F_\beta r}{8\pi\rho c_s^2}\right)$$

or, when we recognize the expressions for wave speeds in terms of moduli, and differentiate,

$$u_\nu(\boldsymbol{x}) = \left(\frac{\delta_{\nu\beta} - x_\nu x_\beta / r^2}{\lambda + 2\mu} + \frac{\delta_{\nu\beta} + x_\nu x_\beta / r^2}{\mu}\right)\frac{F_\beta}{8\pi r} = \frac{(\lambda + 3\mu)F_\nu + (\lambda+\mu)x_\nu x_\beta F_\beta / r^2}{8\pi r\mu(\lambda + 2\mu)} \ .$$

(II) Moment Tensor Sources, Transformation Strain Approach

Transformation strain:

Rapid processes of fault slippage, crack opening, dislocation motion, phase changes or local heating generate waves. Such sources of elastic displacement fields can generally be represented, kinematically, by distributions of *transformation strain* $\varepsilon^T_{\alpha\beta}(\boldsymbol{x},t)$; for cracks, faults and dislocations, these are singular distributions (see below). The transformation strain describes an alteration of the *stress free* configuration of a solid. The usual relation between stress and strain, for a solid with elastic moduli $C_{\alpha\beta\gamma\delta}$, is $\sigma_{\alpha\beta} = C_{\alpha\beta\gamma\delta}\varepsilon_{\gamma\delta}$. If the stress-free configuration of an elementary volume of the solid at $\boldsymbol{x}$ is altered to a new shape, described by $\varepsilon^T_{\alpha\beta}(\boldsymbol{x},t)$, the stress-strain relation is then altered to

$$\sigma_{\alpha\beta} = C_{\alpha\beta\gamma\delta}(\varepsilon_{\gamma\delta} - \varepsilon^T_{\gamma\delta}) \qquad (1)$$

[which is $\sigma_{\alpha\beta} = \lambda\delta_{\alpha\beta}(\varepsilon_{\gamma\gamma} - \varepsilon^T_{\gamma\gamma}) + 2\mu(\varepsilon_{\alpha\beta} - \varepsilon^T_{\alpha\beta})$ for an isotropic material].

This assumes no alteration of the elastic moduli due to the transformation strain. Here the strain $\varepsilon_{\gamma\delta}$ is defined in the usual way, by $\varepsilon_{\gamma\delta} = (1/2)(\partial u_\gamma / \partial x_\delta + \partial u_\delta / \partial x_\gamma)$ where $u_\gamma(\boldsymbol{x},t)$ is the displacement field. Thus, if we cut an element of the source region free from its surroundings, and remove stress from it, it would take on the strain $\varepsilon_{\gamma\delta} = \varepsilon^T_{\gamma\delta}$.

The common model of a crack or fault as a surface of displacement discontinuity is achieved as a limit of a distribution of $\varepsilon^T_{\alpha\beta}(\boldsymbol{x},t)$ over a narrow zone. One lets the transformation zone thickness go to zero, with appropriate components of $\varepsilon^T_{\alpha\beta}(\boldsymbol{x},t)$ then going to infinity, such that there is a net displacement across the rupture; this limit is best taken a little later in the theoretical development, after reaching the stage of the integral involving function $H_{\nu\alpha\beta}$ below. In that limit, we write

$$\varepsilon^T_{\alpha\beta} = (1/2)(n_\alpha \Delta u_\beta + n_\beta \Delta u_\alpha)\delta_{Dirac}(S) \tag{2}$$

Here S denotes the fault surface; $\boldsymbol{\Delta u} = \boldsymbol{u}^+ - \boldsymbol{u}^-$ on S ; $+$ and $-$ denote sides of S ; $\boldsymbol{n}$ is the unit normal to S, pointing from $-$ towards $+$; and $\delta_{Dirac}(S)$ is the surface Dirac function, having the property that, if volume ΔV contains a portion ΔS of the surface S, then

$$\int_{\Delta V} f(\boldsymbol{x})\delta_{Dirac}(S)\, dV = \int_{\Delta S} f(\boldsymbol{x})\, dS\,.$$

This provides an alternative approach to one based on applying the elastodynamic reciprocal theorem to a solid with a cut, for representation of the field generated by a crack or fault; that latter approach is discussed in Aki and Richards, Chp. 3.

Two problems:

To understand how to calculate the response to $\varepsilon^T_{\alpha\beta}(\boldsymbol{x},t)$, consider two problems:

Problem 1, response to given body and surface forces: A solid is subjected to some distribution of body force $f_\beta = f_\beta(\boldsymbol{x},t)$ in the region V that it occupies, and to some distribution of surface tractions (or "surface force") $T_\beta = T_\beta(\boldsymbol{x},t)$ on the boundary S of V, but is subjected to *zero transformation strains*, $\varepsilon^T_{\alpha\beta}(\boldsymbol{x},t)=0$. This is the is the classical problem of elastodynamics.

Problem 2, response to given distribution of transformation strain: The same solid discussed above is now subjected to *zero body and surface forces*, $f_\beta(\boldsymbol{x},t)=0$ and $T_\beta(\boldsymbol{x},t)=0$, but is subjected to some arbitrary *non-zero* distribution of transformation strain $\varepsilon^T_{\alpha\beta}(\boldsymbol{x},t)$.

