PERGAMON

Journal of the Mechanics and Physics of Solids
47 (1999) 1207–1218

JOURNAL OF THE
MECHANICS AND
PHYSICS OF SOLIDS

Stability of quasi-static slip in a single degree of freedom elastic system with rate and state dependent friction

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Received 20 April 1998; accepted 29 October 1998

Abstract

The stability of quasi-static frictional slip of a single degree of freedom elastic system is studied for a Dieterich–Ruina rate and state dependent friction law, showing steady-state velocity weakening, and following the ageing (or slowness) version of the state evolution law. Previous studies have been done for the slip version.

Analytically determined phase plane trajectories and Liapunov function methods are used in this work. The stability results have an extremely simple form: (1) When a constant velocity is imposed at the load point, slip motion is always periodic when the elastic stiffness, K, has a critical value, K_{cr}. Slip is always stable when $K > K_{cr} > 0$, with rate approaching the load-point velocity, and unstable (slip rates within the quasi-static model become unbounded) when $K < K_{cr}$. This is unlike results based on the slip version of the state evolution law, in which instability occurs in response to sufficiently large perturbations from steady sliding when $K > K_{cr}$. An implication of this result for slip instabilities in continuum systems is that a critical nucleation size of coherent slip has to be attained before unstable slip can ensue. (2) When the load point is stationary, the system stably evolves towards slip at a monotonically decreasing rate whenever $K \geqslant K_{cr} > 0$. However, when $K < K_{cr}$, initial conditions leading to stable and unstable slip motion exist. Hence self-driven creep modes of instability exist, but only in the latter case. © 1999 Elsevier Science Ltd. All rights reserved.

Keywords: A. Dynamics; B. Friction; Constitutive behavior; C. Stability and bifurcation

1. Introduction

Consider a rigid block attached to a linear spring (Fig. 1). The block slides frictionally with velocity V when a constant velocity V_0 is imposed at the other end of

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PII: S0022–5096(98)00113–6

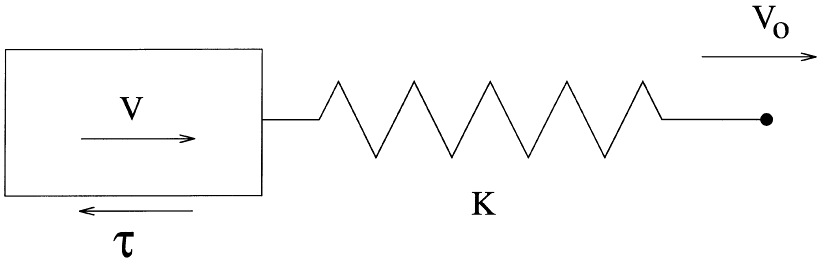

Fig. 1. Single degree of freedom elastic system sliding frictionally under imposed load-point motion.

the spring. Normal stress on the contact interface is assumed to be constant. At steady state, the block slips with the velocity of the imposed motion. Away from steady state, the equation of motion of the block, assuming quasi-static slippage, is

$$\dot{\tau} = K(V_0 - V) \tag{1}$$

where τ is the frictional shear stress on the block and K is the spring stiffness per unit area of sliding contact.

Motivated by experimental studies of rock friction, Dieterich (1979) and Ruina (1983) proposed an empirical frictional constitutive law in which friction depends both on the slip rate, V, and the state of the surface in the form

$$\tau = \tau_* + A\ln(V/V_*) + B\ln(V_*\theta/L) \tag{2}$$

where A, B and L are constants, τ_* and V_* are reference values of friction stress and sliding velocity, respectively, and θ is a state variable. Generally, τ_*, A and B are considered to be proportional to effective normal stress. Also, for V_* chosen in the range of imposed slip rates in the experiments mentioned (e.g., 10^{-9}–10^{-3} m/s), A and B are typically of order 2% to 4% of τ_*, and $|A-B|$ of order 1% or less.

