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DISORDERING OF A DYNAMIC PLANAR CRACK FRONT IN A MODEL ELASTIC MEDIUM OF RANDOMLY VARIABLE TOUGHNESS

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ABSTRACT

RICE *et al.* (1994, *J. Mech. Phys. Solids* **42,** 813–843) analyse the propagation of a planar crack with a nominally straight front in a model elastic solid with a single displacement component. Using the form of their results for a strictly linearized perturbation from a straight crack front which moves at uniform speed, we give the corresponding first-order expression for the deviation of a crack front from straightness as a direct integral expression in the deviation of the material toughness from uniformity in the crack plane. We then use this expression to analyse the autocorrelation of the crack front position when the toughness deviations are random. We find that the root mean square deviation in position diverges logarithmically with travel distance across the random toughness region, as do the variances of the propagation velocity and slope of the crack front. That is, according to strictly linearized analysis, perturbed about the solution for a uniformly moving crack front, the perturbations from straightness and from uniform propagation speed should grow without bound in the presence of random deviations in toughness. What is remarkable about this result is that, according to the same strictly linearized analysis, if the toughness is completely uniform over the remaining part of the fracture plane, after encounter with a region of nonuniform toughness, the moving crack front becomes asymptotically straight with increase of time. Nonlinearities, not considered here, must control how statistically disordered the crack front can ultimately become as it propagates through a region of random toughness variation. Also, because of the logarithmic nature of the growth, significant disorder can occur in response to small perturbations only when the crack moves over a great distance compared to the correlation length scale in the fracture toughness.

1. INTRODUCTION

FOLLOWING RICE *et al.* (1994), consider a half-plane crack propagating in an unbounded solid in a nominal direction x along the plane $y = 0$. This takes place in their model 3D elastodynamic theory involving a single displacement variable u representing tensile opening or shear slippage, and associated tensile or shear stress

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$\sigma = Mu_{,y}$ across planes parallel to the crack, where M is the elastic modulus. The field u is a solution to the scalar wave equation $c^2\Delta u = u_{,tt}$, where $c^2 = M/\rho$ with ρ denoting mass density. The boundary conditions on the plane $y = 0$ are: $u = 0$ ahead of the front, and $\sigma = 0$ behind it. The crack front at time t lies along the curve $x = a(z, t)$. The crack propagation velocity in the x direction $v(z, t) = a_{,t}(z, t)$, is assumed to be close to a constant velocity v_0. More precisely, RICE *et al.* (1994) write $a(z, t) = v_0 t + \varepsilon f(z, t)$, where f is an arbitrary function decribing the deviation from straightness of the front and ε is a "small parameter" (in other words, only the first order perturbation from the $\varepsilon = 0$ case is considered).

The preceding authors show that the local dynamic energy release rate $G(z, t)$ along the crack front is given, to first order in $v(z, t) - v_0$, in an expression for $G^{1/2}$ by

$$\sqrt{G(z, t)} = \sqrt{G_0}\left\{1 - \frac{c[v(z, t) - v_0]}{2(c^2 - v_0^2)} + I(z, t)\right\} \tag{1}$$

with

$$I(z, t) = \frac{1}{2\pi}\int_0^{\infty}\left\{PV\int_{-\alpha_0 c\theta}^{\alpha_0 c\theta}\frac{c\theta}{\sqrt{\alpha_0^2 c^2\theta^2 - \xi^2}}\,\frac{v(z-\xi, t-\theta) - v(z, t-\theta)}{\xi}\,\frac{\mathrm{d}\xi}{\xi}\right\}\mathrm{d}\theta \tag{2}$$

[compare with equations (33b) and (35) in Rice *et al.* (1994)] and where $\alpha_0 c = \sqrt{c^2 - v_0^2}$ is the velocity of information traveling along the crack; that is, two points of the crack front with z coordinate differing by Δz do not influence each other before the time delay $\Delta z/\alpha_0 c$. G_0 is the energy release rate for the reference problem, and is considered as constant. This expression in (1) and (2) is based on what RICE *et al.* (1994) call their *strictly linearized* analysis; they also provide a version of (1), as in their equation (33a), which remains accurate for arbitrarily large, but subsonic, perturbations of velocity when the crack front is straight.

At this step, the authors consider the inverse problem to (1) and (2): suppose that the material rupture is ruled by the local energy release rate criterion

$$G(z, t) = G_{\text{crit}}[a(z, t), z], \tag{3}$$

the function $G_{\text{crit}}(x, z)$ being known, and solve numerically for the crack front velocity. One notable result they obtain deals with a crack with initial straight front which crosses a region of variable toughness of finite width (along axis x) and afterwards enters a constant toughness region forever: although the crack front becomes wavy in the variable toughness region, it gradually straightens up in the following constant toughness region. This shows that straight fronts are in some sense stable.

One possible extension of this analysis is to consider a crack propagating in a region of randomly variable toughness, and ask how much the crack front deviates, statistically, from straightness. We shall suppose that some statistical properties of the random toughness are known, namely the autocorrelation function of

$$\tau(z, t) = \sqrt{G_{\text{crit}}(v_0 t, z)/G_0} - 1 \tag{4}$$

and we shall aim at getting the spatial autocorrelation function of the crack deviation

$A(z,t) = a(z,t) - v_0 t$ from the reference straight form. Notice that at the first order in A and τ, (1), (2), (3) and (4) yield the linear relationship between A and τ

$$\tau(z,t) + \frac{A_{,t}(z,t)}{2\alpha_0^2 c} = \frac{1}{2\pi}\int_0^{\infty}\left\{PV\int_{-\alpha_0 c\theta}^{\alpha_0 c\theta} \frac{c\theta}{\sqrt{\alpha_0^2 c^2\theta^2-\xi^2}} \frac{A_{,t}(z-\xi,t-\theta)-A_{,t}(z,t-\theta)}{\xi}\frac{\mathrm{d}\xi}{\xi}\right\}\mathrm{d}\theta, \tag{5}$$

where the lowest-order neglected term, compared to the system (1), (2) and (3), is of the form

$$O\big(\tau_{,t}(z,t)A(z,t)\big).$$

In Section 2, we attempt to relate the autocorrelation functions of τ and A by the easiest apparent way: use time and space Fourier transform of (5) and relate the power spectra of τ and A. This method appears to be in general not reliable, because it yields a power spectrum for A which is not integrable, and one cannot deduce the autocorrelation function from it. That is, the linearized relation (5) between $A(z,t)$ and $\tau(z,t)$ has the property that neither $A(z,t)$ nor $V(z,t) \equiv A_{,t}(z,t)$ will be stationary random functions of z and t, despite the fact that $G_{\mathrm{crit}}(x,z)$, and hence $\tau(z,t)$, are stationary random functions.

We thus resort to computing the autocorrelation function of A directly. To this end, we need an explicit expression of $A(z,t)$, the material property $\tau(z,t)$ being known from (4). This is achieved in Section 3 [see (9) below]. This result appears to be similar to a well-known equation of antiplane elastodynamics, and Section 4 develops the analogy.