For both problems the governing set of equations are:

(I) The *equations of motion* of a continuum:
$\dfrac{\partial \sigma_{\alpha\beta}}{\partial x_\alpha} + f_\beta = \rho \dfrac{\partial^2 u_\beta}{\partial t^2}$ in the region V, with boundary conditions $n_\alpha \sigma_{\alpha\beta} = T_\beta$ on the surface S.

(II) The *strain - displacement gradient* relations: $\varepsilon_{\gamma\delta} = \dfrac{1}{2}(\dfrac{\partial u_\gamma}{\partial x_\delta} + \dfrac{\partial u_\delta}{\partial x_\gamma})$.

(III) The *(elastic) stress-strain relations* of eq. (1): $\sigma_{\alpha\beta} = C_{\alpha\beta\gamma\delta}(\varepsilon_{\gamma\delta} - \varepsilon^T_{\gamma\delta})$. [Note that since $C_{\alpha\beta\gamma\delta}$ is chosen symmetric in its last two indices, use of (II) shows $C_{\alpha\beta\gamma\delta}\varepsilon_{\gamma\delta} = C_{\alpha\beta\gamma\delta}\partial u_\gamma / \partial x_\delta$.]

Let us now formulate both problems *in terms of displacements as variables*:

Formulation, problem 1, response to given body and surface forces:

(II) and (III) above with $\varepsilon^T_{\alpha\beta} = 0$ lead to $\sigma_{\alpha\beta} = C_{\alpha\beta\gamma\delta}\partial u_\gamma / \partial x_\delta$ for the stresses. Inserting that into (I) we obtain the following statement of the problem of determining the displacement field:

$$\frac{\partial}{\partial x_\alpha}(C_{\alpha\beta\gamma\delta}\frac{\partial u_\gamma}{\partial x_\delta}) + f_\beta = \rho \frac{\partial^2 u_\beta}{\partial t^2} \quad \text{in } V, \text{ subject to } n_\alpha C_{\alpha\beta\gamma\delta}\frac{\partial u_\gamma}{\partial x_\delta} = T_\beta \text{ on } S. \qquad (3)$$

where we regard $f_\beta = f_\beta(\boldsymbol{x},t)$ and $T_\beta = T_\beta(\boldsymbol{x},t)$ as given functions.

Note that since the governing equations are linear in the $u_\gamma(\boldsymbol{x},t)$, their solution for the u_γ must involve some linear summation over all space and prior time of the effects of the given f_β and T_β. The *Green's function* $G_{\nu\beta}(\boldsymbol{x},\boldsymbol{x}',t)$ may then be *defined* (consistently with the understanding in part **(I)** above) as the weighting coefficient in such a linear response, so that we can write the solution to the above set of equations as

$$u_\nu(\boldsymbol{x},t) = \int_{-\infty}^{t} \int_V G_{\nu\beta}(\boldsymbol{x},\boldsymbol{x}',t-t')f_\beta(\boldsymbol{x}',t')\, dV'dt' + \int_{-\infty}^{t} \int_S G_{\nu\beta}(\boldsymbol{x},\boldsymbol{x}',t-t')T_\beta(\boldsymbol{x}',t')\, dS'dt' \quad (4)$$

The coefficient of $G_{\nu\beta}(\boldsymbol{x},\boldsymbol{x}',t-t')$ is an infinitesimal impulse, $f_\beta(\boldsymbol{x}',t')dV'dt'$ or $T_\beta(\boldsymbol{x}',t')dS'dt'$, applied at position $\boldsymbol{x}'$ and time t'. We may thus say that $G_{\nu\beta}(\boldsymbol{x},\boldsymbol{x}',t)$ is the displacement response in the ν direction at place $\boldsymbol{x}$ and time t due to a unit impulse applied in the β direction at place $\boldsymbol{x}'$ and time 0.

Formulation, problem 2, response to given distribution of transformation strain:

(II) and (III) now lead to $\sigma_{\alpha\beta} = C_{\alpha\beta\gamma\delta}(\frac{\partial u_\gamma}{\partial x_\delta} - \varepsilon^T_{\gamma\delta})$ for the stresses, and we insert that into (I) with f_β=0 and T_β=0 to get the following equations governing the displacement field:

$$\frac{\partial}{\partial x_\alpha}[C_{\alpha\beta\gamma\delta}(\frac{\partial u_\gamma}{\partial x_\delta} - \varepsilon^T_{\gamma\delta})] = \rho\frac{\partial^2 u_\beta}{\partial t^2} \text{ in } V, \text{ with } n_\alpha C_{\alpha\beta\gamma\delta}(\frac{\partial u_\gamma}{\partial x_\delta} - \varepsilon^T_{\gamma\delta}) = 0 \text{ on } S. \qquad (5a)$$

Introducing the notation

$$m_{\alpha\beta}(\boldsymbol{x},t) = C_{\alpha\beta\gamma\delta}(\boldsymbol{x})\varepsilon^T_{\gamma\delta}(\boldsymbol{x},t) \ ,$$

where $m_{\alpha\beta}$ is called the *moment (volume) density tensor*, we therefore see that *problem 2* can be restated as

$$\frac{\partial}{\partial x_\alpha}(C_{\alpha\beta\gamma\delta}\frac{\partial u_\gamma}{\partial x_\delta}) - \frac{\partial m_{\alpha\beta}}{\partial x_\alpha} = \rho\frac{\partial^2 u_\beta}{\partial t^2} \text{ in } V, \text{ with } n_\alpha C_{\alpha\beta\gamma\delta}\frac{\partial u_\gamma}{\partial x_\delta} = n_\alpha m_{\alpha\beta} \text{ on } S. \qquad (5b)$$