Based on the work of Dieterich (1979), Ruina (1983) introduced two widely used empirical laws for the evolution of the state variable. In one form, called the Dieterich–Ruina ageing (or slowness) law, the state variable is interpreted as an effective time of contact of surface asperities. State evolution by this law is described by the equation

$$\dot{\theta} = 1 - V\theta/L. \tag{3}$$

An important feature of this law is that friction evolves logarithmically with time even under stationary contact ($V = 0$). Hence it is called an ageing law. Another widely used state evolution law, referred to as the Ruina–Dieterich slip law is given by

$$\dot{\theta} = -(V\theta/L)\ln(V\theta/L). \tag{4}$$

Here, state evolves only during slip ($V \neq 0$). See Beeler et al. (1994), Perrin et al. (1995) and Rice and Ben-Zion (1996) for further discussions and comparisons of slip predictions based on these laws. The latter also discuss a physically based regularization (based on Arrhenius rate process model) of the $\ln(V)$ term near $V = 0$ which is sometimes required, although not in the cases discussed here.

For both the Dieterich–Ruina ageing (or slowness) law and the Ruina–Dieterich slip law, at steady state,

$$V^{ss} = V_0, \quad \tau^{ss} = \tau_* - (B - A)\ln(V_0/V_*). \tag{5}$$

It is clear that the frictional stress at steady state decreases with increasing velocity when $B > A$.

A linear stability analysis of the steady state solution for both evolution laws was done by Ruina (1983), and more generally, for linear frictional constitutive laws with instantaneous velocity dependence and fading memory of prior history of velocity, by Rice and Ruina (1983). They show that quasi-static steady state slip is stable ($V \to V_0$) or unstable ($V \to \infty$) as the spring stiffness is greater than or less than a critical value given by

$$K_{cr} = (B - A)/L. \tag{6}$$

When $K = K_{cr}$, the linearized slip motion is periodic.

Gu et al. (1984) have done a non-linear analysis of the stability of steady slip with state evolution according to the Ruina–Dieterich slip law (4). For this law, they show that when $K = K_{cr}$, initial conditions leading both to periodic stick-slip motions as well as unstable motions exist. When $K > K_{cr}$, initial conditions sufficiently close to the steady state lead to stable slip while others cause unstable slip. When $K < K_{cr}$, slip is always unstable.

In the present work, a non-linear stability analysis of steady quasi-static slip, with evolution described by the Dieterich–Ruina ageing law (3), is carried out. The analysis, similar to the work of Gu et al. (1984), is carried out using analytically determined phase plane trajectories and Liapunov function techniques. The implications of the stability results for slip in continuum systems is also discussed.

This paper is organized as follows. In Section 2, the governing equations are non-dimensionalized and a phase plane analysis is carried out. Analytical phase plane trajectories are derived for two particular cases: (1) the case when $K = K_{cr}$ with arbitrary, non-zero (constant) load point velocity; (2) the case of a stationary load point with arbitrary spring stiffness. In Section 3, the stability of steady state slip with a non-stationary load point is studied using the phase plane trajectories obtained in Section 2 and by constructing a Liapunov function. Stability of sliding with a stationary load point is studied in Section 4. The conditions under which unstable self-driven creep modes can exist are established and compared with those for the Ruina–Dieterich slip law. The results are finally summarized in Section 5.

2. Phase plane trajectories

The governing equations of the problem are (1), (2) and (3). We introduce the dimensionless quantities:

$$T = V_* t/L, \quad k = KL/A, \quad \psi = (\tau - \tau_*)/A,$$

$$\phi = \ln(V/V_*), \quad \beta = B/A, \quad v_0 = V_0/V_*. \tag{7}$$

Combining (1), (2) and (3) to eliminate the state variable θ and using (7), we get

$$\frac{d\phi}{dT} = k(v_0 - e^{\phi}) - \beta[e^{(\phi-\psi)/\beta} - e^{\phi}] \tag{8}$$

$$\frac{d\psi}{dT} = k(v_0 - e^{\phi}). \tag{9}$$

Now, (8) and (9) are the governing equations in non-dimensional form.

