MECHANICS OF SOLIDS

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INTRODUCTION

The application of the principles of mechanics to bulk matter is conventionally divided into the *mechanics of fluids* and the *mechanics of solids*. The entire subject is often called *continuum mechanics*, particularly when we adopt the useful model of matter as being continuously divisible, making no reference to its discrete structure at microscopic length scales well below those of the application or phenomenon of interest. *Solid mechanics* is concerned with the stressing, deformation and failure of solid materials and structures. What, then, is a solid? Any material, fluid or solid, can support *normal* forces. These are forces directed perpendicular, or normal, to a material plane across which they act. The force per unit of area of that plane is called the *normal stress*. Water at the base of a pond, air in an automobile tire, the stones of a Roman arch, rocks at base of a mountain, the skin of a pressurized airplane cabin, a stretched rubber band and the bones of a runner all support force in that way (some only when the force is compressive). We call a material *solid* rather than *fluid* if it can also support a substantial *shearing force* over the time scale of some natural process or technological application of interest. Shearing forces are directed parallel, rather than perpendicular, to the material surface on which they act; the force per unit of area is called *shear stress*. For example, consider a vertical metal rod that is fixed to a support at its upper end and has a weight attached at its lower end. If we consider a horizontal surface through the material of the rod, it will be evident that the rod supports normal stress. But it also supports shear stress, and that becomes evident when we consider the forces carried across a plane through the rod that is neither horizontal nor vertical. Thus, while water and air provide no long term support of shear stress, normally granite, steel, and rubber do so, and are called solids. Materials with tightly bound atoms or molecules, like the crystals formed below melting temperature by most substances or simple compounds, or the amorphous structures formed in glass and many polymer substances at sufficiently low temperature, are usually considered solids.

The distinction between solids and fluids is not precise and in many cases will depend on the time scale. Consider the hot rocks of the Earth's mantle. When a large earthquake occurs, an associated deformation disturbance called a seismic wave propagates through the adjacent rock and the whole earth is set into vibrations which, following a sufficiently large earthquake, may remain detectable with precise instruments for several weeks. We would then describe the rocks of the mantle as solid. So would we on the time scale of, say, tens to thousands of years, over which stresses rebuild enough in the source region to cause one or a few repetitions of the earthquake. But on a significantly longer time scale, say of order of a million years, the hot rocks of the mantle are unable to support shearing stresses and flow as a fluid. Also, many children will be familiar with a substance called *silly putty*, a polymerized silicone gel. If a ball of it is left to sit on a table at room temperature, it flows and flattens on a time scale of a few minutes to an hour. But if picked up and tossed as a ball against a wall, so that large forces act only over the short time of the impact, it bounces back and retains its shape like a highly elastic solid.

In the simple but very common case when such a material is loaded at sufficiently low temperature and/or short time scale, and with sufficiently limited stress magnitude, its deformation is fully recovered upon unloading. We then say that the material is *elastic*. But substances can also deform permanently, so that not all deformation is recovered. For example, if you bend a metal coat hanger substantially and then release the loading, it springs back only partially towards its initial shape, but does not fully recover and remains bent. We say that the metal of the coat hanger has been permanently deformed and in this case, for which the permanent deformation is not so much a consequence of long time loading at sufficiently high temperature, but more a consequence of subjecting the material to large stresses (above the yield stress), we describe the permanent deformation as *plastic* deformation, and call the material *elastic-plastic*. Permanent deformation of a sort that depends mainly on time of exposure to a stress, and that tends to increase significantly with time of exposure, is called *viscous* or *creep* deformation and materials which exhibit that, as well as tendencies for elastic response, are called *viscoelastic* solids (or sometimes *visco-plastic* solids when we focus more on the permanent strain than on the tendency for partial recovery of strain upon unloading).

Who uses solid mechanics? All those who seek to understand natural phenomena involving the stressing, deformation, flow and fracture of solids, and all those who would have knowledge of such phenomena to improve our living conditions and accomplish human objectives, have use for solid mechanics. The latter activities are, of course, the domain of engineering and many important modern subfields of solid mechanics have been actively developed by engineering scientists concerned, for example, with mechanical, structural, materials, civil or aerospace engineering. Natural phenomena involving solid mechanics are studied in geology, seismology and tectonophysics, in materials science and the physics of condensed matter, and in parts of biology and physiology. Further, because solid mechanics poses challenging mathematical and computational problems, it (as well as fluid mechanics) has long been an important topic for applied mathematicians concerned, for example, with partial differential equations and with numerical techniques for digital computer formulations of physical problems.

Here is a sampling of some of the issues addressed using solid mechanics concepts: How do flows develop in the earth's mantle and cause continents to move and ocean floors to slowly subduct beneath them? How do mountains form? What processes take place along a fault during an earthquake, and how do the resulting disturbances propagate through the earth as seismic waves, and shake, and perhaps collapse, buildings and bridges? How do landslides occur? How does a structure on a clay soil settle with time, and what is the maximum bearing pressure which the footing of a building can exert on a soil or rock foundation without rupturing it? What materials do we choose, and how do we proportion and shape them and control their loading, to make safe, reliable, durable and economical structures, whether airframes, bridges, ships, buildings, chairs, artificial heart valves, or computer chips, and to make machinery such as jet engines, pumps, bicycles, and the like? How do vehicles (cars, planes, ships) respond by vibration to the irregularity of surfaces or media along which they move, and how are vibrations controlled for comfort, noise reduction and safety against fatigue failure? How rapidly does a crack grow in a cyclically loaded structure, whether a bridge, engine, or airplane wing or fuselage, and when will it propagate catastrophically? How do we control the deformability of structures during impact so as to design crashworthiness into vehicles? How do we form the materials and products of our technological civilization, e.g., by extruding metals or polymers through dies, rolling material into sheets, punching out complex shapes, etc.? By what microscopic processes do plastic and creep strains occur in polycrystals? How can we fashion different materials together, like in fiber reinforced composites, to achieve combinations of stiffness and strength needed in applications? What is the combination of material properties and overall response needed in downhill skis or in a tennis racket? How does the human skull respond to impact in an accident? How do heart muscles control the pumping of blood in the human body, and what goes wrong when an aneurism develops?

HISTORICAL SKETCH

Solid mechanics developed in the outpouring of mathematical and physical studies following the great achievement of Isaac Newton (1642-1727) in stating the laws of motion, although it has earlier roots. The need to understand and control the fracture of solids seems to have been a first motivation. Leonardo da Vinci (1452-1519) sketched in his notebooks a possible test of the tensile strength of a wire. The Italian experimental scientist Galileo Galilei (1564-1642), who died in the year of Newton's birth, had investigated the breaking loads of rods in tension and concluded that the load was independent of length and proportional to the cross section area, this being a first step towards a concept of stress. He also investigated the breaking loads on beams which were suspended horizontally from a wall into which they were built in.