This statement of *problem 2* can be re-written (to emphasize the analogy to *problem 1*) as:

$$\frac{\partial}{\partial x_\alpha}(C_{\alpha\beta\gamma\delta}\frac{\partial u_\gamma}{\partial x_\delta}) + f^{eff}_\beta = \rho\frac{\partial^2 u_\beta}{\partial t^2} \text{ in } V, \text{ with } n_\alpha C_{\alpha\beta\gamma\delta}\frac{\partial u_\gamma}{\partial x_\delta} = T^{eff}_\beta \text{ on } S. \qquad (5c)$$

where the effective body and surface force terms thus introduced are

$$f^{eff}_\beta = -\partial m_{\alpha\beta}/\partial x_\alpha \text{ and } T^{eff}_\beta = n_\alpha m_{\alpha\beta} \ . \qquad (6)$$

Comparing *problem 2* as formulated in eq. (5c) and (6) with *problem 1* as formulated in eq. (3), we see that both have an identical mathematical statement (but with f_β^{eff} and T_β^{eff} of *problem 2* replacing f_β and T_β of *problem 1*). Hence *problem 2* must have an identical form of solution which can be can be written out directly from eq. (4), in terms of the Green's function, as

$$u_\nu(\boldsymbol{x},t) = -\int_{-\infty}^{t} \int_V G_{\nu\beta}(\boldsymbol{x},\boldsymbol{x}',t-t') \frac{\partial m_{\alpha\beta}(\boldsymbol{x}',t')}{\partial x'_\alpha}\, dV'dt' \\ + \int_{-\infty}^{t} \int_S G_{\nu\beta}(\boldsymbol{x},\boldsymbol{x}',t-t') n_\alpha(\boldsymbol{x}') m_{\alpha\beta}(\boldsymbol{x}',t')\, dS'dt' \,. \tag{7}$$

Use of the divergence theorem to transform the surface integral to a volume integral, then shows that the solution to *problem 2* is

$$u_\nu(\boldsymbol{x},t) = \int_{-\infty}^{t} \int_V \frac{\partial G_{\nu\beta}(\boldsymbol{x},\boldsymbol{x}',t-t')}{\partial x'_\alpha} m_{\alpha\beta}(\boldsymbol{x}',t')\, dV'dt' \tag{8}$$

Since $m_{\alpha\beta}$ is symmetric in α and β (that is because $C_{\alpha\beta\gamma\delta}$ is symmetric in its first two indices, which follows from $\sigma_{\alpha\beta} = \sigma_{\beta\alpha}$), it is preferable to rewrite this solution as

$$u_\nu(\boldsymbol{x},t) = \frac{1}{2} \int_{-\infty}^{t} \int_V H_{\nu\alpha\beta}(\boldsymbol{x},\boldsymbol{x}',t-t') m_{\alpha\beta}(\boldsymbol{x}',t')\, dV'dt' \tag{9}$$

where the moment response function is

$$H_{\nu\alpha\beta}(\boldsymbol{x},\boldsymbol{x}',t-t') = \frac{\partial G_{\nu\alpha}(\boldsymbol{x},\boldsymbol{x}',t-t')}{\partial x'_\beta} + \frac{\partial G_{\nu\beta}(\boldsymbol{x},\boldsymbol{x}',t-t')}{\partial x'_\alpha} \tag{10}$$

$H_{\nu\alpha\beta}(\boldsymbol{x},\boldsymbol{x}',t-t')$ can be interpreted as the displacement u_ν at $\boldsymbol{x}$ in response to the application at $\boldsymbol{x}'$ of a *pair of impulsive force dipoles with zero net moment*. One such dipole is arrayed along the β direction with its impulses in the α direction, the other arrayed along the α direction with its impulses in the β direction. $H_{\nu\alpha\beta}(\boldsymbol{x},\boldsymbol{x}',t-t')$ is called the *double couple* response when α and β differ, and the *linear vector dipole* response when they agree.

Surface source region *S*:

Suppose that the source is a crack or fault so that the transformation strain is a singular distribution over some internal surface *S*, like in equation (2). (This *S* is not to be confused with alternative use of the same symbol to denote the external surface of the body considered.) Then the volume moment density is

$$m_{\alpha\beta}(\boldsymbol{x},t) = \hat{m}_{\alpha\beta}(\boldsymbol{x},t)\delta_{Dirac}(S) \text{ , where } \hat{m}_{\alpha\beta}(\boldsymbol{x},t) = C_{\alpha\beta\gamma\delta}(\boldsymbol{x})n_{\gamma}(\boldsymbol{x})\Delta u_{\delta}(\boldsymbol{x},t) \qquad (11)$$

is the moment *surface* density tensor. The only change in expression above for $\boldsymbol{u}$ is that we replace $m_{\alpha\beta}(\boldsymbol{x}',t')\,dV'$ in equation (9) with $\hat{m}_{\alpha\beta}(\boldsymbol{x}',t')\,dS'$, and integrate over the surface *S* rather than over volume *V*, making analogous changes in subsequent formulae.