The steady state solution (5) can be rewritten non-dimensionally as

$$\phi^{ss} = \ln(V_0/V_*), \quad \psi^{ss} = -(\beta-1)\phi^{ss}. \tag{10}$$

This describes a straight line with slope $-(\beta-1)$ in the (ψ, ϕ) plane. In particular, if V_* is chosen to equal $V_0 \neq 0$, the steady state of the system is at the origin of the phase plane.

The critical spring stiffness for linear stability given by (6) can be written using (7) as

$$k_{cr} = \beta - 1 \tag{11}$$

and the velocity-weakening condition $B > A$ translates to

$$\beta > 1. \tag{12}$$

To commence the phase plane analysis, T is first eliminated from the governing eqns (8) and (9) to get an equation of the form

$$P(\psi, \phi)\, d\psi + Q(\psi, \phi)\, d\phi = 0 \tag{13}$$

where

$$P(\psi, \phi) = k(v_0 - e^{\phi}) - \beta[e^{(\phi-\psi)/\beta} - e^{\phi}] \tag{14}$$

$$Q(\psi, \phi) = -k(v_0 - e^{\phi}). \tag{15}$$

An integrating factor of the form $e^{q(\psi,\phi)}$ is now sought such that

$$dU = [P(\psi, \phi)\, d\psi + Q(\psi, \phi)\, d\phi]e^{q(\psi,\phi)} \tag{16}$$

is a perfect differential. This requires that

$$\frac{\partial[e^{q(\psi,\phi)}P(\psi, \phi)]}{\partial\phi} = \frac{\partial[e^{q(\psi,\phi)}Q(\psi, \phi)]}{\partial\psi}. \tag{17}$$

Substituting for P and Q and simplifying, we get

$$kv_0\left[\frac{\partial q}{\partial\psi} + \frac{\partial q}{\partial\phi}\right] - ke^{\phi}\left[\frac{\partial q}{\partial\psi} + \frac{\partial q}{\partial\phi} + 1\right] - \beta e^{(\phi-\psi)/\beta}\left[\frac{\partial q}{\partial\phi} + \frac{1}{\beta}\right] + \beta e^{\phi}\left[\frac{\partial q}{\partial\phi} + 1\right] = 0. \tag{18}$$

The most general solution of (18) for arbitrary k and v_0 could not be found. However, solutions in which q is linear in its variables could be found when $k = k_{cr}$ with arbitrary v_0 and when $v_0 = 0$ with arbitrary k. We consider the two cases separately below.

2.1. *Case* 1: $k = k_{cr}$, *arbitrary* $v_0 \neq 0$

First, the case $k = k_{cr} \equiv \beta - 1$ (i.e. $K = K_{cr}$) with a non-stationary load point is studied. The phase plane trajectories obtained here are used in Section 3 to construct a Liapunov function and hence establish results on the stability of steady-state sliding for arbitrary perturbations from steady state.

When $k = k_{cr}$ and $v_0 \neq 0$, it can be shown that

$$q = (\psi - \phi)/\beta \tag{19}$$

is a solution to (18). On integrating (16), the trajectories in phase plane are found to be given by

$$U = k\beta\left[v_0 + \frac{e^{\phi}}{\beta - 1}\right]e^{(\psi-\phi)/\beta} - \psi\beta = \text{constant}; \quad k = k_{cr} \equiv \beta - 1. \tag{20}$$

2.2. *Case* 2: $v_0 = 0$, *arbitrary* k

Explicit phase plane trajectories could also be determined for the case of a stationary load point, with the spring stiffness being arbitrary. This situation may be taken to model, for instance, the stressing of a stationary fault segment by a large earthquake in its vicinity. It is of interest to know whether the stress change associated with the earthquake can trigger a delayed instability (aftershock) in the fault segment. Such an instability mechanism is referred to as inducing a state of accelerating self-driven creep (see Rice and Gu (1983) and Dieterich (1994)). The results obtained below will be used in section 4 to derive a simple condition for the existence of such an instability.