In Section 5, we resume the analysis of the random deviation from straightness by considering the following well-posed problem: a crack with a straight front enters a variable toughness region at time 0, and we analyse the autocorrelation of its deviation from straightness at any finite time t, as a functional of the autocorrelation function of the random energy release rate. This expression appears to be tractable for a set of realistic, representative distributions of random toughness, and predicts the standard deviation, $\Delta A(t)$, of crack advance to increase like the logarithm of the crack advance $a(t)$. The unboundedness of ΔA at infinite time explains our failed attempt of Section 2. The divergence obtained there stems from the use of time-wise Fourier transform, implicitly assuming that the problem can be made invariant by time translation through considering a crack that has propagated in the random medium from an asymptotically large negative time on. A modal analysis of the crack front autocorrelation function is carried out in Sections 6 and 7. The variance of each Fourier component of $A(z,t)$ and $V(z,t)$, of non-zero mode number, is shown to diverge logarithmically with time. These results indicate that, according to the strictly linearized perturbation analysis, *the front of a crack that can run forever in a random unbounded medium grows overwhelmingly wavy*. However, the logarithmic time dependence means that the amount of disorder actually predicted will be quite limited in many practical cases for which the total growth distance is not vastly larger than the correlation length of the toughness distribution.

2. ANALYSIS IN FOURIER SPACE

In this Section, we aim at relating the power spectra of τ and A. To do so, we need the Fourier transform of the linear relationship (5) between $v-v_0 = A_{,t}$ and τ. It is advantageous to reduce the singularity of the kernel of (5) via integration by parts on the variable ξ (integrating the kernel and differentiating the velocity difference). The contribution of the bounds around $\xi = 0$ stemming from the principal value integral must be handled carefully; they cancel provided that $v_{,z}$ be a continuous function of z. The result is

$$\tau(z,t)+\frac{A_{,t}(z,t)}{2\alpha_0^2 c} = -\frac{1}{2\pi\alpha_0^2 c}\int_0^t \left\{ PV\int_{-\alpha_0 c\theta}^{\alpha_0 c\theta} A_{,tz}(z-\xi,t-\theta)\sqrt{\alpha_0^2c^2\theta^2-\xi^2}\,\frac{\mathrm{d}\xi}{\xi}\right\}\frac{\mathrm{d}\theta}{\theta} \tag{6}$$

which implies in Fourier space

$$\hat{A}(k,\omega) = 2\alpha_0^2 c h(k,\omega)\hat{\tau}(k,\omega) \text{ with } h(k,\omega) = \begin{cases} -(k^2\alpha_0^2c^2-\omega^2)^{-1/2} \text{ if } |\omega| < |\alpha_0 ck| \\ \mathrm{i}\,\mathrm{sign}\,(\omega)(\omega^2-k^2\alpha_0^2c^2)^{-1/2} \text{ if } |\omega| > |\alpha_0 ck|, \end{cases} \tag{7}$$

where $\omega/2\pi$ is the frequency, k the wave-number, and ^ denotes the space-time Fourier transform, e.g.

$$\hat{A}(k,\omega) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} A(z,t)\,\mathrm{e}^{-\mathrm{i}kz}\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}z\,\mathrm{d}t.$$

The calculation of h is involved and offers little interest.

If we assume that the two point autocorrelation functions of τ and A both depend only on the relative position of the two points, which implicitly assumes that both random functions are stationary in space and time, we are able to use the classical ergodic definition of the autocorrelation function of a stationary random function $X(z,t)$ as

$$\Gamma_X(z,t) = \lim_{L\to\infty, T\to\infty}\frac{1}{4LT}\int_{-T}^{+T}\int_{-L}^{+L} X(\xi,\theta)X(\xi+z,\theta+t)\,\mathrm{d}\xi\,\mathrm{d}\theta$$

and of its power spectrum as $S_X(k,\omega) = \hat{\Gamma}_X(k,\omega)$. Power spectra S_τ and S_A being defined that way, (7) provides

$$S_A(k,\omega) = 4\alpha_0^4 c^2\frac{S_\tau(k,\omega)}{|\omega^2-k^2\alpha_0^2c^2|}. \tag{8}$$

Since a realistic power spectrum S_τ would not vanish identically on the cone $\omega^2 = k^2(\alpha_0 c)^2$, it is obvious from (8) that the autocorrelation function of A would be nowhere finite.

The solution to this apparently absurd conclusion is that the hypothesis that $A(z,t)$ be a stationary random variable is not fulfilled, as will become obvious from the results of Section 5. In that case, the power spectrum of A cannot be defined, and the relationship (8) breaks down. Instead, we must answer our question in two steps: in Section 3, we express the deviation from straightness of the crack front, $A(z,t)$, as a linear functional of $\tau(z,t)$, and then in Section 5 we perform statistical analysis of the deviation from straightness A. Similarly, the purported power spectrum $S_V(k,\omega)$ of the velocity fluctuation, $V(z,t) = a_{,t}(z,t) - v_0$, can be written, and we find likewise that $V(z,t)$ cannot be a stationary random variable; note that the purported spectra are related by $S_V(k,\omega) = \omega^2 S_A(k,\omega)$.

3. Straightness Deviation as an Explicit Functional of Toughness Perturbation

The purpose of this Section is to invert the integral equation (6) for $A(z,t)$ with the additional condition that both A and τ are uniformly zero before time $t = 0$.

The first step is to deduce from (7) for imaginary $\omega = -\mathrm{i}s$, $s > 0$, the relation between the z-Fourier and t-Laplace transforms of A and τ.

$$\hat{A}(k,-\mathrm{i}s) = -\frac{2\alpha_0^2 c}{\sqrt{k^2\alpha_0^2c^2+s^2}}\,\hat{\tau}(k,-\mathrm{i}s).$$

This expression is suitable for the Laplace convolution theorem [the transfer function decreases as $O(1/s)$ for large s] and results in

$$\tilde{A}(k,t) = -2\alpha_0^2 c\int_0^t J_0(k\alpha_0 c\theta)\tilde{\tau}(k,t-\theta)\,\mathrm{d}\theta,$$

where $\tilde{A}(k,t)$ and $\tilde{\tau}(k,t)$ are spacewise Fourier transforms, e.g.

$$\tilde{A}(k,t) = \int_{-\infty}^{+\infty} A(z,t)\,\mathrm{e}^{-\mathrm{i}kz}\,\mathrm{d}z.$$

The inverse Fourier transform provides

$$A(z,t) = -\frac{2\alpha_0^2 c}{\pi}\int_0^t\int_{-\alpha_0 c\theta}^{\alpha_0 c\theta}\frac{\tau(z-\xi,t-\theta)}{\sqrt{\alpha_0^2c^2\theta^2-\xi^2}}\,\mathrm{d}\xi\,\mathrm{d}\theta. \tag{9}$$

Notice that the integration kernel is integrable at point $\theta = \xi = 0$.

Since the velocity fluctuation $V(z,t) = v(z,t) - v_0 = A_{,t}(z,t)$, the solutions for $\tilde{V}(k,t)$ and $V(z,t)$ are given by the last two expressions, with $\tilde{\tau}$ and τ replaced, respectively, with $\tilde{\tau}_{,t}$ and $\tau_{,t}$, and with additional terms that arise from the instantaneously induced velocity heterogeneity when the random region is first encountered at $t = 0$:

$$\tilde{V}(k,t) = -2\alpha_0^2 c\int_0^t J_0(k\alpha_0 c\theta)\tilde{\tau}_{,t}(k,t-\theta)\,\mathrm{d}\theta - 2\alpha_0^2 c J_0(k\alpha_0 ct)\tilde{\tau}(k,0),$$

$$V(z,t) = -\frac{2\alpha_0^2 c}{\pi}\int_0^t\int_{-\alpha_0 c\theta}^{\alpha_0 c\theta}\frac{\tau_{,t}(z-\xi,t-\theta)}{\sqrt{\alpha_0^2c^2\theta^2-\xi^2}}\,\mathrm{d}\xi\,\mathrm{d}\theta - \frac{2\alpha_0^2 c}{\pi}\int_{-\alpha_0 ct}^{\alpha_0 ct}\frac{\tau(z-\xi,0)}{\sqrt{\alpha_0^2c^2t^2-\xi^2}}\,\mathrm{d}\xi.$$

The first of these is a simple extension of a result given by equations (52) and (53) of RICE *et al.* (1994).