Fourier transform version:

It is convenient for some purposes to have the Fourier transform of the displacement, which is given as

$$\tilde{u}_{\nu}(\boldsymbol{x},\omega) = \frac{1}{2}\int_{V} \tilde{H}_{\nu\alpha\beta}(\boldsymbol{x},\boldsymbol{x}',\omega)\tilde{m}_{\alpha\beta}(\boldsymbol{x}',\omega)\,dV' \,. \qquad (12a)$$

Observe also that the transform of the moment rate $\dot{m}_{\alpha\beta}(\boldsymbol{x},t)$ is

$$\tilde{\dot{m}}_{\alpha\beta}(\boldsymbol{x},\omega) = i\omega\tilde{m}_{\alpha\beta}(\boldsymbol{x},\omega) \,,$$

and that if we define $\tilde{E}_{\nu\alpha\beta}(\boldsymbol{x},\boldsymbol{x}',\omega) = \tilde{H}_{\nu\alpha\beta}(\boldsymbol{x},\boldsymbol{x}',\omega)/\,i\omega$, then

$$E_{\nu\alpha\beta}(\boldsymbol{x},\boldsymbol{x}',t) = \int_0^t H_{\nu\alpha\beta}(\boldsymbol{x},\boldsymbol{x}',t'')\,dt''$$

is the function analogous to the moment response $H_{\nu\alpha\beta}(\boldsymbol{x},\boldsymbol{x}',t)$, but based on a step-function rather than impulsive time history. That is, $E_{\nu\alpha\beta}(\boldsymbol{x},\boldsymbol{x}',t)$ can be calculated like in equation (10), but replacing $G_{\alpha\beta}(\boldsymbol{x},\boldsymbol{x}',t)$ with the function which gives the displacement $u_{\alpha}(\boldsymbol{x},t)$ in response to the body force density $f_{\nu}(\boldsymbol{x},t) = \delta_{\nu\beta}U_{step}(t)\delta_{Dirac}(\boldsymbol{x}-\boldsymbol{x}')$ where $U_{step}(t)$ is the unit step function; recall that $G_{\alpha\beta}(\boldsymbol{x},\boldsymbol{x}',t)$ corresponds to $f_{\nu}(\boldsymbol{x},t) = \delta_{\nu\beta}\delta_{Dirac}(t)\delta_{Dirac}(\boldsymbol{x}-\boldsymbol{x}')$.

Thus an equivalent form for the displacement field is

$$\tilde{u}_V(\boldsymbol{x},\omega) = \frac{1}{2}\int_V \tilde{E}_{V\alpha\beta}(\boldsymbol{x},\boldsymbol{x}',\omega)\tilde{\dot{m}}_{\alpha\beta}(\boldsymbol{x}',\omega)\, dV' \ , \tag{12b}$$

which corresponds in the time domain to the equivalent version of equation (9) as

$$u_V(\boldsymbol{x},t) = \frac{1}{2}\int_{-\infty}^{t}\int_V E_{V\alpha\beta}(\boldsymbol{x},\boldsymbol{x}',t-t')\dot{m}_{\alpha\beta}(\boldsymbol{x}',t')\, dV'dt' \tag{12c}$$

Low-frequency response and point source approximation:

Let a be a typical dimension of the source region, such that $t_w = a/c$, where c = wave speed, is a typical time for waves to traverse the source. If the elastic properties and geometry of the body are relatively uniform through the source region, such that the response $E_{V\alpha\beta}(\boldsymbol{x},\boldsymbol{x}',t)$ at a distant receiver site $\boldsymbol{x}$ is essentially independent of the location of $\boldsymbol{x}'$ in the source region (ignoring travel time differences of the order of t_w), then the source can be considered as if it was a *point source*. This neglect of accuracy of order t_w in the time dependence can be characterized, in the frequency domain, as being valid at low frequencies ω, such that $|\omega| t_w << 1$. We may then write the response as given in equations (9b,c) as

$$\tilde{u}_V(\boldsymbol{x},\omega) \approx \frac{1}{2}\tilde{E}_{V\alpha\beta}(\boldsymbol{x},\boldsymbol{x}',\omega)\tilde{\dot{M}}_{\alpha\beta}(\omega) \ , \text{ or } \ u_V(\boldsymbol{x},t) \approx \frac{1}{2}\int_{-\infty}^{t} E_{V\alpha\beta}(\boldsymbol{x},\boldsymbol{x}',t-t')\dot{M}_{\alpha\beta}(t')dt' ,$$

where $\boldsymbol{x}'$ is any location in the source region, and where

$$M_{\alpha\beta}(t) = \int_V m_{\alpha\beta}(\boldsymbol{x}',t)\, dV'$$

is the *total moment* of the source. In general, if r is distance from source to receiver, and if there is enough uniformity in the source region for the independence of the exact source locations $\boldsymbol{x}'$, then the point source model is valid when

$$r >> a \quad \text{and} \quad c/|\omega| >> a \ ,$$

where the latter expresses the condition $|\omega| t_w << 1$.

The time scale t_s of the source process is the time over which the $\dot{\varepsilon}^T_{\alpha\beta}(\boldsymbol{x}',t)$, and hence $\dot{m}_{\alpha\beta}(\boldsymbol{x}',t)$, are non-zero. Suppose t_s is short compared to the precision with which we want to know the time history. Such is the case when we consider low frequencies such that $|\omega| t_s << 1$. In that frequency range,

$$\tilde{m}_{\alpha\beta}(\boldsymbol{x}',\omega) = \int_{-\infty}^{+\infty} e^{-i\omega t} \dot{m}_{\alpha\beta}(\boldsymbol{x}',t)\, dt \approx m_{\alpha\beta}(\boldsymbol{x}',t_s) \quad (|\omega| t_s << 1)$$

where $m_{\alpha\beta}(\boldsymbol{x}',t_s)$ is the final moment density at the termination of the source process. Hence (9b,c) become, in that low-frequency range,

$$\tilde{u}_\nu(\boldsymbol{x},\omega) \approx \frac{1}{2}\int_V \tilde{E}_{\nu\alpha\beta}(\boldsymbol{x},\boldsymbol{x}',\omega) m_{\alpha\beta}(\boldsymbol{x}',t_s)\, dV', \text{ or } u_\nu(\boldsymbol{x},t) \approx \frac{1}{2}\int_V E_{\nu\alpha\beta}(\boldsymbol{x},\boldsymbol{x}',t) m_{\alpha\beta}(\boldsymbol{x}',t_s)\, dV'.$$