When $v_0 = 0$, it can be shown that

$$q = \frac{(\beta-1)(\beta-k)}{k\beta}\psi - \frac{\phi}{\beta} \tag{21}$$

is a solution to (18). The trajectories, obtained by integrating (16), are

$$U = \left[\beta e^{\phi} e^{(\psi-\phi)/\beta} - \frac{(\beta-1)\beta}{\beta-1-k}\right]\frac{k}{\beta-1}e^{(\beta-1-k)\psi/k} = \text{constant}; \quad k \neq k_{cr} \tag{22}$$

$$U = \beta e^{\phi} e^{(\psi-\phi)/\beta} - \psi\beta = \text{constant}; \quad k = k_{cr}. \tag{23}$$

3. Stability results with non-stationary load point

In this section, the phase plane trajectories obtained in Section 2.1 for the case of critical spring stiffness are used to study the stability of steady state slip when the load point is non-stationary (moving at constant velocity $V_0 > 0$).

3.1. *Case* 1: k = k_{cr}

We show that slip motion is always periodic when $k = k_{cr}$ and the velocity weakening condition, $\beta > 1$ is satisfied. In other words, the phase plane trajectories (20) always form closed contours when $\beta > 1$. We establish this result by showing that for given values of U and ψ, there exist either two or no values of ϕ that satisfy (20) and, similarly, for specified values of U and ϕ, there exist either two or no values of ψ satisfying (20). First, choose $V_* = V_0$ (i.e. $v_0 = 1$) without loss of generality. For given values of U and ψ, (20) may be written in the form

$$e^{-\phi/\beta} + \frac{e^{\phi(\beta-1)/\beta}}{\beta-1} = \text{constant}. \tag{24}$$

Now, from a graphical construction of the left and right hand sides of the above equation, it is easily seen that there are either two or no values of ϕ that satisfy the above equation when $\beta > 1$. Similarly, for specified values of U and ϕ, (20) gives

$$(\text{constant})e^{\psi/\beta} = \psi\beta + U. \tag{25}$$

As before, it may be shown that the above equation is satisfied by either two or no values of ψ. Hence, the trajectories form closed contours when $\beta > 1$ and slip is always periodic. A typical plot of the phase plane trajectories for this case is shown in Fig. 2, with $\beta = 5/4$.

This result may be contrasted with the one obtained by Gu et al. (1984) for the Ruina–Dieterich slip law. They show that, when $k = k_{cr}$ and $\beta > 1$, with friction

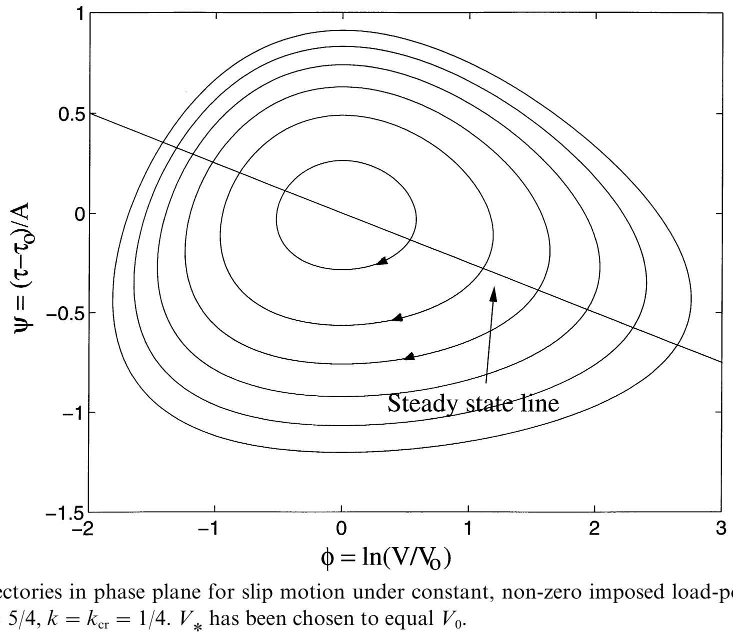

Fig. 2. Trajectories in phase plane for slip motion under constant, non-zero imposed load-point velocity V_0, with $\beta = 5/4$, $k = k_{cr} = 1/4$. V_* has been chosen to equal V_0.