The equations just given seem to suggest that the straight crack front is configurationally stable. For example, suppose that after some compactly supported zone of non-uniform G_{crit} has been traversed, that G_{crit} reverts to the uniform value G_0; i.e. $\tau(z, t) = 0$, and hence $\tilde{\tau}(k, t) = 0$, for all t greater than some maximum traverse t^*. Then the crack front is perturbed from straightness as it leaves that zone, but it is straightforward to show that, as $t \to \infty$, $\tilde{V}(k, t) \to 0$, for all k and $\tilde{A}(k, t) \to 0$ for all $k \neq 0$. Thus the crack recovers its straight shape and uniform propagation speed v_0. However, stability is a subtle issue here. While the straight shape is indeed recovered for propagation through a zone of precisely uniform G_{crit}, we shall see that the straight crack front becomes increasingly more disordered as it propagates through a region of arbitrarily small but sustained random variation in G_{crit} (i.e. in τ) and, to the extent that we regard such heterogeneity of properties as being inevitable, we then conclude that the straight crack front is *configurationally unstable*. Such has already been anticipated by the lack of existence of stationary solutions for A and V in response to stationary τ.

4. ANALOGY WITH AN ANTIPLANE FRICTIONAL PROBLEM

In this section, we note an interesting mathematical analogy between our 3D perturbed crack front problem and the 2D problem of an "antiplane frictional fault". For the latter, a linear elastic unbounded body contains a weak plane ($y = 0$) on which frictional sliding can occur. Only displacement along direction z is allowed, and this is uniform in z, i.e. a function only of x, y and t. The elastic shear modulus is μ and the shear wave velocity is c_s. Time and space boundary integral equation methods allow us to condense the problem into a two-dimensional problem in terms of two functions, the displacement discontinuity, δ, across the plane $y = 0$ and the alteration of shear stress, $\sigma(=\sigma_{yz})$, on the plane $y = 0$ as a function of time t, and of the position along the fault, x. For details, see KOSTROV (1966, 1975), KOSTROV and DAS (1988; page 264), or FREUND [1990; page 66, equations (2.3.4) and (2.3.5) with his $y_0 = 0$].

An analogy between the two corresponding quasistatic problems was uncovered by RICE (1988). It appears here that a similar analogy can be constructed in the dynamic case: indeed, (6) and (9) hold for the frictional fault, provided we make the identification:

Crack front	Frictional fault
$\alpha_0 c$	c_s
A	δ
$\alpha_0\left(\sqrt{\dfrac{G_{crit}(v_0 t + A(z,t), z)}{G_0}} - 1\right)$	$\dfrac{\sigma(x,t)}{\mu}$

In other words, the solution to the dynamic crack front problem is also the solution to a frictional problem where the friction may vary along the fault, and depend at each point of time on the accumulated displacement discontinuity δ only through a combination of form $v_0 t + \delta(x, t)$. This analogy shows, incidentally, that the long-time response to the action of a stationary random shear stress $\sigma(x, t)$ of zero mean on the boundary of a half-space is a non-stationary random surface displacement and particle velocity.

5. Effect of Random Energy Release Rate

From this section on, we resume the analysis of deviation from straightness of a crack front within the scalar model elasticity framework of Rice *et al.* (1994). We now introduce the following well posed problem: we consider the case of a material with critical energy release rate $G_{crit}(x, z)$, constant for $x < 0$, and random for $x > 0$. We assume that the expected value $E[\tau(z, x)]$ is zero everywhere. Here it is sometimes convenient to use z and x, and sometimes z and t, as arguments of τ, $\tau(z, x) = \tau(z, t)$ of (4), once we make the substitution $v_0 t = x$. Notice that if the deviations δG from G_0 in the $x > 0$ plane are supposed to be small, so that the approximation $\tau = \delta G/2G_0$ holds, this assumption is equivalent to assuming that the expected value of G is everywhere G_0. Let us consider the two point autocorrelation function of τ, $E[\tau(z_1, x_1)\tau(z_2, x_2)]$, where $E[X]$ denotes the mathematical expectation, as an ensemble average, of random variable X. It is reasonable to assume that this function, which describes the local properties of a macroscopically homogeneous material, is stationary and hence depends only on the differences $z_2 - z_1$ and $x_2 - x_1$, and is an even function of these quantities. Accordingly, we denote

$$R_\tau(z_2 - z_1, x_2 - x_1) = E[\tau(z_1, x_1)\tau(z_2, x_2)].$$

The corresponding power spectral density is

$$P_\tau(k_1, k_2) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} R_\tau(z, x)\, e^{-ik_1 z}\, e^{-ik_2 x}\, dz\, dx.$$

In a particular case which we will sometimes use to illustrate formulae, we consider

$$R_\tau(z_2 - z_1, x_2 - x_1) = \sigma^2 \exp[-\sqrt{(x_2 - x_1)^2 + (z_2 - z_1)^2}/b],$$

where b is a correlation length and the statistical distribution of $\tau(x, z)$ is isotropic in the plane; then

$$P_\tau(k_1, k_2) = 2\pi\sigma^2 b^2/[1 + b^2(k_1^2 + k_2^2)]^{3/2}.$$

Also, we shall need the correlation function and spectrum of $\tau_{,t}(z, t)$ which, in the notation $\tau = \tau(z, x)$ with $x = v_0 t$, is the same as $v_0 \tau_{,x}(z, x)$. Thus, by standard methods for stationary distributions, we have

$$R_{\tau_{,t}}(z, x) = -v_0^2 \frac{\partial^2}{\partial x^2} R_\tau(z, x) \quad \text{and} \quad P_{\tau_{,t}}(k_1, k_2) = v_0^2 k_2^2 P_\tau(k_1, k_2).$$

For every realization of the material, in the probabilistic sense, Section 3 provides explicitly the deviation from straightness. A formal expression of the mathematical expectation $E[A(z_1, t)A(z_2, t)]$, which by statistical invariance on coordinate z depends only on the difference $z_1 - z_2$, is straightforwardly deduced from (9): choosing $(z_1, z_2) = (0, z)$, linearity of the mathematical expectation $E[\cdot]$ and of (9) yields:

$$E[A(0,t)A(z,t)] =$$

$$\frac{4\alpha_0^4 c^2}{\pi^2}\int_0^t \int_0^t \left\{ \int_{-\alpha_0 c\theta_1}^{\alpha_0 c\theta_1} \int_{-\alpha_0 c\theta_2}^{\alpha_0 c\theta_2} \frac{R_\tau[z-(z_1-z_2), v_0(\theta_2-\theta_1)]\,\mathrm{d}z_1\,\mathrm{d}z_2}{\sqrt{\alpha_0^2 c^2\theta_1^2 - z_1^2}\sqrt{\alpha_0^2 c^2\theta_2^2 - z_2^2}} \right\} \mathrm{d}\theta_1\,\mathrm{d}\theta_2, \quad (10)$$

where we used the fact that the unperturbed crack front lies at abscissa $x_i = v_0\theta_i$ at time θ_i.