In general, for dynamic rupture processes, one expects t_w to be less than t_s. Thus frequencies which are low enough to justify the approximation just made will also be low enough for validity of the point source model, at least if we also meet the condition of enough uniformity in the source region for the independence of the exact locations $\boldsymbol{x}'$. In that case we may write

$$\tilde{u}_\nu(\boldsymbol{x},\omega) \approx \frac{1}{2}\tilde{E}_{\nu\alpha\beta}(\boldsymbol{x},\boldsymbol{x}',\omega) M_{\alpha\beta}(t_\mathrm{s}) \text{ , or } u_\nu(\boldsymbol{x},t) \approx \frac{1}{2} E_{\nu\alpha\beta}(\boldsymbol{x},\boldsymbol{x}',t) M_{\alpha\beta}(t_\mathrm{s}) \text{ .}$$

At such level of approximation, the expressions show no effects of the actual time dependence of the source process, but only the time dependence embedded in $E_{\nu\alpha\beta}(\boldsymbol{x},\boldsymbol{x}',t)$, possibly reflecting multiple wave reflections, scattering and dispersion on the route from source to receiver.

(III) Moment Tensor Response, Isotropic and Homogeneous Solid

The displacement in response to a distribution of transformation strain $\varepsilon^T_{\alpha\beta}(\boldsymbol{x},t)$, with property that the stress-strain relations of the medium are altered to

$$\sigma_{\alpha\beta} = \lambda\delta_{\alpha\beta}(\varepsilon_{\gamma\gamma} - \varepsilon^T_{\gamma\gamma}) + 2\mu(\varepsilon_{\alpha\beta} - \varepsilon^T_{\alpha\beta}),$$

is given in terms of the moment tensor source density

$$m_{\alpha\beta}(\mathbf{x},t) = \lambda\delta_{\alpha\beta}\varepsilon^T_{\gamma\gamma}(\mathbf{x},t) + 2\mu\varepsilon^T_{\alpha\beta}(\mathbf{x},t)$$

by the expressions above, i.e., equation (9) with the Green's function of part **(I)** for the unbounded isotropic and homogeneous medium used in equation (10). Because, in the present case, we consider an unbounded homogeneous solid, $G_{\nu\alpha}(\mathbf{x},\mathbf{x}',t) = G_{\nu\alpha}(\mathbf{x}-\mathbf{x}',t)$. Thus $H_{\nu\alpha\beta}(\boldsymbol{x},\boldsymbol{x}',t) = H_{\nu\alpha\beta}(\boldsymbol{x}-\boldsymbol{x}',t)$, which means that $\partial G_{\nu\alpha}(\mathbf{x}-\mathbf{x}',t)/\partial x'_\beta = -\partial G_{\nu\alpha}(\mathbf{x}-\mathbf{x}',t)/\partial x_\beta$. Thus the moment response is

$$H_{\nu\alpha\beta}(\mathbf{x},t) = -\frac{\partial}{\partial x_\alpha} G_{\nu\beta}(\mathbf{x},t) - \frac{\partial}{\partial x_\beta} G_{\nu\alpha}(\mathbf{x},t) .$$

Using the expression above for the Green's function $G_{\nu\beta}$,

$$\frac{1}{2} H_{\nu\alpha\beta}(\mathbf{x},t) = (L^p_{\nu\alpha\beta} + L^s_{\nu\alpha\beta})\left(\frac{R(t)}{4\pi\rho r}\right) - L^p_{\nu\alpha\beta}\left(\frac{R(t-r/c_p)}{4\pi\rho r}\right) - L^s_{\nu\alpha\beta}\left(\frac{R(t-r/c_s)}{4\pi\rho r}\right)$$

where

$$L^p_{\nu\alpha\beta} = \frac{\partial^3}{\partial x_\nu \partial x_\alpha \partial x_\beta} \quad , \quad L^s_{\nu\alpha\beta} = \frac{1}{2}(\delta_{\nu\alpha}\frac{\partial}{\partial x_\beta} + \delta_{\nu\beta}\frac{\partial}{\partial x_\alpha})\nabla^2 - L^p_{\nu\alpha\beta} \quad .$$

Note that the first term of $H_{\nu\alpha\beta}$ will make no contribution outside the source; $(L^p + L^s)(R/4\pi\rho r) = 0$ there because $\nabla^2(1/r) = 0$ for $r>0$.