evolving according to the slip law, there exists initial conditions leading to both periodic as well as unstable slip motions. Perturbations that displace the system sufficiently away from the steady state solution cause the instability.

3.2. *Case* 2: $k \neq k_{cr}$

The stability of steady-state slip for generic $k \neq k_{cr}$ is determined by finding a Liapunov function for the problem. It is shown that with a velocity weakening Dieterich–Ruina ageing law, slip is always stable when $k > k_{cr}$ and always unstable when $k < k_{cr}$.

Consider the function generated by adding a certain function of ψ to U of (20):

$$U_1 = k\beta\left[v_0 + \frac{e^{\phi}}{\beta-1}\right]e^{(\psi-\phi)/\beta} - \psi\beta - v_0^{(\beta-1)/\beta}\beta^2\left[\frac{k}{\beta-1} - 1\right]e^{\psi/\beta}. \tag{26}$$

It is easily established that U_1 is a Liapunov function when $\beta > 1$ since:

1. Trajectories of constant U_1, when they exist, are closed contours around the steady state solution (10) when $\beta > 1$. This may be shown by graphical construction similar to the ones presented earlier. The global minimum of U_1 occurs at the steady state solution (10).
2. The derivative of U_1 along a solution trajectory is given by

$$\frac{dU_1}{dT} = \frac{\partial U_1}{\partial \psi}\frac{d\psi}{dT} + \frac{\partial U_1}{\partial \phi}\frac{d\phi}{dT}. \tag{27}$$

Using (26) and the governing eqns (8) and (9), this may be evaluated as

$$\frac{dU_1}{dT} = -k\left[\frac{k}{\beta-1} - 1\right]\beta e^{\psi/\beta}(e^{\phi} - v_0)(e^{(\beta-1)\phi/\beta} - v_0^{(\beta-1)/\beta}) \tag{28}$$

Clearly, the factor $(e^{\phi} - v_0)(e^{(\beta-1)\phi/\beta} - v_0^{(\beta-1)/\beta})$ in (28) is always of positive sign when $\beta > 1$. Hence, when $k > \beta - 1(= k_{cr})$ and $\beta > 1$, $dU_1/dT < 0$ and every solution trajectory evolves with monotonically decreasing values of U_1. Since the global minimum of U_1 occurs at the steady state solution when $\beta > 1$, it follows that every initial condition evolves towards the steady-state solution. Therefore, all slip motions are stable when $k > k_{cr}$ and $\beta > 1$. Similarly, when $k < \beta - 1(= k_{cr})$ and $\beta > 1$, $dU_1/dT > 0$. Every initial condition evolves towards ever increasing values of U_1. Hence, all slip motions are unstable with slip velocity becoming unbounded.

These results may be compared with the analogous results obtained by Gu et al. (1984) for the Ruina–Dieterich slip law. They show that when $k > k_{cr}$, initial conditions corresponding to a sufficiently small perturbation from the steady state solution give rise to stable slip, while others cause unstable slip. When $k < k_{cr}$ slip is always unstable, as for the Dieterich–Ruina ageing law.