We now want to handle the right hand side of (10). We first get rid of one integration by means of the Parseval bilinear identity for real valued functions of two variables,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(z_1, z_2)g(z_1, z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2 = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{f}(k_1, k_2)\hat{g}(-k_1, -k_2)\,\mathrm{d}k_1\,\mathrm{d}k_2,$$

which allows the curly bracket of (10) to be transformed into

$$\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 2\pi\delta_0(k_1+k_2)\exp\left(\mathrm{i}z\frac{k_1-k_2}{2}\right)\tilde{R}_\tau\left[\frac{k_2-k_1}{2}, v_0(\theta_2-\theta_1)\right]$$

$$\times \pi J_0(k_1\alpha_0 c\theta_1)\pi J_0(k_2\alpha_0 c\theta_2)\,\mathrm{d}k_1\,\mathrm{d}k_2 = \frac{\pi}{2}\int_{-\infty}^{\infty} J_0(k\alpha_0 c\theta_1)J_0(k\alpha_0 c\theta_2)$$

$$\times \exp(-\mathrm{i}kz)\,\tilde{R}_\tau[k, v_0(\theta_2-\theta_1)]\,\mathrm{d}k$$

where the notation $\tilde{R}_\tau$ denotes the z-Fourier transform of R_τ, and δ_0 denotes the Dirac impulse function. Noticing that J_0 and $\tilde{R}_\tau$ are even functions of k, we obtain an expression for the autocorrelation function which is equivalent to, but simpler than, (10):

$$E[A(0,t)A(z,t)] = \frac{4\alpha_0^4 c^2}{\pi}\int_0^t\int_0^t\int_0^\infty J_0(k\alpha_0 c\theta_1)J_0(k\alpha_0 c\theta_2)$$

$$\times \cos(kz)\tilde{R}_\tau[k, v_0(\theta_2-\theta_1)]\,\mathrm{d}k\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2. \quad (11)$$

We now focus on the asymptotic value of (11) for large time t. We will show that, under some reasonable assumptions on function R_τ, this expression diverges at large time as $(\log t)^2$. In order to have a preliminary insight towards such a result, one might consider the case where the dependency of R_τ (thus of $\tilde{R}_\tau$ as well) on $x_2 - x_1$ is a Dirac function, $R_\tau[z, x] = R_\tau[z]\delta_0(x)$, modeling the random variable τ as a white noise in the x direction (the notation R_τ is kept for the new function of variable z only). Mathematically, this assumption results in replacing the (θ_1, θ_2) integration over the square $(\theta_1, \theta_2) \in [0, t]^2$ with one along the line segment $0 \leqslant \theta_1 = \theta_2 \leqslant t$ and (11) simplifies into

$$E[A(0,t)A(z,t)] = \frac{4\alpha_0^4 c^2}{\pi v_0}\int_0^t\int_0^\infty (J_0(k\alpha_0 c\theta))^2 \cos(kz)\tilde{R}_\tau[k]\,\mathrm{d}k\,\mathrm{d}\theta. \tag{12}$$

The next step is to use the well-known behavior of Bessel functions for large argument,

$$J_0(k\alpha_0 c\theta) \sim \sqrt{\frac{2}{\pi k\alpha_0 c\theta}}\cos\left(k\alpha_0 c\theta - \frac{\pi}{4}\right), \tag{13}$$

so that the large θ integration of its square generates a log t term, accounting for the averaging of the oscillating part. It is difficult to pursue the treatment, specifically, to explain why integration over k yields a *squared* log t in the final result (14), without being more rigorous. Yet, dominance of the diagonal $\theta_1 = \theta_2$ in integration over the square $(\theta_1, \theta_2) \in [0, t]^2$, and asymptotic behavior of Bessel Function J_0 are the two basic elements which we use in the Appendix to derive the following asymptotic expression of (11) for large times:

$$\text{for any fixed } z,\quad E[A(0,t)A(z,t)] = \frac{2\alpha_0^3 c}{\pi^2 v_0} P_\tau(0,0)(\log t)^2 + O(\log t). \tag{14}$$

In the process, we must assume that the following conditions (H1), (H2), (H3) and (H4) are fulfilled:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x\tilde{R}_\tau(k,x)|\,\mathrm{d}k\,\mathrm{d}x < \infty, \tag{H1}$$

$$\max_k \int_{-\infty}^{\infty} |\tilde{R}_\tau[k,x]|\,\mathrm{d}x < \infty, \tag{H2}$$

$$\int_{k=-\infty}^{\infty}\int_{y=x}^{\infty} |\tilde{R}_\tau[k,y]|\,\mathrm{d}k\,\mathrm{d}y < \frac{C_2}{x^{1+\varepsilon}}, \quad x > 0, \tag{H3}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\tilde{R}_{\tau,k}[k,x]|\,\mathrm{d}k\,\mathrm{d}x < \infty, \tag{H4}$$

where C_2 and ε are positive numbers. These conditions are quite realistic; indeed they are fulfilled for an autocorrelation function $R_\tau(z,x)$ with absolutely convergent integral on the (z,x) plane, which is continuous in z and which decreases faster than $1/x^{2+\varepsilon}$. Expression (14) calls for a comment about dimensionality of the argument of the logarithm: notice that whatever the time unit, it merely adds to (14) a term of order $O(\log t)$; the time unit can thus not be determined by the first-order asymptotics. For practical applications, it is reasonable to guess that the time unit involved in (14) is of the order of the coherence length, like b, of critical toughness G_{crit} divided by a relevant velocity (v_0 for direction x and $\alpha_0 c$ for direction z).

The striking fact about the result (14) is that it grows unboundedly for large times, although $E[A(z,t)] = 0$. This seems to indicate that the crack front develops random waviness of amplitude proportional to log t. This consideration is tempered by the remark that, since the result does not depend on z, at least at this order, we have

$$E\{[A(0,t)-A(z,t)]^2\} = E[A^2(0,t)]+E[A^2(z,t)]-2E[A(0,t)A(z,t)] = O(\log t),$$

a conclusion to be reinforced by later results for the power spectrum of $A(z,t)$ with respect to z, suggesting that (14) evidences a translational perturbation, and that waviness is to be looked for with amplitude of order $(\log t)^{1/2}$. Yet, the idea that the perturbation of order $\log t$ is merely a translation is too simple; indeed, it can be seen directly in (10) that, for any ζ with absolute value greater than 2,

$$E[A(0,t)A(\zeta\alpha_0 ct,t)] = O\left(t\int_{-\infty}^{\infty}\left|\max_{z>\alpha_0 ct(|\zeta|-2)} R_\tau(z,x)\right|\mathrm{d}x\right), \tag{15}$$

in which the integral is a bounded quantity for almost any physically reasonable function R_τ. This means that the length of coherent crack segments after traveling for duration t in the random material always remain smaller than $2\alpha_0 ct$. The physical argument that explains this feature consists of recalling that $\alpha_0 c$ is the velocity of information along the crack front.