Concepts of stress, strain and elasticity. The English scientist Robert Hooke discovered in 1660, but published only in 1678, the observation that for many materials that displacement under a load was proportional to force, thus establishing the notion of (linear) elasticity but not yet in a way that was expressible in terms of stress and strain. E. Mariotte in France published similar discoveries in 1680 and, also, reached an understanding of how beams like those studied by Galileo resisted transverse loadings or, more precisely, resist the torques caused by those transverse loadings, by developing extensional and compressional deformations, respectively, in material fibers along their upper and lower portions. It was for Swiss mathematician and mechanician James Bernoulli (1654-1705) to observe, in the final paper of his life, in 1705, that the proper way of describing deformation was to give force per unit area, or stress, as a function of the elongation per unit length, or strain, of a material fiber under tension. Swiss mathematician and mechanician Leonhard Euler (1707-1783), who was taught mathematics by James' brother John Bernoulli (1667-1748), proposed, among many contributions, a linear relation between stress σ and strain ε in 1727, of form $\sigma = E\,\varepsilon$ where the coefficient E is now generally called Young's modulus after English naturalist Thomas Young who developed a related idea in 1807.

The notion that there is an internal tension acting across surfaces in a deformed solid was expressed by German mathematician and physicist Gottfried Wilhelm Leibniz in 1684 and James Bernoulli in 1691. Also, Bernoulli and Euler (see below) introduced the idea that at a given section along the length of a beam there were internal tensions amounting to a net force and a net torque. Euler introduced the idea of compressive normal stress as the pressure in a fluid in 1752. The French engineer and physicist Charles-Augustine Coulomb (1736-1806) was apparently the first to relate the theory of a beam as a bent elastic line to stress and strain in an actual beam, in a way never quite achieved by Bernoulli and, although possibly recognized, never published by Euler. He developed the famous expression $\sigma = M\,y\,/\,I$ for the stress due to the pure bending of a homogeneous linear elastic beam; here M is the torque, or *bending moment*, y is the distance of a point from an axis that passes through the section centroid, parallel to the torque axis, and I is the integral of y^2 over the section area. The French mathematician Parent introduced the concept of shear stress in 1713, but Coulomb was the one who extensively developed the idea in connection with beams and with the stressing and failure of soil in 1773, and studies of frictional slip in 1779. It was the great French mathematician Augustin Louis Cauchy (1789-1857), originally educated as an engineer, who in 1822 formalized the stress concept in the context of a general three-dimensional theory, showed its properties as consisting of a 3 by 3 symmetric array of numbers that transform as a tensor, derived the equations of motion for a continuum in terms of the components of stress, and gave the specific development of the theory of linear elastic response for isotropic solids. As part of this work, Cauchy also introduced the equations which express the six components of strain, three extensional and three shear, in terms of derivatives of displacements for the case when all those derivatives are much smaller than unity; similar expressions had been given earlier by Euler in expressing rates of straining in terms of the derivatives of the velocity field in a fluid.

Beams, columns, plates, shells. The 1700's and early 1800's were a productive period in which the mechanics of simple elastic structural elements were developed well before the beginnings in the 1820's of the general three-dimensional theory. The development of beam theory by Euler, who generally modeled beams as elastic lines which resist bending, and by several members of the Bernoulli family and by Coulomb, remains among the most immediately useful aspects of solid mechanics, in part for its simplicity and in part because of the pervasiveness of beams and columns in structural technology. James Bernoulli proposed in his final paper of 1705 that the curvature of a beam was proportional to bending moment. Euler in 1744 and John's son, Daniel Bernoulli (1700-1782) in 1751 used the theory to address the transverse vibrations of beams, and Euler gave in 1757 his famous analysis of the buckling of an initially straight beam subjected to a compressive loading; the beam is then commonly called a column. Following a suggestion of Daniel Bernoulli in 1742, Euler in 1744 introduced the strain energy per unit length for a beam, proportional to the square of its curvature, and regarded the total strain energy as the quantity analogous to the potential energy of a discrete mechanical system. By adopting procedures that were becoming familiar in analytical mechanics, and following from the principle of virtual work as introduced by John Bernoulli for discrete systems such as pin-connected rigid bodies in 1717, Euler rendered the energy stationary and in this way developed the *calculus of variations* as an approach to the equations of equilibrium and motion of elastic structures.

That same variational approach played a major role in the development by French mathematicians in the early 1800's of a theory of small transverse displacements and vibrations of elastic plates. This theory was developed in preliminary form by Sophie Germain and partly improved upon by Simeon Denis Poisson in the early 1810's; they considered a flat plate as an elastic plane which resists curvature. Navier gave a definitive development of the correct energy expression and governing differential equation a few years later. An uncertainty of some duration in the theory arose from the fact that the final partial differential equation for the transverse displacement is such that it is impossible to prescribe, simultaneously, along an unsupported edge of the plate, both the twisting moment per unit length of middle surface and the transverse shear force per unit length. This was finally resolved in 1850 by German physicist Gustav Robert Kirchhoff in an application of virtual work and variational calculus procedures, in the framework of simplifying kinematic assumptions that fibers initially perpendicular to the plate middle surface remain so after deformation of that surface. As first steps in the theory of thin shells, in the 1770's Euler addressed the deformation of an initially curved beam, as an elastic line, and provided a simplified analysis of vibration of an elastic bell as an array of annular beams. John's grandson, through a son and mathematician also named John, James Bernoulli "the younger" (1759-1789) further developed this model in the last year of his life as a two dimensional network of elastic lines, but could not develop an acceptable treatment. Shell theory was not to attract attention for a century after Euler's work, as the outcome of many researches following the first consideration of shells from a three-dimensional elastic viewpoint by H. Aron in 1873. Acceptable thin-shell theories for general situations, appropriate for cases of small deformation, were developed by English mathematician, mechanician and geophysicist A. E. H. Love in 1888 and mathematician and physicist Horace Lamb in 1890 (there is no uniquely correct theory as the Dutch applied mechanician and engineer W. T. Koiter and Russian mechanician V. V. Novozhilov were to clarify in the 1950's; the difference between predictions of acceptable theories is small when the ratio of shell thickness to a typical length scale is small). Shell theory remained of immense interest well beyond the mid-1900's in part because so many problems lay beyond the linear theory (rather small transverse displacements often dramatically alter the way that a shell supports load by a combination of bending and membrane action), and in part because of the interest in such light-weight structural forms for aeronautical technology.

Elasticity, general theory. Linear elasticity as a general three-dimensional theory was in hand in the early 1820's based on Cauchy's work. Simultaneously, Navier had developed an elasticity theory based on a simple corpuscular, or particle, model of matter in which particles interacted with their neighbors by a central-force attractions between particle pairs. As was gradually realized following works by Navier, Cauchy and Poisson in the 1820's and 1830's, the particle model is too simple and makes predictions concerning relations among elastic moduli which are not met by experiment. In the isotropic case it predicts that there is only one elastic constant and that the Poisson ratio has the universal value of 1/4. Most subsequent development of the subject was in terms of the continuum theory. Controversies concerning the maximum possible number of independent elastic moduli in the most general anisotropic solid were settled by English mathematician George Green in 1837, through pointing out that the existence of an elastic strain energy required that of the 36 elastic constants, relating the six stress components to the six strains, at most 21 could be independent. Scottish physicist Lord Kelvin (William Thomson) put this consideration on sounder ground in 1855 as part of his development of macroscopic thermodynamics, in much the form as it is known today, showing that a strain energy function must exist for reversible isothermal or adiabatic (isentropic) response, and working out such results as the (very modest) temperature changes associated with isentropic elastic deformation.