3.3. *Discussion*

The stability results obtained in this section have an important implication for nucleation of slip instabilities in continuum systems. Consider a patch of linear dimension D slipping frictionally in elastic surroundings with shear modulus μ. An effective stiffness can be identified in this case as

$$K \sim \mu/D. \tag{29}$$

For the Dieterich–Ruina ageing law, we know from the present analysis that unstable slip occurs only when

$$K < K_{cr} \Leftrightarrow D > D_{cr}. \tag{30}$$

Hence, a critical nucleation size, D_{cr}, has to be attained before unstable slip can occur. On the other hand, for the Ruina–Dieterich slip law, such a nucleation size can be defined for linearized stability analysis but is not strictly defined in general since unstable slip can occur at any value of K if the initial condition is sufficiently perturbed from the steady state.

The results of this section may conveniently be visualized in terms of a stability diagram in the K–Δ plane, where Δ is a perturbation in stress or sliding velocity from steady state. We have seen that in this plane, the straight line $K = K_{cr}$ forms the stability boundary dividing regions of stable and unstable slip. The effect of inertia of the block on the stability boundary has been studied numerically in work the details of which are not reported here. An additional parameter is introduced into the problem by inclusion of inertia. This maybe taken to be the ratio of an inertial to a frictional time scale:

$$r = \left(\frac{T/2\pi}{L/V_*}\right), \tag{31}$$

where T is the period of free vibrations of the spring-mass system. An interesting feature of the results is that for small values of r, the stability boundary initially bends towards higher values of K from K_{cr} and then bends back towards values of $K < K_{cr}$. This implies that in response to finite perturbation, periodic stick-slip motion can occur even for values of $K > K_{cr}$, albeit in a very narrow range. For larger values of r, the stability boundary lies throughout in the region $K \leqslant K_{cr}$.

4. Stability results for stationary load point

In this section, the phase plane trajectories obtained in Section 2.2 are used to establish conditions for the existence of self-driven creep modes of instability. We shall show that the spring stiffness $K = K_{cr}$ plays a critical role in dividing types of response. This is remarkable because K_{cr} arose in the context of a linearized stability analysis of steady sliding, which is not a mode of response when the load point is stationary.

We observe that when $v_0 = 0$, the only long time behaviors of the system, consistent with the governing eqns (8) and (9) are

$$\phi \to \infty \quad \text{and} \quad \psi \to -\infty, \tag{32}$$

or

$$\phi \to -\infty \quad \text{and} \quad \psi \to -\infty.$$

We consider the two cases $k < k_{cr}$ and $k \geqslant k_{cr}$ separately below.

4.1. *Case* 1: $k < k_{cr}$

First, consider the case when $k < k_{cr}$. The trajectories corresponding to $U = 0$ are straight lines given by

$$\psi = -(\beta-1)\phi+\beta\ln[(\beta-1)/(\beta-1-k)]. \tag{33}$$

These are parallel to the steady state line in the (ϕ, ψ) plane, but located above it (at higher ψ, for given ϕ).

When $U > 0$, it follows from (22) that

$$e^{\phi}e^{(\psi-\phi)/\beta} > (\beta-1)/(\beta-1-k). \tag{34}$$

Using this condition in (8), it can be shown that

$$\mathrm{d}\phi/\mathrm{d}T > ke^{\phi}/(\beta-1). \tag{35}$$

Now, since $\int_{\phi}^{\infty} \mathrm{e}^{-\phi'}\,\mathrm{d}\phi'$ is bounded, $\phi \to \infty$ in finite time. Therefore, the slip velocity becomes unbounded in finite time when $U > 0$. The unboundedness of slip velocity in this case is due to the neglect of inertial effects.

From (22), we can write

$$e^{(\beta-1)\phi/\beta} = \left[\frac{U}{\beta k}e^{-(\beta-1-k)\psi/k}+\frac{1}{\beta-1-k}\right](\beta-1)e^{-\psi/\beta}. \tag{36}$$

When $U < 0$, with $k < \beta-1(=k_{cr})$, $\beta > 1$ and $\psi \to -\infty$ according to (32), it is clear that ϕ decreases at long times along every trajectory. Hence slip is stable when $U < 0$.