6. Spatial Power Spectrum of the Crack Front Position, Slope and Velocity

To get some more insight about the growth of random waviness of the crack front, let us consider (11) again. When we replace $\cos kz$ with e^{ikz}, extending the integral on k from $-\infty$ to $+\infty$ at the expense of a factor of 2, the correlation $E[A(0,t)A(z,t)]$ is then expressed as an inverse Fourier transform of a quantity which we may call $\psi_A(k,t)$. That quantity is thus the power spectrum for the spatial dependence of crack front position at time t, and is given by

$$\psi_A(k,t) \equiv \int_{-\infty}^{+\infty} E[A(0,t)A(z,t)]\,e^{-ikz}\,\mathrm{d}z$$

$$= 4\alpha_0^4 c^2\int_0^t\int_0^t J_0(k\alpha_0 c\theta_1)J_0(k\alpha_0 c\theta_2)\tilde{R}_\tau[k, v_0(\theta_2-\theta_1)]\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2.$$

For $k>0$, following a method analogous to, but much simpler than, that used in Section 5, we are able to show that

$$\text{for any fixed } k,\quad \psi_A(k,t) = \frac{4\alpha_0^3 c P_\tau(k,\alpha_0 ck/v_0)\log(|k|\alpha_0 ct)}{\pi v_0|k|}+O(1), \tag{16}$$

this holding for $t \gg 1/|k|\alpha_0 c$.

To see the origin of (16), we use the association $x = v_0 t$, writing $x_1 = v_0\theta_1$ and $x_2 = v_0\theta_2$, and then make the change of variables $r = (x_1+x_2)/2$, $s = x_2-x_1$, so that the boundaries of the square over which one integrates extend from $-s_1(r,x)$ to $+s_1(r,x)$ where $s_1(r,x) = 2\min(r, x-r)$. Thus

$$\psi_A(k,t) = \frac{4\alpha_0^4 c^2}{v_0^2}\int_0^{v_0 t}\int_{-s_1(r,v_0 t)}^{+s_1(r,v_0 t)} J_0(\beta r-\beta s/2)J_0(\beta r+\beta s/2)\tilde{R}_\tau(k,s)\,\mathrm{d}s\,\mathrm{d}r$$

where $\beta = \alpha_0 kc/v_0$. Notice now that since $\tilde{R}_\tau(k, s)$ becomes effectively zero for $|s|$ greater than some correlation scale b, the effective domain of this integration is, for $v_0 t \gg b$, a narrow strip of length $v_0 t$ in the r direction but only of order $2b$ in the s direction. Considering now values of t such that $\beta v_0 t = \alpha_0 kct \gg 1$, the asymptotic behavior of J_0 then leads to

$$J_0(\beta r - \beta s/2) J_0(\beta r + \beta s/2) \approx \frac{\sin 2\beta r + \cos \beta s}{\pi \beta r}$$

at large βr and $s \ll r$. The $\sin 2\beta r$ term makes only a bounded contribution to the integral as $v_0 t \to \infty$ and can be neglected. Also, the effective limits on s can now be extended to infinity and, integrating in r,

$$\psi_A(k, t) \sim \frac{4\alpha_0^4 c^2}{v_0^2} \int_{-\infty}^{+\infty} \frac{\cos \beta s}{\pi \beta} \tilde{R}_\tau(k, s) \, \mathrm{d}s \log(\alpha_0 |k| ct),$$

the lower limit in r being chosen as that for which $\beta r = 1$. The $\cos \beta s$ can be replaced by $e^{-i\beta s}$, and this second Fourier transform of $R_\tau(z, s)$ [the first changed $R_\tau(z, s)$ to $\tilde{R}_\tau(k, s)$] now produces the power spectrum $P_\tau(k, \beta)$, and we obtain the formula (16) when we recognize that $\beta = \alpha_0 ck/v_0$.

In the particular case of an isotropic fracture plane discussed earlier, with the exponential decay of correlation over length scale b, the divergent part of the spectrum is thus

$$\psi_A(k, t) \sim \frac{8\sigma^2 b^2 \alpha_0^3 c v_0^2}{[v_0^2 + (kbc)^2]^{3/2} |k|} \log(\alpha_0 |k| ct).$$

The coefficient of $\log(\alpha_0 |k| ct)$ is divergent near $k = 0$ and its decay with increasing $|k|$, as $1/|k|$ for small $|k|$, becomes much more rapid for $|k| > v_0/bc$.

Since we always expect the spectrum P_τ to be relatively constant for large wavelengths (i.e. small k) compared to lengthscales in the correlation function, result (16) suggests that the long wavelength wiggles grow proportionally to their wavelength. This is consistent with the following naive view of their development: on one hand, random wiggles are generated at small lengthscales, of the same order as the coherence length of the toughness homogeneities. On the other hand, one must remember that, at all scales, the crack front tends to straighten back. This can be viewed as a tendency to average locally the perturbation A. But averaging the perturbation of A on a patch of some length L takes some time, during which new perturbations are generated on lower scales; on the other hand, the averaging creates a relatively straight fault segment of length L, one among many uncorrelated ones, all of them appearing as perturbations on larger scales. This view recalls vaguely the model of turbulent flow in which energy from bigger eddies feeds smaller eddies, down to a scale where viscosity dominates inertia. In the crack front case though, disorder from smaller scales feeds that on larger scales. This feature is not surprising, since perturbation from the straight crack front is created on the small scale of the toughness inhomogeneities.

The autocorrelation function for crack front slope $S(z, t) \equiv \partial a(z, t)/\partial z = \partial A(z, t)/\partial z$ is obtained by differentiating $E[A(z_1, t)A(z_2, t)] = E[A(0, t)A(z_2 - z_1, t)]$ with respect to z_1 and z_2. We thus have

$$E[S(0,t)S(z,t)] = -\frac{\partial^2}{\partial z^2} E[A(0,t)A(z,t)]$$

and hence the power spectral density $\psi_S(k,t)$ for the slope is $k^2\psi_A(k,t)$. This is logarithmically divergent in t, from (16), but the spectrum for S is integrable over $k = 0$.

Expressions for the velocity perturbation $V(z,t)$ and $\tilde{V}(k,t)$ were given earlier. At long times the transient terms, representing the initial effect of entering the randomly heterogeneous part $x > 0$ of the fracture plane, become statistically uncorrelated with the driving terms $\tilde{\tau}_d$ or τ_d. Thus to calculate the long time divergent part of the spatial power spectrum $\psi_V(k,t)$, defined by

$$\psi_V(k,t) = \int_{-\infty}^{+\infty} E[V(0,t)V(z,t)]\,\mathrm{e}^{-ikz}\,\mathrm{d}z,$$

we can use the result (16) for $\psi_A(k,t)$, changing the forcing power spectrum from that of τ to that of τ_d as discussed earlier, so that P_τ of (16) gets replaced by $v_0^2(\alpha_0 ck/v_0)^2 P_\tau$. Thus

$$\psi_V(k,t) \sim \frac{4\alpha_0^5 c^3 |k| P_\tau(k, \alpha_0 ck/v_0)}{\pi v_0} \log(\alpha_0|k|ct)$$

when $\alpha_0|k|ct \gg 1$, and in the case of the statistically isotropic fracture plane with exponential correlation decay over lengthscale b,

$$\psi_V(k,t) \sim \frac{8\sigma^2 b^2 \alpha_0^5 c^3 v_0^2 |k|}{[v_0^2 + (kbc)^2]^{3/2}} \log(\alpha_0|k|ct).$$

The coefficient of the logarithm is as large as possible at the wavenumber $|k| = v_0/2bc$. Hence, writing $|k| = 2\pi/L$, the wavelength L on which the strongest variations of velocity are expected is $L \approx 4\pi bc/v_0 \approx 25b$ when $v_0/c = 1/2$. Thus the most active dynamical scale is greatly enlarged over the heterogeneity scale in the material. Note that the logarithmically divergent parts of $\psi_V(k,t)$ and $\psi_S(k,t)$ are proportional to one another, $\psi_V \sim \alpha_0^2 c^2 \psi_S$.