The middle and late 1800's were a period in which many basic elastic solutions were derived and applied to technology and to the explanation of natural phenomena. French mathematician Barre de Saint-Venant derived in the 1850's solutions for the torsion of non-circular cylinders, which explained the necessity of warping displacement of the cross section in the direction parallel to the axis of twisting, and for flexure of beams due to transverse loadings; the latter allowed understanding of approximations inherent in the simple beam theory of Bernoulli, Euler and Coulomb. The German physicist Heinrich Rudolph Hertz developed solutions for the deformation of elastic solids as they are brought into contact, and applied these to model details of impact collisions. Solutions for stress and displacement due to concentrated forces acting at an interior point of a full space were derived by Kelvin, and on the surface of a half space by mathematicians J. V. Bousinesq (French) and V. Cerruti (Italian). The Prussian mathematician L. Pochhammer analyzed the vibrations of an elastic cylinder and Lamb and the Prussian physicist P. Jaerisch derived the equations of general vibration of an elastic sphere in the 1880's, an effort that was continued by many seismologists in the 1900's to describe the vibrations of the Earth. Kelvin derived in 1863 the basic form of the solution of the static elasticity equations for a spherical solid, and these were applied in following years to such problems as deformation of the Earth due to rotation and to tidal forcing, and to effects of elastic deformability on the motions of the Earth's rotation axis.

The classical development of elasticity never fully confronted the problem of finite elastic straining, in which material fibers change their lengths by other than very small amounts. Possibly this was because the common materials of construction would remain elastic only for very small strains before exhibiting either plastic straining or brittle failure. However, natural polymeric materials show elasticity over a far wider range (usually also with enough time or rate effects that they would more accurately be characterized as viscoelastic), and the widespread use of natural rubber and like materials motivated the development of finite elasticity. While many roots of the subject were laid in the classical theory, especially in the work of Green, G. Piola and Kirchhoff in the mid-1800's, the development of a viable theory with forms of stress-strain relations for specific rubbery elastic materials, and an understanding of the physical effects of the nonlinearity in simple problems like torsion and bending, is mainly the achievement of British-American engineer and applied mathematician Ronald S. Rivlin in the 1940's and 1950's.

Waves. Poisson, Cauchy and George G. Stokes showed that the equations of the theory predicted the existence of two types of elastic deformation waves which could propagate through isotropic elastic solids. These are called *body waves*. In the faster type, called *longitudinal*, or *dilational*, or *irrotational* waves, the particle motion is in the same direction as that of wave propagation; in the slower, called *transverse*, or *shear*, or *rotational* waves, it is perpendicular to the propagation direction. No analog of the shear wave exists for propagation through a fluid medium, and that fact led seismologists in the early 1900's to understand that the Earth has a liquid core (at the center of which there is a solid inner core).

Lord Rayleigh (John Strutt) showed in 1887 that there is a wave type that could propagate along surfaces, such that the motion associated with the wave decayed exponentially with distance into the material from the surface. This type of *surface wave*, now called a Rayleigh wave, propagates typically at slightly more than 90% of the shear wave speed, and involves an elliptical path of particle motion that lies in planes parallel to that defined by the normal to the surface and the propagation direction. Another type of surface wave, with motion transverse to the propagation direction and parallel to the surface, was found by Love for solids in which a surface layer of material sits atop an elastically stiffer bulk solid; this defines the situation for the Earth's crust. The shaking in an earthquake is communicated first to distant places by body waves, but these spread out in three-dimensions and must diminish in their displacement amplitude as r^{-1}, where r is distance from the source, to conserve the energy propagated by the wave field. The surface waves spread out in only two dimensions and must, for the same reason, diminish only as fast as $r^{-1/2}$. Thus shaking in surface waves is normally the more sensed, and potentially damaging, effect at moderate to large distances from a crustal earthquake. Indeed, well before the theory of waves in solids was in hand, Thomas Young had suggested in his 1807 Lectures on Natural Philosophy that the shaking of an earthquake "is probably propagated through the earth in the same manner as noise is conveyed through air". (It had been suggested by American mathematician and astronomer Jonathan Winthrop, following his experience of the "Boston" earthquake of 1755, that the ground shaking was due to a disturbance propagated like sound through the air.)

With the development of ultrasonic transducers operated on piezoelectric principles, the measurement of the reflection and scattering of elastic waves has developed into an effective engineering technique for the non-destructive evaluation of materials for detection of potentially dangerous defects such as cracks. Also, very strong impacts, whether from meteorite collision, weaponry, or blasting and the like in technological endeavors, induce waves in which material response can be well outside the range of linear elasticity, involving any or all of finite elastic strain, plastic or viscoplastic response, and phase transformation. These are called *shock waves*; they can propagate much beyond the speed of linear elastic waves and are accompanied with significant heating.

Stress concentrations and fracture. In 1898 G. Kirsch derived the solution for the stress distribution around a circular hole in a much larger plate under remotely uniform tensile stress. The same solution can be adapted to the tunnel-like cylindrical cavity of circular section in a bulk solid. His solution showed a significant concentration of stress at the boundary, by a factor of three when the remote stress was uniaxial tension. Then in 1907 the Russian mathematician G. Kolosov, and independently in 1914 the British engineer Charles Inglis, derived the analogous solution for stresses around an elliptical hole. Their solution showed that the concentration of stress could become far greater as the radius of curvature at an end of the hole becomes small compared to the overall length of the hole. These results provided the insight to sensitize engineers to the possibility of dangerous stress concentrations at, for example, sharp re-entrant corners, notches, cut-outs, keyways, screw threads, and the like in structures for which the nominal stresses were at otherwise safe levels. Such stress concentration sites are places from which a crack can nucleate.

The elliptical hole of Kolosov and Inglis defines a crack in the limit when one semi-axis goes to zero, and the Inglis solution was adopted in that way by British aeronautical engineer A. A. Griffith in 1921 to describe a crack in a brittle solid. In that work Griffith made his famous proposition that spontaneous crack growth would occur when the energy released from the elastic field just balanced the work required to separate surfaces in the solid. Following a hesitant beginning, in which Griffith's work was initially regarded as important only for very brittle solids such as glass, there developed, largely under the impetus of American engineer and physicist George R. Irwin, a major body of work on the mechanics of crack growth and fracture, including fracture by fatigue and stress corrosion cracking, starting in the late 1940's and continuing into the 1990's. This was driven initially by the cracking of American fleet of Liberty ships during the Second World War, by the failures of the British Comet airplane, and by a host of reliability and safety issues arising in aerospace and nuclear reactor technology. The new complexion of the subject extended beyond the Griffith energy theory and, in its simplest and most widely employed version in engineering practice, used Irwin's *stress intensity factor* as the basis for predicting crack growth response under service loadings in terms of laboratory data that is correlated in terms of that factor. That stress intensity factor is the coefficient of a characteristic singularity in the linear elastic solution for the stress field near a crack tip, and is recognized as providing a proper characterization of crack tip stressing in many cases, even though the linear elastic solution must be wrong in detail near the crack tip due to non-elastic material response, large strain, and discreteness of material microstructure.