To evaluate the expression for $u_\nu(\boldsymbol{x},t)$ it is simplest to solve first for $\partial^2 u_\nu(\boldsymbol{x},t)/\partial t^2$, in the representation of eq. (9), observing that

$$\frac{\partial^2}{\partial t^2}\left(\frac{1}{2}H_{\nu\alpha\beta}\right) = \text{ same expression as for } \frac{1}{2}H_{\nu\alpha\beta} \text{ above, but with } R(t) \to \ddot{R}(t) = \delta_{\text{Dirac}}(t) \ .$$

Note now that when performing the convolution on t' of $\dfrac{\partial^2}{\partial t^2}\left(\dfrac{1}{2}H_{\nu\alpha\beta}(\mathbf{x},\mathbf{x}',t-t')\right)$ with $m_{\alpha\beta}(\mathbf{x}',t')$, there will arise integrals $\int_{-\infty}^{t} \delta_{\text{Dirac}}(t-t'-r/c) m_{\alpha\beta}(\mathbf{x}',t')dt' = m_{\alpha\beta}(\mathbf{x}',t-r/c)$, where $r = |\boldsymbol{x}-\boldsymbol{x}'|$. Thus

$$\frac{\partial^2}{\partial t^2} u_\nu(\mathbf{x},t) = -L^p_{\nu\alpha\beta} \int\limits_{\substack{source\\ volume}} \frac{m_{\alpha\beta}(\mathbf{x}', t - r/c_p)}{4\pi\rho r} dV' - L^s_{\nu\alpha\beta} \int\limits_{\substack{source\\ volume}} \frac{m_{\alpha\beta}(\mathbf{x}', t - r/c_s)}{4\pi\rho r} dV' \ .$$

This gives the solution for $u_\nu(\boldsymbol{x},t)$ using initial conditions $u_\nu = \partial u_\nu / \partial t = 0$ before onset of the source process. To formally write the expression for $u_\nu(\boldsymbol{x},t)$, let $W_{\alpha\beta}(\boldsymbol{x},t)$ be defined by

$$\frac{\partial^2}{\partial t^2} W_{\alpha\beta}(\boldsymbol{x},t) = m_{\alpha\beta}(\boldsymbol{x},t) \text{, with } W_{\alpha\beta}(\boldsymbol{x},0) = \frac{\partial}{\partial t} W_{\alpha\beta}(\boldsymbol{x},0) = 0 \ .$$

Such $W_{\alpha\beta}(\boldsymbol{x},t)$ is given by $W_{\alpha\beta}(\boldsymbol{x},t) = \int_0^t (t - t'') m_{\alpha\beta}(\boldsymbol{x},t'') dt''$. In terms of it,

$$u_\nu(\boldsymbol{x},t) = -L^p_{\nu\alpha\beta} \int\limits_{\substack{source\\ volume}} \frac{W_{\alpha\beta}(\boldsymbol{x}', t - r/c_p)}{4\pi\rho r} dV' - L^s_{\nu\alpha\beta} \int\limits_{\substack{source\\ volume}} \frac{W_{\alpha\beta}(\boldsymbol{x}', t - r/c_s)}{4\pi\rho r} dV' \ .$$

Evidently, the displacement field can be written in the form

$$u_\nu(\boldsymbol{x},t) = u^p_\nu(\boldsymbol{x},t) + u^s_\nu(\boldsymbol{x},t) \ ,$$

and in terms of these notations the last expression would, e.g., be written

$$u^{p \text{ or } s}_\nu(\boldsymbol{x},t) = -L^{p \text{ or } s}_{\nu\alpha\beta} \int\limits_{\substack{source\\ volume}} \frac{W_{\alpha\beta}(\boldsymbol{x}', t - r/c_{p \text{ or } s})}{4\pi\rho r} dV' \ ,$$

with the Fourier transform version being

$$\tilde{u}^{p \text{ or } s}_\nu(\boldsymbol{x},\omega) = L^{p \text{ or } s}_{\nu\alpha\beta} \int\limits_{\substack{source\\ volume}} \frac{\exp(-i\omega r / c_{p \text{ or } s}) \tilde{m}_{\alpha\beta}(\boldsymbol{x}',\omega)}{4\pi\rho\omega^2 r} dV' \ .$$

Rupture on a surface *S*:

In this case the transformation strain is given by equation (2),

$$\varepsilon^T_{\alpha\beta}(\boldsymbol{x},t) = (1/2)[n_\alpha(\boldsymbol{x})\Delta u_\beta(\boldsymbol{x},t) + n_\beta(\boldsymbol{x})\Delta u_\alpha(\boldsymbol{x},t)]\delta_{Dirac}(S)$$

where $\boldsymbol{n}$ is the unit normal to *S*, pointing from the – to + side, and $\Delta\boldsymbol{u} = \boldsymbol{u}^+ - \boldsymbol{u}^-$ is the displacement discontinuity on *S*. Thus the *surface* moment density tensor $\hat{m}_{\alpha\beta}(\boldsymbol{x},t)$ is given by

$$\hat{m}_{\alpha\beta}(\boldsymbol{x},t) = \lambda\delta_{\alpha\beta}n_\gamma(\boldsymbol{x})\Delta u_\gamma(\boldsymbol{x},t) + \mu[n_\alpha(\boldsymbol{x})\Delta u_\beta(\boldsymbol{x},t) + n_\beta(\boldsymbol{x})\Delta u_\alpha(\boldsymbol{x},t)]\,.$$