Using $\psi_V(k,t)$ above for the statistically isotropic surface with exponential correlation decay, let us now estimate the variance $E[V^2(z,t)]$ of the deviation of propagation velocity from the mean (note that this variance is independent of z) by

$$E[V^2(z,t)] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \psi_V(k,t)\,\mathrm{d}k.$$

Thus, with the ψ_V above at large t, and with the substitution $y = kbc/v_0$ in the integration,

$$E[V^2(z,t)] \sim (4\alpha_0^4 c^2\sigma^2)\frac{2\alpha_0 v_0}{\pi c}\int_0^{\infty} \frac{y \log(\alpha_0 y v_0 t/b)}{(1+y^2)^{3/2}}\,\mathrm{d}y = (4\alpha_0^4 c^2\sigma^2)\frac{2\alpha_0 v_0}{\pi c}\log\left(\frac{2\alpha_0 v_0 t}{b}\right).$$

The factor $(4\alpha_0^4 c^2\sigma^2)$ has been isolated because this is what we would have predicted for $E[V^2(z,t)]$ from a purely local model, e.g. neglecting the 3D effects of wave propagation along the moving crack front as contained on the right side of (5), and instead of just writing (5) as $V(z,t) = -2\alpha_0^2 c^2 \tau(z,t)$. That procedure corresponds to predicting V in terms of G_{crit} at each moment of time as one would do for a purely 2D, straight-front crack model with $G_{\text{crit}}(x,z)$ having no dependence on z.

Thus the coefficient of $4\alpha_0^4 c^2 \sigma^2$ is an amplification factor on the variance due to the 3D and wave effects. That amplification factor is determined by our present considerations only to within an additive constant at long time, and thus has the asymptotic form: constant$+(2\alpha_0 v_0/\pi c)\log(2\alpha_0 v_0 t/b)$. The amplification does ultimately diverge but that happens very slowly in time and the growth actually realized may be small in many practical cases. That is, if t_1 and t_2 are times within the range for which the asymptotic form of the amplification factor is valid, then in order for the variance to increase between t_1 and t_2 by an amount equal to its unamplified value, $4\alpha_0^4 c^2 \sigma^2$, it is necessary that

$$E[V^2(z,t_2)] - E[V^2(z,t_1)] = (4\alpha_0^4 c^2 \sigma^2)(2\alpha_0 v_0/\pi c)\log(t_2/t_1) = 4\alpha_0^4 c^2 \sigma^2,$$

and thus that $t_2/t_1 = \exp(\pi c/2\alpha_0 v_0)$. This t_2/t_1 ratio becomes unbounded as $v_0 \to 0$ and as $v_0 \to c$, whereas $t_2/t_1 = \exp(2\pi/\sqrt{3}) \approx 38$ when $v_0 = c/2$ and the lowest possible ratio, $t_2/t_1 = \exp(\pi) \approx 23$, results when $v_0 = c\sqrt{2} \approx 0.71c$. Recalling that the coefficient of the log term in ψ_V is greatest when $k = v_0/2bc$, and that the expression with the log term is valid when $\alpha_0 kct \gg 1$, we require $\alpha_0 v_0 t/2b \gg 1$ for validity of the log term in the asymptotic expression for $E[V^2(z,t)]$. If we interpret that as, say, $\alpha_0 v_0 t/2b > 5$ and thus choose t_1 as the minimum time to meet that, $\alpha_0 v_0 t_1/2b = 5$, then for $v_0 = 0.5c$, t_1 corresponds to a crack travel distance $v_0 t_1 \approx 11.5b$ and the further travel distance, for $E[V^2(z,t_2)]$ to have increased by an amount equal to the unamplified variance, is $v_0(t_2 - t_1) \approx 425b$. To achieve yet another increase by the unamplified variance requires travel $v_0(t_3 - t_2) \approx 16{,}170b$, and one more increase requires $v_0(t_4 - t_3) \approx 614{,}400b$. Thus, even for very small correlation lengths b, the required growth to amplify significantly, or perhaps even noticeably, the variance of propagation speed may become greater than the size of the cracked body or of the range of crack growth through it for which the half-plane crack model can be justified.

Note that since $\psi_S(k,t) \sim \psi_V(k,t)/\alpha_0^2 c^2$ at long time, the variance $E[S^2(z,t)]$ in crack front slope will behave asymptotically like $E[V^2(z,t)]/\alpha_0^2 c^2 \sim (8\alpha_0^3 \sigma^2 v_0/\pi c)\log(2\alpha_0 v_0 t/b)$ at large t.

7. Fourier Series Representation of Crack Growth

A computational approach to crack growth through regions of locally variable G_{crit} is given by Rice *et al.* (1994) based on a Fourier series representation of crack front position $a(z,t)$ with respect to z. We note that there are four differences between those computations and our treatment of the linear case here: (i) they consider that the random physical characteristic is G_{crit}, but that merely changes the average value of τ to a non-zero value; (ii) we replaced $\tau(z, v_0 t + A(z,t))$ by $\tau(z, v_0 t)$, which is consistent with their *strictly linearized* first-order approximation; (iii) we have neglected terms of order $(v - v_0)^2$ and higher when linearizing their formulae, which are chosen to replicate exactly the solution for a straight crack front with an arbitrary growth history; and (iv) for numerical reasons, they use a periodic crack front. Difference (i) is not serious. On the other hand differences (ii) and (iii) might hinder the present results from the beginning if the perturbation τ is not small enough. Even if τ is small, our linear analysis predicts that A and V will grow unboundedly; at some time, $(v-v_0)^2$ and products of the form $A\tau$ will not be negligible any more. It seems that disorder may begin to grow and then saturate because of the nonlinear terms. If this is confirmed, it might be possible to look for an average level of disorder on the crack front as a balance between the chaogenic influence of the random toughness and the truncating effects of the nonlinear terms.

Difference (iv) has to be carefully taken into account. Let us represent the toughness variation on $x > 0$ and its correlation in the Fourier series

$$\tau(z,x) = \sum_{m=-N}^{+N} e^{2i\pi mz/\lambda}\,\tau_m(x), \quad R_\tau(z,x) = \sum_{m=-N}^{+N} e^{2i\pi mz/\lambda}\,r_m(x),$$

where the $\tau_m(x)$ are a set of statistically independent stationary random functions, with independent but identically distributed real and imaginary parts, with $\tau_{-m} = \bar{\tau}_m$ (the over-bar denotes complex conjugate), and with $r_m(x) = E[\tau_m(\xi)\bar{\tau}_m(\xi+x)]$. Thus the toughness variation is truly random in a strip along the x direction with width λ in the z direction, but is periodically replicated into adjoining strips. In this circumstance $R_\tau(z,x)$ is necessarily periodic in λ but we will generally want to choose $\lambda \gg b$, where b is a correlation scale in the underlying non-periodic random toughness variation. Then, by letting $N \to \infty$ and using standard methods of Fourier series, we may choose the $r_m(x)$ to make the $R_\tau(z,x)$ given by the series agree exactly with the underlying $R_\tau(z,x)$ in the strip $-\lambda/2 < z < \lambda/2$. We let

$$p_m(k) = \int_{-\infty}^{+\infty} e^{-ikx}\,r_m(x)\,dx$$

denote the power spectral density of $\tau_m(x)$, i.e. of mode m of the toughness variation. One then readily shows that the Fourier series coefficients $r_m(x)$ are such that $p_m(k) = P_\tau(2\pi m/\lambda, k)/\lambda$, provided λ is much greater than b so integrals over $(-\lambda/2, +\lambda/2)$ of $R_\tau(z,x)$ times trigonometric functions of z may be replaced by integrals over $(-\infty, +\infty)$.