Dislocations. The Italian elastician and mathematician V. Volterra introduced in 1905 the theory of the elastostatic stress and displacement fields created by dislocating solids. This involves making a cut in a solid, displacing its surfaces relative to one another by some fixed amount, and joining the sides of the cut back together, filling in with material as necessary. The initial status of this work was simply as an interesting way of generating elastic fields but, in the early 1930's, Geoffrey Ingram Taylor, Egon Orowan and Michael Polanyi realized that just such a process could be going on in ductile crystals and could provide an explanation of the low plastic shear strength of typical ductile solids, much like Griffith's cracks explained low fracture strength under tension. In this case the displacement on the dislocated surface corresponds to one atomic lattice spacing in the crystal. It quickly became clear that this concept provided the correct microscopic description of metal plasticity and, starting with Taylor in the 1930's and continuing into the 1990's, the use of solid mechanics to explore dislocation interactions and the microscopic basis of plastic flow in crystalline materials has been a major topic, with many distinguished contributors.

The mathematical techniques advanced by Volterra are now in common use by Earth scientists in explaining ground displacement and deformation induced by tectonic faulting. Also, the first elastodynamic solutions for the rapid motion of a crystal dislocations by South African materials scientist F. R. N. Nabarro, in the early 1950's, were quickly adapted by seismologists to explain the radiation from propagating slip distributions on faults. Japanese seismologist H. Nakano had already shown in 1923 how to represent the distant waves radiated by an earthquake as the elastodynamic response to a pair of force dipoles amounting to zero net torque. (All of his manuscripts were destroyed in the fire in Tokyo associated with the great Kwanto earthquake in that same year, but some of his manuscripts had been sent to Western colleagues and the work survived.)

Continuum plasticity theory. The macroscopic theory of plastic flow has a history nearly as old as that of elasticity. While in the microscopic theory of materials, the word “plasticity” is usually interpreted as denoting deformation by dislocation processes, in macroscopic continuum mechanics it is taken to denote any type of permanent deformation of materials, especially those of a type for which time or rate of deformation effects are are not the most dominant feature of the phenomenon (the terms viscoplasticity or creep or viscoelasticity are usually used in such cases). Coulomb’s work of 1773 on the frictional yielding of soils under shear and normal stress has been mentioned; yielding denotes the occurrence of large shear deformations without significant increase in applied stress. This work found applications to explaining the pressure of soils against retaining walls and footings in work of the French mathematician and engineer J. V. Poncelot in 1840 and Scottish engineer and physicist W. J. M. Rankine in 1853. The inelastic deformation of soils and rocks often takes place in situations for which the deforming mass is infiltrated by groundwater, and Austrian-American civil engineer Karl Terzaghi in the 1920’s developed the concept of effective stress, whereby the stresses which enter a criterion of yielding or failure are not the total stresses applied to the saturated soil or rock mass, but rather the effective stresses, which are the difference between the total stresses and those of a purely hydrostatic stress state with pressure equal to that in the pore fluid. Terzaghi also introduced the concept of consolidation, in which the compression of a fluid-saturated soil can take place only as the fluid slowly flows through the pore space under pressure gradients, according to the law of Darcy; this effect accounts for the time-dependent settlement of constructions over clay soils.

Apart from the earlier observation of plastic flow at large stresses in the tensile testing of bars, the continuum plasticity of metallic materials begins with Henri Edouard Tresca in 1864. His experiments on the compression and indentation of metals led him to propose that this type of plasticity, in contrast to that in soils, was essentially independent of the average normal stress in the material and dependent only on shear stresses, a feature later rationalized by the dislocation mechanism. Tresca proposed a yield criterion for macroscopically isotropic metal polycrystals based on the maximum shear stress in the material, and that was used by Saint-Venant to solve a first elastic-plastic problem, that of the partly plastic cylinder in torsion, and also to solve for the stresses in a completely plastic tube under pressure. German applied mechanician Ludwig Prandtl developed the rudiments of the theory of plane plastic flow in 1920 and 1921, with an analysis of indentation of a ductile solid by a flat-ended rigid indenter, and the resulting theory of *plastic slip lines* was completed by H. Hencky in 1923 and Hilda Geiringer in 1930. Additional developments include the methods of plastic limit analysis, which allowed engineers to directly calculate upper and lower bounds to the plastic collapse loads of structures or to forces required in metal forming. Those methods developed gradually over the early 1900’s on a largely intuitive basis, first for simple beam structures and later for plates, and were put on a rigorous basis within the rapidly developing mathematical theory of plasticity around 1950 by Daniel C. Drucker and William Prager in the United States and Rodney Hill in England.

German applied mathematician Richard von Mises proposed in 1913 that a mathematically simpler theory of plasticity than that based on the Tresca yield criterion could be based on the *second tensor invariant* of the *deviatoric stresses* (that is, of the total stresses minus those of a hydrostatic state with pressure equal to the average normal stress over all planes). An equivalent yield condition had been proposed independently by Polish engineer M.-T. Huber. The Mises theory incorporates a proposal by M. Levy in 1871 that components of the plastic strain increment tensor are in proportion to one another just as are the components of deviatoric stress. This criterion was found to generally provide slightly better agreement with experiment than did that of Tresca, and most work on the application of plasticity theory uses this form. Following a suggestion of Prandtl, E. Reuss completed the theory in 1930 by adding an elastic component of strain increments, related to stress increments in the same way as for linear elastic response. This formulation was soon generalized to include strain hardening, whereby the value of the second invariant for continued yielding increases with ongoing plastic deformation, and was extended to high-temperature creep response in metals or or other hot solids by assuming that the second invariant of the plastic (now generally called “creep”) strain rate is a function of that same invariant of the deviatoric stress, typically a power law type with Arrhenius temperature dependence. This formulation of plastic and viscoplastic or creep response has been applied to all manner of problems in materials and structural technology and in flow of geological masses. Representative problems addressed include the large growth to coalescence of microscopic voids in the ductile fracture of metals, the theory of the indentation hardness test, the extrusion of metal rods and rolling of metal sheets, the auto-frettage of gun tubes, design against collapse of ductile steel structures, estimation the thickness of the Greenland ice sheet, and modeling the geologic evolution of the Tibetian plateau. Other types of elastic-plastic theories intended for analysis of ductile single crystals originate from the work of G. I. Taylor and Hill, and bases the criterion for yielding on E. Schmid’s concept from the 1920’s of a critical resolved shear stress along a crystal slip plane, in the direction of an allowed slip on that plane; this sort of yield condition has approximate support from the dislocation theory of plasticity.