Since in such cases we replace $m_{\alpha\beta}(\boldsymbol{x}',t')\,dV'$ in equation (9) with $\hat{m}_{\alpha\beta}(\boldsymbol{x}',t')\,dS'$, the displacement field is given by

$$u_\nu^{p \text{ or } s}(\boldsymbol{x},t) = -L_{\nu\alpha\beta}^{p \text{ or } s}\int_S \frac{\hat{W}_{\alpha\beta}(\boldsymbol{x}',t-r/c_{p \text{ or } s})}{4\pi\rho r}\,dS'\ ,$$

where

$$\frac{\partial^2}{\partial t^2}\hat{W}_{\alpha\beta}(\mathbf{x},t) = \hat{m}_{\alpha\beta}(\mathbf{x},t) \ \text{ with } \ \hat{W}_{\alpha\beta}(\mathbf{x},0) = \frac{\partial}{\partial t}\hat{W}_{\alpha\beta}(\mathbf{x},0) = 0\,,$$

and the Fourier transform version is

$$\tilde{u}_\nu^{p \text{ or } s}(\boldsymbol{x},\omega) = L_{\nu\alpha\beta}^{p \text{ or } s}\int_S \frac{\exp(-i\omega r/c_{p \text{ or } s})\tilde{\hat{m}}_{\alpha\beta}(\boldsymbol{x}',\omega)}{4\pi\rho\omega^2 r}\,dS'.$$

(IV) High-Frequency Response and Far-Field Approximation

Assume $r >> a$ (a = source dimension), and $r >> c/\omega$. Then for such range of (high) frequencies

$$\frac{\partial}{\partial x_\mu}\left(\frac{e^{-i\omega r/c}}{r}\right) = -\frac{i\omega}{c}\gamma_\mu\left(\frac{e^{-i\omega r/c}}{r}\right)\left[1-\frac{ic}{\omega r}\right] \approx -\frac{i\omega}{c}\gamma_\mu\left(\frac{e^{-i\omega r/c}}{r}\right)$$

where $\boldsymbol{\gamma} = \dfrac{\partial r}{\partial \mathbf{x}} = \dfrac{\mathbf{x} - \mathbf{x}'}{r} \approx \dfrac{\mathbf{x}}{r_o}$ for a coordinate origin in the source region and observation point at distance r_o from the origin. Hence, each time we take a derivative with respect to some x_μ, which we do successively three times in operating with the $L_{\nu\alpha\beta}^{p \text{ or } s}$ above, that is equivalent in the high frequency limit to multiplying by $-i\omega\gamma_\mu / c_{p \text{ or } s}$. Thus the Fourier transform expression above becomes

$$\tilde{u}_\nu^{p \text{ or } s}(\boldsymbol{x},\omega) \approx R_{\nu\alpha\beta}^{p \text{ or } s} \int\limits_{\substack{source\\volume}} \frac{\exp(-i\omega r / c_{p \text{ or } s})}{4\pi\rho r c_{p \text{ or } s}^3} \tilde{\dot{m}}_{\alpha\beta}(\boldsymbol{x}',\omega)\ dV' \ ,$$

(using $\tilde{\dot{m}}_{\alpha\beta}(\boldsymbol{x}',\omega) = i\omega\tilde{m}_{\alpha\beta}(\boldsymbol{x}',\omega)$) where the *radiation patterns* $R_{\nu\alpha\beta}^{p}$ and $R_{\nu\alpha\beta}^{s}$, descended from the two differential operators, are

$$R_{\nu\alpha\beta}^{p} = \gamma_\nu\gamma_\alpha\gamma_\beta, \ \text{ and } \ R_{\nu\alpha\beta}^{s} = \frac{1}{2}(\delta_{\nu\alpha}\gamma_\beta + \delta_{\nu\beta}\gamma_\alpha) - \gamma_\nu\gamma_\alpha\gamma_\beta .$$

By noting further that $r \approx r_o - \boldsymbol{\gamma}\cdot\boldsymbol{x}'$, when $|\boldsymbol{\gamma}\cdot\boldsymbol{x}'| << r_o$, which is the case here, this can also be written as

$$\tilde{u}_\nu^{p \text{ or } s}(\boldsymbol{x},\omega) \approx R_{\nu\alpha\beta}^{p \text{ or } s} \frac{\exp(-i\omega r_o / c_{p \text{ or } s})}{4\pi\rho r_o c_{p \text{ or } s}^3} \int\limits_{\substack{source\\volume}} \exp(i\omega\boldsymbol{\gamma}\cdot\boldsymbol{x}' / c_{p \text{ or } s})\tilde{\dot{m}}_{\alpha\beta}(\boldsymbol{x}',\omega)\ dV' .$$

The last two expressions for $\tilde{u}_\nu^{p \text{ or } s}(\boldsymbol{x},\omega)$ invert to the time domain as

$$\begin{aligned} u_\nu^{p \text{ or } s}(\boldsymbol{x},t) &\approx R_{\nu\alpha\beta}^{p \text{ or } s} \int\limits_{\substack{source\\volume}} \frac{\dot{m}_{\alpha\beta}(\boldsymbol{x}',t - r / c_{p \text{ or } s})}{4\pi\rho r c_{p \text{ or } s}^3}\ dV' \\ &\approx R_{\nu\alpha\beta}^{p \text{ or } s} \frac{1}{4\pi\rho r_o c_{p \text{ or } s}^3} \int\limits_{\substack{source\\volume}} \dot{m}_{\alpha\beta}(\boldsymbol{x}',t - r_o / c_{p \text{ or } s} + \boldsymbol{\gamma}\cdot\boldsymbol{x}' / c_{p \text{ or } s})\ dV' \ , \end{aligned}$$

where now it is understood that the slower (low frequency) parts of the radiated field may not be well represented (for example, these expressions predict no long-term static displacement field).

The equations do show that the high-frequency part of the radiated signal is a direct linear map, corrected for travel-time differences, of the rate of transformation strain in the source region.