A typical plot of the phase plane trajectories when $k < k_{cr}$ is shown in Fig. 3. As has been shown in the analysis above, the straight line trajectory corresponding to $U = 0$ divides the phase plane into stable and unstable regions. Slip is stable when initial conditions cause $U < 0$ and unstable when $U > 0$. Gu et al. (1984) found a similar division of the phase plane in the stationary load-point case, but for their analysis, with the slip law (4), such division exists for all $k > 0$, and here it exists only for $k < k_{cr}$.

4.2. *Case* 2: $k \geqslant k_{cr}$

Next, we show that when $k \geqslant k_{cr}$, slip is always stable. It is easily seen that, when $k \geqslant k_{cr}$ and $\beta > 1$, $\phi \to \infty$ is inconsistent with U being a constant along a trajectory.

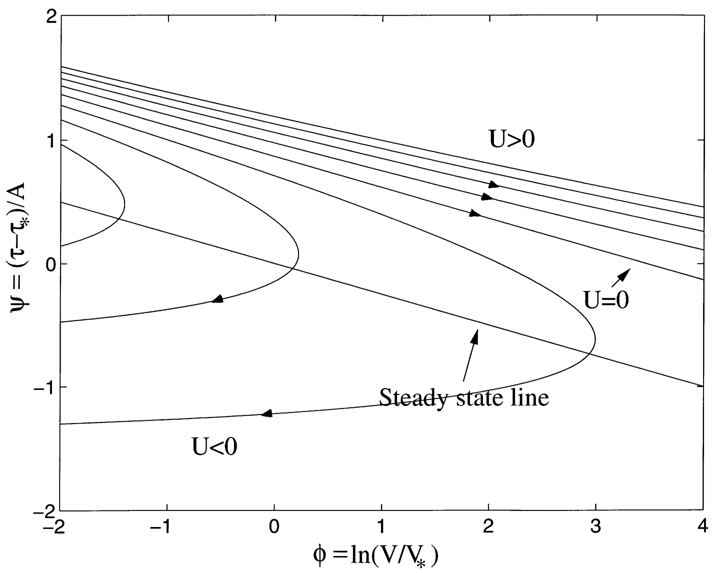

Fig. 3. Trajectories in phase plane for slip motion with stationary load point and $\beta = 5/4$, $k = 1/8 < k_{cr}$. Self-driven creep occurs when $U > 0$.

Hence, according to (32), the only long time behavior permissible is that $\phi \rightarrow -\infty$ (i.e. $V \rightarrow 0$). In other words slip is always stable when $k \geqslant k_{cr}$. Typical plots of trajectories for $k = k_{cr}$ and $k > k_{cr}$ are shown in Figs 4 and 5 respectively.

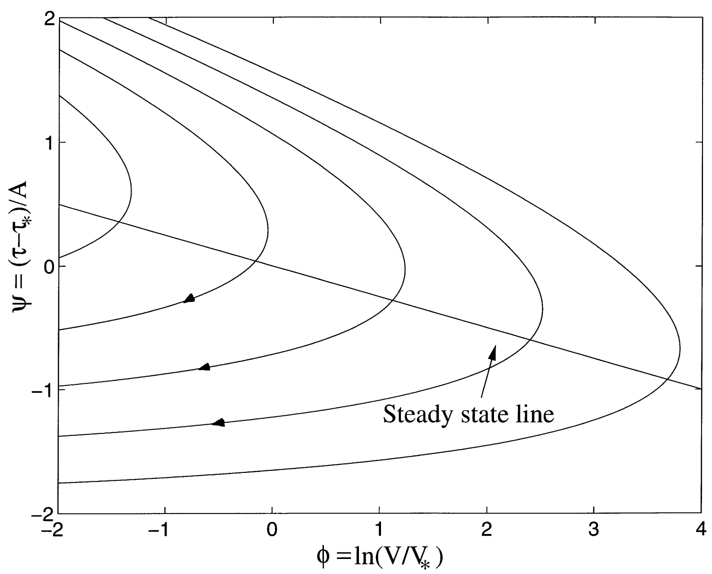

Fig. 4. Trajectories in phase plane for slip motion with stationary load point and $\beta = 5/4$, $k = k_{cr} = 1/4$.