Viscoelasticity. The German physicist Wilhelm Weber noticed in 1835 that a load applied to a silk thread produced not only an immediate extension but also a continuing elongation of the thread with time. This type of *viscoelastic* response is especially notable in polymeric solids but is present to some extent in all types of solids and often does not have a clear separation from what could be called viscoplastic or creep response. In general, if all the strain is ultimately recovered when a load is removed from a body, the response is termed viscoelastic, but the term is also used in cases for which sustained loading leads to strains which are not fully recovered. The Austrian physicist Ludwig Boltzmann developed in 1874 the theory of linear viscoelastic stress-strain relations. In their most general form these involve the notion that a step loading (suddenly imposed stress, subsequently maintained constant) causes an immediate strain followed by a time-dependent strain which, for different materials, may either have a finite long time limit or may increase indefinitely with time. Within the assumption of linearity, the strain at time t in response to a general time dependent stress history $\sigma(t)$ can then be written as the sum (or integral) of terms that involve the step-loading strain response at time t-t' due to a step loading dt' $d\sigma(t')/dt'$ at time t'. The theory of viscoelasticity is important for consideration of the attenuation of stress waves and the damping of vibrations.

A new class of problems arose with the mechanics of very long molecule polymers, without significant cross-linking, existing either in solution or as a melt. These are fluids in the sense that they cannot long support shear stress but have, at the same time, remarkable properties like those of finitely deformed elastic solids. A famous demonstration is to pour one of these fluids slowly from a bottle and to suddenly cut the flowing stream with scissors; if the cut is not too far below the place of exit from the bottle, the stream of falling fluid immediately contracts elastically and returns to the bottle. The molecules are elongated during flow but tend to return to their thermodynamically preferred coiled configuration when forces are removed. The theory of such materials came under intense development in the 1950's after British applied mathematician James Gardner Oldroyd showed in 1950 how viscoelastic stress-strain relations of a memory type could be generalized to a flowing fluid. This involves subtle issues on assuring that the *constitutive relation*, or *rheological relation*, between the stress history and the deformation history at a material "point" is properly invariant to a superposed history of rigid rotation , which should not affect the local physics determining that relation (the resulting Coriolis and centrifugal effects are quite negligible at the scale of molecular interactions). Important contributions on this issue were made by S. Zaremba and G. Jaumann in the first decade of the 1900's; they showed how to make tensorial definitions of stress rate that were invariant to superposed spin and thus were suitable for use in constitutive relations. But it was only during the 1950's that these concepts found their way into the theory of constitutive relations for general viscoelastic materials and, independently and a few years later, properly invariant stress rates were adopted in continuum formulations of elastic-plastic response.

Computational mechanics. The digital computer revolutionized the practice of many areas of engineering and science, and solid mechanics was among the first fields to benefit from its impact. Many computational techniques have been used in that field, but the one which emerged by the end of the 1970's as, by far, the most widely adopted is the *finite element method*. This method was outlined by the mathematician Richard Courant in 1943 and was developed independently, and put to practical use on computers, in the mid-1950's by aeronautical structures engineers M. J. Turner, R. W. Clough, H. C. Martin and L. J. Topp in the United States and by J. H. Argyris and S. Kelsey in Britain. Their work grew out of earlier attempts at systematic structural analysis for complex frameworks of beam elements. The method was soon recast in a variational framework and related to earlier efforts at approximate solution of problems described by variational principles, by substituting trial functions of unknown amplitude into the variational functional which is then rendered stationary as an algebraic function of the amplitude coefficients. In the most common version of the finite element method, the domain to be analyzed is divided into cells, or *elements*, and the displacement field within each element is interpolated in terms of displacements at a few points around the element boundary, and sometimes within it, called *nodes*. The interpolation is done so that the displacement field is continuous across element boundaries for any choice of the nodal displacements. The strain at every point can thus be expressed in terms of nodal displacements, and it is then required that the stresses associated with these strains, through the stress-strain relations of the material, satisfy the principle of virtual work for arbitrary variation of the nodal displacements. This generates as many simultaneous equations as there are degrees of freedom in the finite element model, and numerical techniques for solving such systems of equations are programmed for computer solution.

BASIC PRINCIPLES

In addressing any problem in continuum or solid mechanics, we need to bring together the following considerations: (1) The Newtonian equations of motion, in the more general form recognized in the subsequent century by Euler, expressing conservation of linear and angular momentum for finite bodies (rather than just for point particles), and the related concept of stress as formalized by Cauchy; (2) Consideration of the geometry of deformation and thus expression of strains in terms of gradients in the displacement field; and (3) Use of relations between stress and strain that are characteristic of the material in question, and of the stress level, temperature and time scale of the problem considered.

These three considerations suffice for most problems. They must be supplemented for solids undergoing diffusion processes in which one material constituent moves relative to another (as of interest sometimes for a fluid-infiltrated soils or petroleum reservoir rocks), and in cases for which the induction of a temperature field by deformation processes and the related heat transfer cannot be neglected. The latter cases require that we supplement the above three considerations with the following: (4) Equations for conservation of mass of diffusing constituents; (5) The first law of thermodynamics, which introduces the concept of heat flux and relates changes in energy to work and heat supply, and (6) Relations which express the diffusive fluxes and heat flow in terms of spatial gradients of appropriate chemical potentials and of temperature. Also, in many important technological devices, electric and magnetic fields affect the stressing, deformation and motion of matter. Examples are provided by piezoelectric crystals and other ceramics for electric or magnetic actuators, and the coils and supporting structures of powerful electromagnets. In these cases, we must add the following: (7) Scottish physicist James Clerk Maxwell's set of equations which interrelate electric and magnetic fields to polarization and magnetization of material media, and to the density and motion of electric charge; and (8) Augmented relations between stress and strain which now, for example, express all of stress, polarization and magnetization in terms of strain, electric filed and magnetic intensity, and of temperature. The second law of thermodynamics, combined with the principles mentioned above, serves to constrain physically allowed relations between stress, strain and temperature in (3), and also constrains the other types of relations described in (6) and (8) above. Such expressions, which give the relationships between stress, deformation and other variables, are commonly referred to as *constitutive relations*.

In general, the stress-strain relations are to be determined by experiment. A variety of mechanical testing machines and geometrical configurations of material specimens have been devised to measure them. These allow, in different cases, simple tensile, compressive, or shear stressing, and sometimes combined stressing with several different components of stress, and the determination of material response over a range of temperature, strain rate and loading history. The testing of round bars under tensile stress, with precise measurement of their extension to obtain the strain, is common for metals and for technological ceramics and polymers. For rocks and soils, which generally carry load in compression, the most common test involves a round cylinder that is compressed along its axis, often while being subjected to confining pressure on its curved face. Often, a measurement interpreted by solid mechanics theory is used to determine some of the properties entering stress-strain relations. For example, measuring the speed of deformation waves or the natural frequencies of vibration of structures can be used to extract the elastic moduli of materials of known mass density, and measurement of indentation hardness of a metal can be used to estimate its plastic shear strength.