The resulting crack front perturbations may be written as

$$A(z,t) = \sum_{m=-N}^{+N} e^{2i\pi mz/\lambda}\,A_m(t), \quad V(z,t) = \sum_{m=-N}^{+N} e^{2i\pi mz/\lambda}\,V_m(t),$$

where $A_{-m} = \bar{A}_m$ and $V_m(t) = dA_m(t)/dt$. One readily shows that $A_m(t)$ and $V_m(t)$ are related to $\tau_m(v_0 t)$ by the same expressions of Section 3 which relate $\tilde{A}(k,t)$ and $\tilde{V}(k,t)$ to $\tilde{\tau}(k,t)$ for $k = 2\pi m/\lambda$. Thus $E[A_m(t)] = E[V_m(t)] = 0$ and one finds

$$E[|A_m(t)|^2] = 4\alpha_0^4 c^2 \int_0^t \int_0^t J_0(2\pi m\alpha_0 c\theta_1/\lambda)\,J_0(2\pi m\alpha_0 c\theta_2/\lambda)\,r_m[v_0(\theta_2-\theta_1)]\,d\theta_1\,d\theta_2.$$

The same equation holds for $E[|V_m(t)|^2]$ at large t, when memory is lost of the initial velocity distribution induced by entry into the heterogeneous region at $t = 0$, provided that we replace $r_m(x)$ by $-v_0^2\,d^2r_m(x)/dx^2$.

For Fourier modes $m \geq 1$ the results are like those obtained for the power spectral densities $\psi_A(k,t)$ and $\psi_V(k,t)$ of the infinite width case in Section 6; the mean square modal amplitudes diverge as $\log t$ for $2\pi m\alpha_0 ct/\lambda \gg 1$:

$$E[|A_m(t)|^2] \sim \frac{2\alpha_0^3 c\lambda}{\pi^2 m v_0} p_m(2\pi m\alpha_0 c/\lambda v_0)\log(2\pi m\alpha_0 ct/\lambda),$$

$$E[|V_m(t)|^2] \sim \frac{8m\alpha_0^5 c^3}{\lambda v_0} \cdot p_m(2\pi m\alpha_0 c/\lambda v_0)\log(2\pi m\alpha_0 ct/\lambda).$$

However, the mode $m = 0$, corresponding to the average of the perturbed motion over the periodic repeat distance λ in the z-direction, has different behavior: first, $V_0(t)$ is a *purely stationary* process for $t > 0$ with statistics scaled to those of the average, $\tau_0(v_0 t)$, of the fracture energy heterogeneity $\tau(z, v_0 t)$ over distance λ;

$$E[V_0^2(t)] = 4\alpha_0^4 c^2 r_0(0).$$

Second, although this averaged-in-z velocity fluctuation $V_0(t)$ from v_0 is stationary, its integral process $A_0(t)$ is not. Rather,

$$E[A_0^2(t)] = 4\alpha_0^4 c^2 \int_0^t \int_0^t r_0[v_0(\theta_2-\theta_1)]\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2 \sim 4\alpha_0^4 c^2 p_0(0) t/v_0.$$

We may note that $E[A_0^2(t)]$ coincides asymptotically, at large t, with $E[A(0,t)A(z,t)]$, since the z-varying fluctuations grow only as $\log t$. Also, for λ large compared to the correlation scale, we have $\lambda p_0(0) = P_\tau(0,0)$ and hence for the λ-periodic case

$$E[A(0,t)A(z,t)] \sim 4\alpha_0^4 c^2 P_\tau(0,0) t/v_0\lambda.$$

This means that $\{E[A(0,t)A(z,t)]\}^{1/2}$ grows proportionally to $\sqrt{t/\lambda}$ instead of $\log t$, for the λ-periodic system versus the infinite system, although the statistics of $[A(z,t)-A(0,t)]$ are essentially identical in both cases.

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APPENDIX

The first step is to show that the contribution to (11) of the diagonal $\theta_1 = \theta_2$ is dominant, and that the integration domain $(0 < \theta_1 < t,\ 0 < \theta_2 < t)$ can be replaced by the more forthcoming one $(0 < \theta_1 + \theta_2 < 2t)$ with merely adding a bounded term. Indeed, let us define

$$\varphi_1(z,t) = \int_0^{\theta=t}\int_0^{\infty}\int_0^{\infty} J_0(k\alpha_0 c(\theta+u))J_0(k\alpha_0 c(\theta-u))\cos(kz)\tilde{R}_\tau(k,2v_0 u)\,\mathrm{d}k\,\mathrm{d}u\,\mathrm{d}\theta. \quad \text{(A1)}$$

The integrals in (11) and (A1) differ in that: (i) the change of variables $\theta_1 = \theta - u$, $\theta_2 = \theta + u$ has been performed; (ii) symmetry in u has been used; and (iii) the integration domain of (A1) consists of that of (11) plus the region $u > \max(\theta, t-\theta)$. Considering the coarse upper bounding of the norm of Bessel and cosine functions by unity, we get the upper bound

$$\left| E[A(0,t)A(z,t)] - \frac{16\alpha_0^4 c^2}{\pi}\varphi_1(z,t)\right| < \frac{16\alpha_0^4 c^2}{\pi}\int_{\theta=0}^{t}\int_{u=\max(\theta,t-\theta)}^{\infty}\int_0^{\infty}|\tilde{R}_\tau(k,2v_0 u)|\,\mathrm{d}k\,\mathrm{d}u\,\mathrm{d}\theta,$$

which is further bounded by

$$\frac{32\alpha_0^4 c^2}{\pi}\int_{\theta=0}^{\infty}\int_{u=\theta}^{\infty}\int_0^{\infty}|\tilde{R}_\tau(k,2v_0 u)|\,\mathrm{d}k\,\mathrm{d}u\,\mathrm{d}\theta < \frac{2\alpha_0^4 c^2}{\pi v_0^2}\int_0^{\infty}\int_0^{\infty}|x\tilde{R}_\tau(k,x)|\,\mathrm{d}k\,\mathrm{d}x.$$

We encounter the first condition on autocorrelation function R for our analysis to be carried on: the last integral must be finite; hereafter, this is called hypothesis (H1).