For the case of rupture with displacement discontinuities $\Delta u_\alpha = u_\alpha^+ - u_\alpha^- = \Delta u_\alpha(\mathbf{x},t)$ on a surface S, the above integrals over the source region become

$$\begin{aligned}
&\int_{\text{source volume}} (1/4\pi r\rho c^3_{p\text{ or }s})\dot{m}_{\alpha\beta}(\boldsymbol{x}',t-r/c_{p\text{ or }s})\,dV' \\
&\quad= \int_S (1/4\pi r\rho c^3_{p\text{ or }s})\{\lambda\delta_{\alpha\beta}n_\gamma(\boldsymbol{x}')\Delta\dot{u}_\gamma(\boldsymbol{x}',t-r/c_{p\text{ or }s}) \\
&\qquad+\mu[n_\alpha(\boldsymbol{x}')\Delta\dot{u}_\beta(\boldsymbol{x}',t-r/c_{p\text{ or }s})+n_\beta(\boldsymbol{x}')\Delta\dot{u}_\alpha(\boldsymbol{x}',t-r/c_{p\text{ or }s})]\}dS' \\
&\quad= \int_S (1/4\pi r c^3_{p\text{ or }s})\{(c_p^2-2c_s^2)\delta_{\alpha\beta}n_\gamma(\boldsymbol{x}')\Delta\dot{u}_\gamma(\boldsymbol{x}',t-r/c_{p\text{ or }s}) \\
&\qquad+c_s^2[n_\alpha(\boldsymbol{x}')\Delta\dot{u}_\beta(\boldsymbol{x}',t-r/c_{p\text{ or }s})+n_\beta(\boldsymbol{x}')\Delta\dot{u}_\alpha(\boldsymbol{x}',t-r/c_{p\text{ or }s})]\}dS'\,.
\end{aligned}$$

Far-field radiation due to slip on a planar fault:

The fault plane S is in the plane x_2=0; slip is entirely in the 1 direction; $\boldsymbol{\Delta u} = (\Delta u_1,0,0)$, $\boldsymbol{n} = (0,1,0)$; $\Delta u_1=\Delta u_1(x_1,x_3,t)$; $\hat{m}_{12}(\boldsymbol{x},t) = \hat{m}_{21}(\boldsymbol{x},t) = \mu\Delta u_1(x_1,x_3,t) = \rho c_s^2\Delta u_1(x_1,x_3,t)$. Then the far-field p and s wave displacements are

$$u_\nu^{p\text{ or }s}(\boldsymbol{x},t) \approx \frac{R_{\nu 12}^{p\text{ or }s}c_s^2}{2\pi r_o c^3_{p\text{ or }s}}\Omega(\boldsymbol{\gamma},t-r_o/c_{p\text{ or }s};c_{p\text{ or }s})$$

where

$$\Omega(\boldsymbol{\gamma},t;c) = \int_S \Delta\dot{u}_1(x_1,x_3,t+\frac{\gamma_1x_1+\gamma_3x_3}{c})dx_1dx_3\ , \quad \text{or}$$

$$\tilde{\Omega}(\boldsymbol{\gamma},\omega;c) = \int_S \exp(i\omega\frac{\gamma_1x_1+\gamma_3x_3}{c})\Delta\tilde{\dot{u}}_1(x_1,x_3,\omega)dx_1dx_3$$

Far-field radiation due to tensile opening on a planar crack:

The crack plane S is in the plane x_2=0; relative displacement of the crack walls is entirely in the opening mode, and hence in the 2 direction; $\boldsymbol{\Delta u} = (0,\Delta u_2,0)$, $\boldsymbol{n} = (0,1,0)$; $\Delta u_2 = \Delta u_2(x_1,x_3,t)$; $\hat{m}_{11}(\boldsymbol{x},t) = \hat{m}_{33}(\boldsymbol{x},t) = \lambda\Delta u_2(x_1,x_3,t) = \rho(c_p^2 - 2c_s^2)\Delta u_2(x_1,x_3,t)$, and $\hat{m}_{22}(\boldsymbol{x},t) = (\lambda + 2\mu)\Delta u_2(x_1,x_3,t) = \rho c_p^2 \Delta u_2(x_1,x_3,t)$. In this case

$$u_\nu^{p \text{ or } s}(\boldsymbol{x},t) \approx \frac{R_{\nu\alpha\alpha}^{p \text{ or } s}(c_p^2 - 2c_s^2) + 2R_{\nu 22}^{p \text{ or } s} c_s^2}{4\pi r_o c_{p \text{ or } s}^3} \Omega(\boldsymbol{\gamma}, t - r_o / c_{p \text{ or } s}; c_{p \text{ or } s}) \ ,$$

where we note that $R_{\nu\alpha\alpha}^p = \gamma_\nu$ and $R_{\nu\alpha\alpha}^s = 0$, and where now

$$\Omega(\boldsymbol{\gamma},t;c) = \int_S \Delta\dot{u}_2(x_1,x_3,t + \frac{\gamma_1 x_1 + \gamma_3 x_3}{c}) dx_1 dx_3 \ , \quad \text{or}$$

$$\tilde{\Omega}(\boldsymbol{\gamma},\omega;c) = \int_S \exp(i\omega \frac{\gamma_1 x_1 + \gamma_3 x_3}{c}) \Delta\tilde{\dot{u}}_2(x_1,x_3,\omega) dx_1 dx_3 \ .$$