In some favorable cases, stress strain relations can be calculated approximately by applying appropriate principles of mechanics at the microscale of the material considered. In a composite material, the microscale could be regarded as the scale of the separate materials making up the reinforcing fibers and matrix. When their individual stress-strain relations are known from experiment, continuum mechanics principles applied at the scale of the individual constituents can be used to predict the overall stress-strain relations for the composite. For rubbery polymer materials, made up of long chain molecules which would randomly configure themselves into coil-like shapes, some aspects of the elastic stress-strain response can be obtained by applying principles of statistical thermodynamics to the partial uncoiling of the array of molecules by imposed strain. In the case of a single crystallite of an element like silicon or aluminum, or simple compound like silicon carbide, the relevant microscale is that of the atomic spacing in the crystals, and quantum mechanical principles governing atomic force laws at that scale can be used to estimate elastic constants. For consideration of plastic flow processes in metals and in sufficiently hot ceramics, the relevant microscale involves the network of *dislocation* lines that move within crystals. These lines shift atom positions relative to one another by one atomic spacing as they move along slip planes. Important features of elastic-plastic and viscoplastic stress-strain relations can be understood by modeling the stress dependence of dislocation generation and motion, and the resulting dislocation entanglement and immobilization processes which account for strain hardening.

To examine the mathematical structure of the theory, considerations (1) to (3) above are now further developed. For this purpose, we adopt a continuum model of matter, making no detailed reference to its discrete structure at molecular, or possibly other larger microscopic, scales that are far below those of the intended application.

Linear and Angular Momentum Principles: Stress, and Equations of Motion

Let $\mathbf{x}$ denote the *position vector* of a point in space as measured relative to the origin of a Newtonian reference frame; $\mathbf{x}$ has the components (x_1, x_2, x_3) relative to a cartesian set of axes, fixed in the reference frame, which we denote as the 1, 2 and 3 axes, Figure 1. Suppose that a material occupies the part of space considered and let $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$ be the velocity vector of the material point which occupies position $\mathbf{x}$ at time t; that same material point will be at position $\mathbf{x} + \mathbf{v}dt$ an infinitesimal time interval dt later. Let $\rho = \rho(\mathbf{x}, t)$ be the mass density of the material. Here $\mathbf{v}$ and ρ are *macroscopic* variables. What we idealize in the continuum model as a material point, moving as a smooth function of time, will correspond on molecular (or larger but still "microscopic") length scales to a region with strong fluctuations of density and velocity. In terms of phenomena at such scales, ρ corresponds to an average of mass per unit of volume, and $\rho\,\mathbf{v}$ to an average of linear momentum per unit volume, as taken over spatial and temporal scales that are large compared to those of the microscale processes but still small compared to those of the intended application or phenomenon under study. Thus $\mathbf{v}$ of the continuum theory is a mass-weighted average velocity, from the microscopic viewpoint.

The linear momentum $\mathbf{P}$, and angular momentum $\mathbf{H}$ relative to the coordinate origin, of the matter instantaneously occupying any volume V of space are then given by summing up the linear and angular momentum vectors of each element of material. Such summation over infinitesimal elements is represented mathematically by the integrals $\mathbf{P} = \int_V \rho\,\mathbf{v}\,dV$, $\mathbf{H} = \int_V \rho\,\mathbf{x} \times \mathbf{v}\,dV$. We limit attention to situations in which relativistic effects can be ignored. Let $\mathbf{F}$ denote the total *force* and $\mathbf{M}$ the total *torque* or *moment* (relative to the coordinate origin) acting instantaneously on the material occupying any arbitrary volume V. The basic laws of Newtonian mechanics are the linear and angular momentum principles that $\mathbf{F} = d\mathbf{P}/dt$ and $\mathbf{M} = d\mathbf{H}/dt$, where time derivatives of $\mathbf{P}$ and $\mathbf{H}$ are calculated following the motion of the matter which occupies V at time t. When either $\mathbf{F}$ or $\mathbf{M}$ vanish, these equations of motion correspond to conservation of linear or angular momentum. An important, very common, and non-trivial class of problems in solid mechanics involves determining the deformed and stressed configuration of solids or structures that are in *static equilibrium*; in that case the relevant basic equations are $\mathbf{F} = \mathbf{0}$ and $\mathbf{M} = \mathbf{0}$. The understanding of such conditions for equilibrium, at least in a rudimentary form, long predates Newton. Indeed, Archimedes of Syracuse (3rd Century BC), the great Greek mathematician and arguably the first theoretically and experimentally minded physical scientist, understood these equations at least in a nonvectorial form appropriate for systems of parallel forces. That is shown by his treatment of the hydrostatic equilibrium of a partially submerged body and his establishment of the principle of the lever (torques about the fulcrum sum to zero) and the concept of center of gravity.

Stress. We now assume that **F** and **M** derive from two type of forces, namely *body forces* **f**, like gravitational attractions, defined such that force **f** dV acts on volume element dV (see Figure 1), and *surface forces* which represent the mechanical effect of matter immediately adjoining that along the surface, S, of the volume V that we consider. Cauchy formalized in 1822 a basic assumption of continuum mechanics that such surface forces could be represented as a *stress vector* **T**, defined so that **T** dS is an element of force acting over the area dS of the surface (Figure 1). Hence the principles of linear and angular momentum take the forms

$$\int_S \mathbf{T}\, dS + \int_V \mathbf{f}\, dV = \int_V \rho\, \mathbf{a}\, dV \ , \qquad \int_S \mathbf{x} \times \mathbf{T}\, dS + \int_V \mathbf{x} \times \mathbf{f}\, dV = \int_V \rho\, \mathbf{x} \times \mathbf{a}\, dV$$

which we now assume to hold good for every conceivable choice of region V. In calculating the right hand sides, which come from d**P**/dt and d**H**/dt, it has been noted that ρ dV is an element of mass and is therefore time-invariant; also, $\mathbf{a} = \mathbf{a}(\mathbf{x}, t) = d\mathbf{v}/dt$ is the *acceleration*, where the time derivative of **v** is taken following the motion of a material point so that **a**(**x**, t) dt corresponds to the difference between **v**(**x** + **v** dt, t + dt) and **v**(**x**, t). Further, a more detailed analysis of this step shows that we must now adjust the understanding of what **T** dS denotes so as to include within it averages, over temporal and spatial scales that are large compared to those of microscale fluctuations, of transfers of momentum across the surface S due to the microscopic fluctuations about the motion described by the macroscopic velocity **v**.

The nine quantities σ_{ij} ($i, j = 1, 2, 3$) are called *stress components*; these will vary with position and time, $\sigma_{ij} = \sigma_{ij}(\mathbf{x}, t)$, and have the following interpretation. Suppose that we consider an element of surface dS through a point **x** with dS oriented so that its outer normal (pointing away from the region V, bounded by S) points in the positive x_i direction, where i is any of 1, 2 or 3. Then σ_{i1}, σ_{i2} and σ_{i3} at **x** are defined as the cartesian components of the stress vector **T** (call it $\mathbf{T}^{(i)}$) acting on this dS. Figure 2 shows the components of such stress vectors for faces in each of the three coordinate directions. To use a vector notation with $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ denoting unit vectors along the coordinate axes (Figure 1), $\mathbf{T}^{(i)} = \sigma_{i1}\, \mathbf{e}_1 + \sigma_{i2}\, \mathbf{e}_2 + \sigma_{i3}\, \mathbf{e}_3$. Thus the stress σ_{ij} at **x** is the stress in the j direction associated with an i-oriented face through point **x**; the physical dimension of the σ_{ij} is [Force]/[Length]^2. The components σ_{11}, σ_{22} and σ_{33} are stresses directed perpendicular, or *normal*, to the faces on which they act and are *normal stresses*; the σ_{ij} with $i \neq j$ are directed parallel to the plane on which they act and are *shear stresses*.