The second step is to consider the derivative of φ_1 with respect to t to get

$$\alpha_0 c\varphi_{1,t}\left(z, t = \frac{x}{\alpha_0 c}\right) = \int_0^{\infty}\int_0^{\infty} J_0(kx+ky)J_0(kx-ky)\cos(kz)\tilde{R}_\tau(k,\lambda y)\,\mathrm{d}k\,\mathrm{d}y,$$

where $\lambda = 2v_0/(\alpha_0 c)$. At this point, we notice that, provided that the following hypothesis (H2)

$$\max_k \int_0^{\infty}|\tilde{R}_\tau(k,x)|\,\mathrm{d}x = C_1 < \infty$$

holds, we can define

$$\varphi_2(z,x) = \int_{k=2x}^{\infty}\int_0^{\infty} J_0(kx+ky)J_0(kx-ky)\cos(kz)\tilde{R}_\tau(k,\lambda y)\,\mathrm{d}k\,\mathrm{d}y \quad \text{(A2)}$$

which verifies

$$|\alpha_0 c\varphi_{1,t}(z,t) - \varphi_2(z,\alpha_0 ct)| < \frac{C_1}{v_0 t}.$$

We shall claim that the quantity φ_2 behaves as $(\log x + O(1))/x$ for large x. We now want to approach the Bessel functions by formulae analogous to (13), which is interesting only if the arguments are bounded from below. For that reason, we need to reduce the interval of integration on variable y so that $kx-ky$ remains greater than, say, 1. For the error introduced in doing so not to affect the result, a sufficient condition is that there exist some exponent $0<\varepsilon<1$ such that

$$\int_{k=-\infty}^{\infty}\int_{y=lx^{\varepsilon}}^{\infty}|\tilde{R}_{\tau}(k,y)|\,\mathrm{d}k\,\mathrm{d}y<\frac{C_2}{x},$$

where l has dimension $(\text{length})^{1-\varepsilon}$ and C_2 is a constant; this is equivalent to hypothesis (H3) in the main text, provided C_2 and ε appearing here be rewritten $C_2 l^{-1-\varepsilon}$ and $1/(1+\varepsilon)$ with $\varepsilon>0$. Then, if we define function $\varphi_3(z,x)$ as the double integral of equation (A2), except for the upper bounds of integration on variable y which are set to lx^{ε}, we have the upper bounding

$$|\varphi_2(z,x)-\varphi_3(z,x)|<\frac{C_2/(2\lambda)}{x}.$$

For x large enough, namely $x>(2l)^{1/(1-\varepsilon)}$, all arguments of Bessel functions appearing in the definition of φ_3 are greater than 1, allowing for the use of upper bounds

$$\left|\{J_0(kx-ky)J_0(kx+ky)\}-\frac{\sin 2kx+\cos 2ky}{\pi k\sqrt{x^2-y^2}}\right|<\frac{C_3}{k^2x^2}\tag{A3}$$

[derived from formulae 1, 7 and 8 of paragraph 8.451 of Gradshteyn and Ryzhik (1987)] where C_3 is a constant, so as to obtain

$$\left|\varphi_3(z,x)-\int_{k=2/x}^{\infty}\int_{y=0}^{lx^{\varepsilon}}\frac{\sin 2kx+\cos 2ky}{\pi kx\sqrt{1-y^2/x^2}}\cos(kz)\tilde{R}_{\tau}(k,\lambda y)\,\mathrm{d}k\,\mathrm{d}y\right|$$
$$<\frac{C_3}{x^2}\int_{k=2/x}^{\infty}\int_{y=0}^{lx^{\varepsilon}}|\tilde{R}_{\tau}(k,\lambda y)|\frac{\mathrm{d}k}{k^2}\,\mathrm{d}y,\tag{A4}$$

in which we partition both integrals to build the upper bound:

$$\left|\varphi_3(z,x)-\int_{k=2/x}^{\infty}\int_{0}^{\infty}\frac{\sin 2kx+\cos 2ky}{\pi kx}\cos(kz)\tilde{R}_{\tau}(k,\lambda y)\,\mathrm{d}k\,\mathrm{d}y\right|$$
$$<\frac{C_3}{x^2}\int_{k=2/x}^{k_1}\int_{y=0}^{lx^{\varepsilon}}|\tilde{R}_{\tau}(k,\lambda y)|\frac{\mathrm{d}k}{k^2}\,\mathrm{d}y+\frac{C_3}{k_1^2x^2}\int_{k=k_1}^{\infty}\int_{y=0}^{lx^{\varepsilon}}|\tilde{R}_{\tau}(k,\lambda y)|\,\mathrm{d}k\,\mathrm{d}y$$
$$+x^{2\varepsilon-3}\int_{k=2/x}^{k_1}\int_{y=0}^{lx^{\varepsilon}}|\tilde{R}_{\tau}(k,\lambda y)|\frac{\mathrm{d}k}{k}\,\mathrm{d}y+\frac{x^{2\varepsilon-3}}{k_1}\int_{k=k_1}^{\infty}\int_{y=0}^{lx^{\varepsilon}}|\tilde{R}_{\tau}(k,\lambda y)|\,\mathrm{d}k\,\mathrm{d}y$$
$$+\int_{k=2/x}^{\infty}\int_{y=lx^{\varepsilon}}^{\infty}|\tilde{R}_{\tau}(k,\lambda y)|\,\mathrm{d}k\,\mathrm{d}y.\tag{A5}$$

More precisely, the first and second integrals of the right hand side appear when splitting the integration over k at point $k=k_1$, and using $1/k<k_1$ in the second one. The third and fourth terms appear when replacing $1/(1-y^2/x^2)^{1/2}$ by 1 in the integral of the left hand side of (A4), and bounding the difference by y^2/x^2; this holds because $y<lx^{\varepsilon}$ and $x>(2l)^{1/(1-\varepsilon)}$, whence $y^2/x^2<1/4$. The k-integration domain is then split at the point $k=k_1$. The fifth term appears when further expanding the integration domain from $(2/x<k; y<lx^{\varepsilon})$ to $(2/x<k; y<\infty)$. To obtain the last three integrals, we also bounded the trigonometric functions by 1 and 2 by π. Hypothesis (H1) guarantees that the second term is $O(1/x^2)$ and that the fourth one is $O(x^{2\varepsilon-3})$, hypothesis (H2) that the first term is $O(1/x)$ and the third term $O(x^{2\varepsilon-3}\log x)$, and hypothesis (H3) that the fifth one is $O(1/x)$. Finally $\varepsilon<1$ makes the right hand side of (A5) $O(1/x)$.

Let us now concentrate on the integral of the left hand side of (A5), and first consider the term containing $\sin 2kx$:

$$\varphi_4(z,x) = \int_{2/x}^{\infty} \frac{\sin 2kx \cos kz}{kx} F(k)\,\mathrm{d}k, \quad \text{with} \quad F(k) = \frac{1}{\pi}\int_0^{\infty} \tilde{R}_\tau(k, \lambda y)\,\mathrm{d}y.$$

Let us define

$$Y(u,w) = \int_0^u \frac{\sin r \cos (rw)}{r}\,\mathrm{d}r = \int_0^u \frac{\sin [r(1+w)] + \sin [r(1-w)]}{2r}\,\mathrm{d}r.$$

The last equality makes it clear that the function Y is bounded, for instance by $3\pi/2$. Integrating the definition of φ_4 by parts yields

$$\varphi_4(z,x) = \left[\frac{Y\big(2kx, z/(2x)\big)F(k)}{x}\right]_{2/x}^{\infty} - \frac{1}{x}\int_{2/x}^{\infty} Y\big(2kx, z/(2x)\big)F'(k)\,\mathrm{d}k = O(1/x)$$

using hypothesis (H2), and provided that the following hypothesis (H4) holds

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\tilde{R}_{\tau,k}(k, x)|\,\mathrm{d}k\,\mathrm{d}x < \infty. \tag{H4}$$

Secondly, we consider the cos $2ky$ of the left hand side of (A5), and easily show that, *for fixed* z, it amounts to

$$\frac{\log x}{2\pi\lambda x} P_\tau(0,0) + O(\log x/x^2)$$

where P_τ denotes the double Fourier transform of R_τ, which is rephrased

$$\varphi_{1,t} = \frac{1}{\alpha_0 c}\frac{\log t}{4\pi v_0 t} P_\tau(0,0) + O(1/t),$$

and yields (14) through integration of t.