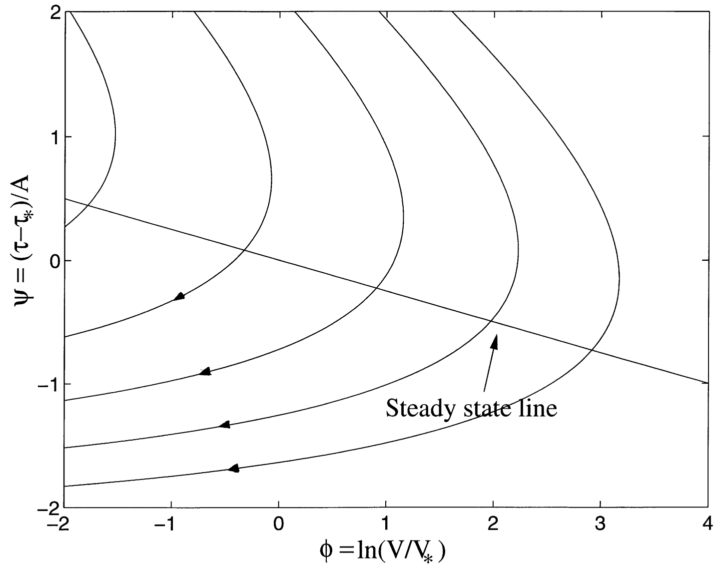

Fig. 5. Trajectories in phase plane for slip motion with stationary load point and $\beta = 5/4$, $k = 1/2 > k_{cr}$.

4.3. *Discussion*

It is clear from the above analysis that self-driven creep modes of instability can exist, when the ageing law (3) applies, only when $k < k_{cr}$. The amount of stress perturbation, $\Delta\psi$ required to drive a system from steady state to instability is the distance of the stability boundary (33) from the steady state line (10):

$$\Delta\psi = \beta \ln[(\beta - 1)/(\beta - 1 - k)]. \qquad (37)$$

In contrast, for the Ruina–Dieterich slip law, Gu et al. (1984) show that stable and unstable regions exist for all $k_{cr} > 0$. Hence, self-driven creep modes exist for all values of k. In the unstable region, slip velocity becomes unbounded in finite time, as in the present study.

5. Conclusions

The stability of quasi-static frictional slip of a rigid block loaded by a linear spring has been studied. A rate- and state-dependent frictional constitutive law (2) with state evolution described by the Dieterich–Ruina ageing law (3) have been adopted in this study. A non-linear stability analysis of the steady state solution (5) using analytically determined phase-plane trajectories and Liapunov function techniques has been carried out. The stability results are shown to have an extremely simple form:

- With non-zero load-point velocities ($V_0 \neq 0$), slip motion is always periodic when

$K = K_{cr} = (B-A)/L > 0.$

When $K > K_{cr} > 0$, sliding is always stable. In other words, the block always approaches the velocity of the imposed motion. When $K < K_{cr}$, slip is always unstable with the sliding velocity becoming unbounded.

- When the load point is held stationary ($V_0 = 0$), the system stably evolves towards ever-slower slip rates whenever $K \geqslant K_{cr} > 0$. However when $K < K_{cr}$, initial conditions leading to stable and unstable slip motion exist. This shows that self-driven creep modes can exist only in the latter case. The unstable motions are shown to be such that the slip velocity becomes unbounded in finite time, corresponding to a delayed instability, or an aftershock.

The preclusion of instabilities when $K > K_{cr}$ has an important implication for slip instabilities in continuum systems. In a continuum system, a critical nucleation size of coherent slip has to be attained before unstable slip can ensue and the nucleation size does not depend on the strength of the perturbation.

Acknowledgements

This study was supported by ONR under grant N00014-96-10777 and USGS under grant 1434-HQ-97-GR-03094.

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