By hypothesis, the linear momentum principle applies for any volume V. Consider a small tetrahedron (Figure 3) at **x** with inclined face having outward unit normal vector **n**, and other three faces oriented perpendicular to the three coordinate axes. Letting its size shrink to zero, we derive the result that the stress vector **T** on a surface element with outward normal **n** can be expressed as a linear function of the σ_{ij} at **x**. The relation is such that the j component of the stress vector **T** is $T_j = n_1\sigma_{1j} + n_2\sigma_{2j} + n_3\sigma_{3j} = (j = 1, 2, 3)$. This relation for **T** (or T_j) also tells us that the σ_{ij} have the mathematical property of being the components of a *second rank tensor*. Suppose that a different set of cartesian reference axes $1'$, $2'$ and $3'$ have been chosen. Let x_1', x_2', x_3' denote the components of the position vector of point **x** and let σ_{kl}' $(k, l = 1, 2, 3)$ denote the 9 stress components relative to that coordinate system. The σ_{kl}' can be written as the 3 by 3 matrix $[\sigma']$, and the σ_{ij} as the matrix $[\sigma]$, where the first index is the matrix row number and the second column number. Then the expression for T_j implies that $[\sigma'] = [\alpha][\sigma][\alpha]^T$, which is the defining equation of a second rank tensor. Here $[\alpha]$ is the *orthogonal transformation matrix*, having components $\alpha_{pq} = \mathbf{e}_p' \cdot \mathbf{e}_q$ satisfying $[\alpha]^T[\alpha] = [\alpha][\alpha]^T = [I]$; superscript T denotes transpose (interchange rows and columns) and [I] denotes the unit matrix, having unity for every diagonal element and zero elsewhere; also, the matrix multiplications are such that if [A] = [B][C], then $A_{ij} = B_{i1}C_{1j} + B_{i2}C_{2j} + B_{i3}C_{3j}$.

Equations of motion. Now let us apply the linear momentum principle to an arbitrary finite body. Using the expression for T_j above and the *divergence theorem* of multivariable calculus, we may derive that

$$\partial\sigma_{1j}/\partial x_1 + \partial\sigma_{2j}/\partial x_2 + \partial\sigma_{3j}/\partial x_3 + f_j = \rho a_j \qquad (j = 1, 2, 3)$$

at least when the σ_{ij} are continuous and differentiable, which is the typical case. These are the *equations of motion* for a continuum. Once the above consequences of the linear momentum principle are accepted, the only further result which can be derived from the angular momentum principle is that $\sigma_{ij} = \sigma_{ji}$ $(i, j = 1, 2, 3)$. Thus the stress tensor is symmetric.

Principal stresses. Symmetry of the stress tensor has the important consequence that, at each point **x**, there exist three mutually perpendicular directions along which there are no shear stresses. These directions are called the *principal directions* and the corresponding normal stresses are called the *principal stresses*. If we order the principal stresses algebraically as σ_I, σ_{II}, σ_{III} (Figure 4), then the normal stress on any face (given as $\sigma_n = \mathbf{n} \cdot \mathbf{T}$) satisfies $\sigma_I \leq \sigma_n \leq \sigma_{III}$. The principal stresses are the *eigenvalues* (or *characteristic values*) s, and the principal directions the *eigenvectors* **n**, of the problem **T** = s**n**, or $[\sigma]\{n\} = s\{n\}$ in matrix notation with the 3-column $\{n\}$ representing **n**. It has solutions when $\det([\sigma] - s[I]) = -s^3 + I_1 s^2 + I_2 s + I_3 = 0$, with $I_1 = \mathrm{tr}\,[\sigma]$, $I_2 = -(1/2) I_1^2 + (1/2)\,\mathrm{tr}\,([\sigma][\sigma])$, $I_3 = \det[\sigma]$. Here "det" denotes *determinant* and "tr" denotes *trace*, or sum of diagonal elements, of a matrix. Since the principal stresses are determined by I_1, I_2, I_3 and can have no dependence on how we chose the coordinate system with respect to which we refer the components of stress, I_1, I_2 and I_3 must be independent of that choice and are therefore called *stress invariants*. One may readily verify that they have the same values when evaluated in terms of σ_{ij}' above as in terms of σ_{ij} by using the tensor transformation law and properties noted for the orthogonal transformation matrix.

Very often, both in nature and technology, there is interest in structural elements in forms that might be identified as *strings, wires, rods, bars, beams*, or *columns*, or as *membranes, plates,* or *shells*. These are usually idealized as, respectively, one- or two-dimensional continua. One possible approach is to develop the consequences of the linear and angular momentum principles entirely within that idealization, working in terms of net axial and shear forces, and bending and twisting torques, at each point along a one-dimensional continuum, or in terms of forces and torques per unit length of surface in a two-dimensional continuum.

Geometry of Deformation: Strain, Strain-Displacement Relations, Compatibility

The shape of a solid or structure changes with time during a deformation process. To characterize deformation, we adopt a certain *reference* configuration which we agree to call *undeformed*. Often, that reference configuration is chosen as an unstressed state, but such is neither necessary nor always convenient. Measuring time from zero at a moment when the body exists in that reference configuration, we may then use the upper case $\mathbf{X}$ to denote the position vectors of material points when $t = 0$. At some other time t, a material point which was at $\mathbf{X}$ will have moved to some spatial position $\mathbf{x}$. We thus describe the deformation as the mapping $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$, with $\mathbf{x}(\mathbf{X}, 0) = \mathbf{X}$. The *displacement vector* $\mathbf{u} = \mathbf{x}(\mathbf{X}, t) - \mathbf{X}$ and, also, $\mathbf{v} = \partial\mathbf{x}(\mathbf{X}, t)/\partial t$ and $\mathbf{a} = \partial^2\mathbf{x}(\mathbf{X}, t)/\partial t^2$.

It is simplest to write equations for strain in a form which, while approximate in general, is suitable for the case when any infinitesimal line element $d\mathbf{X}$ of the reference configuration undergoes extremely small rotations and fractional change in length, in deforming to the corresponding line element $d\mathbf{x}$. These conditions are met when $|\partial u_i/\partial X_j| << 1$. The solids with which we deal are very often sufficiently rigid, at least under the loadings typically applied to them, that these conditions are realized in practice. Linearized expressions for strain in terms of $[\partial u/\partial X]$, appropriate to this situation, are called *small strain* or *infinitesimal strain*. Expressions for strain will also be given that are valid for rotations and fractional length changes of arbitrary magnitude; such expressions are called *finite strain*.

Two simple types of strain are *extensional* strain and *shear* strain. Consider a rectangular parallelpiped, a brick-like block of material with mutually perpendicular planar faces, and let the edges of the block be parallel to the 1, 2 and 3 axes. If we deform the block homogeneously, so that each planar face moves perpendicular to itself and such that the faces remain orthogonal (i.e., the parallelpiped is deformed into another rectangular parallelpiped), then we say that the block has undergone extensional strain relative to each of the 1, 2 and 3 axes, but no shear strain relative to these axes. Denote the edge lengths of the undeformed parallelpiped as ΔX_1, ΔX_2 and ΔX_3, and those of the deformed parallelpiped as Δx_1, Δx_2 and Δx_3; see Figure 5, where the dashed-line figure represents the reference configuration and solid-line the deformed configuration. Then the quantities $\lambda_1 = \Delta x_1/\Delta X_1$, $\lambda_2 = \Delta x_2/\Delta X_2$, and $\lambda_3 = \Delta x_3/\Delta X_3$ are called *stretch ratios*. There are various ways that extensional strain can be defined in terms of them. Note that the change in displacement in, say, the x_1 direction between points at one end of the block and those at the other is $\Delta u_1 = (\lambda_1 - 1)\, \Delta X_1$. For example, if E_{11} denotes the extensional strain along the x_1 direction, then the most commonly understood definition of strain is E_{11} = (change in length)/(initial length) = $(\Delta x_1 - \Delta X_1)/ \Delta X_1 = \Delta u_1/ \Delta X_1 = \lambda_1 - 1$. A variety of other measures of extensional strain can be defined by $E_{11} = g(\lambda_1)$ where the function $g(\lambda)$ satisfies $g(1) = 0$ and $g'(1) = 1$, so as to agree with the above definition when λ_1 is very near 1. Two such in common use are the strain $E^M_{11} = (\lambda_1^2 - 1) / 2$ based on the change of *metric* tensor, and the *logarithmic* strain $E^L_{11} = \ln(\lambda_1)$.

To define a *simple shear* strain, consider the same rectangular parallelpiped but now deform it so that every point on a plane of type X_2 = constant moves only in the x_1 direction, and by an amount that increases linearly with X_2. Thus the deformation $x_1 = \gamma X_2 + X_1$, $x_2 = X_2$, $x_3 = X_3$ defines a homogeneous simple shear strain of amount γ, and is illustrated in Figure 6. Note that this strain causes no change of volume. For small strain we can identify the shear strain γ as the reduction in angle between two initially perpendicular lines.

Small strain tensor. The *small strains*, or *infinitesimal strains*, ε_{ij} are appropriate for situations with $|\partial u_k/\partial X_l| << 1$ for all k and l. Two infinitesimal material fibers, one initially in the 1 direction and the other in the 2 direction, are shown in Figure 7 as dashed lines in the reference configuration and as solid lines in the deformed configuration. To first order accuracy in components of $[\partial u/\partial X]$, the extensional strains of these fibers are $\varepsilon_{11} = \partial u_1/\partial X_1$ and $\varepsilon_{22} = \partial u_2/\partial X_2$, and the reduction of the angle between them is $\gamma_{12} = \partial u_2/\partial X_1 + \partial u_1/\partial X_2$. For the shear strain denoted ε_{12}, however, we use half of γ_{12}. Thus, considering all extensional and shear strains associated with infinitesimal fibers in the 1, 2 and 3 direction at a point of the material, the set of strains is given by

$$\varepsilon_{ij} = (1/2)\,(\partial u_j/\partial X_i + \partial u_i/\partial X_j) \qquad (i,\ j = 1,\ 2,\ 3).$$

The ε_{ij} are symmetric, $\varepsilon_{ij} = \varepsilon_{ji}$, and form a second rank tensor (that is, if we chose cartesian reference axes 1´, 2´, 3´ instead, and formed ε_{kl}', then the ε_{kl}' are related to the ε_{ij} by the same equations which relate the stresses σ_{kl}' to the σ_{ij}). These mathematical features require that there exist principal strain directions; at every point of the continuum it is possible to identify three mutually perpendicular directions along which there is purely extensional strain, with no shear strain between these special directions. The directions are the principal directions and we denote the corresponding strains include the least and greatest extensional strains experienced by fibers through the material point considered. Invariants of the strain tensor may be defined in a way paralleling those for the stress tensor.

An important fact to note is that the strains cannot vary in an arbitrary manner from point to point in the body. That is because the six strain components are all derivable from three displacement components. Restrictions on strain resulting from such considerations are called *compatibility relations*; the body would not fit together after deformation unless they were satisfied. Consider, for example, a state of *plane strain* in the 1, 2 plane (so that $\varepsilon_{33} = \varepsilon_{23} = \varepsilon_{31} = 0$). The non-zero strains ε_{11}, ε_{22} and ε_{12} cannot vary arbitrarily from point to point but must satisfy $\partial^2\varepsilon_{22}/\partial X_1^2 + \partial^2\varepsilon_{11}/\partial X_2^2 = 2\partial^2\varepsilon_{12}/\partial X_1\partial X_2$, as may be verified by directly inserting the relations for strains in terms of displacements.

When the smallness of stretch and rotation of line elements allows use of the infinitesimal strain tensor, a derivative $\partial/\partial X_i$ will be very nearly identical to $\partial/\partial x_i$. Frequently, but not always, it will then be acceptable to ignore the distinction between the deformed and undeformed configurations in writing the governing equations of solid mechanics. For example, the differential equations of motion in terms of stress are rigorously correct only with derivatives relative to the deformed configuration but, in the circumstances considered, the equations of motion can be written relative to the undeformed configuration. This is what is done in the most widely used variant of solid mechanics, in the form of the theory of *linear elasticity*. The procedure can go badly wrong in some important cases, like for columns under compressive loadings so that buckling occurs, or for elastic-plastic materials when the slope of the stress versus strain relation is of the same order as existing stresses; these cases are best approached through finite deformation theory.

Finite deformation and strain tensors. In the theory of finite deformations, with extension and rotations of line elements are unrestricted as to size. The *deformation gradient* is defined by $F_{ij} = \partial x_i(\mathbf{X}, t)/\partial X_j$, and the 3 by 3 matrix [F], with components F_{ij}, may be represented as a pure deformation, characterized by a symmetric matrix [U], followed by a rigid rotation [R]. This result is called the *polar decomposition theorem*, and takes the form, in matrix notation [F] = [R][U]. For an arbitrary deformation, there exist three mutually orthogonal principal stretch directions at each point of the material; call these directions in the reference configuration $\mathbf{N}^{(I)}$, $\mathbf{N}^{(II)}$, $\mathbf{N}^{(III)}$ and let the stretch ratios be λ_I, λ_{II}, λ_{III}. Fibers in these three principal directions undergo extensional strain but have no shearing between them. Those three fibers in the deformed configuration remain orthogonal but are rotated by the operation [R]. An extensional strain may be defined by $E = g(\lambda)$ where $g(1) = 0$ and $g'(1) = 1$, with examples for $g(\lambda)$ given above. We may then define a *finite strain tensor* E_{ij} based on any particular function $g(\lambda)$ by $E_{ij} = g(\lambda_I)\, N_i^{(I)} N_j^{(I)} + g(\lambda_{II})\, N_i^{(II)} N_j^{(II)} + g(\lambda_{III})\, N_i^{(III)} N_j^{(III)}$. Usually, it is rather difficult to actually solve for the λ's and $\mathbf{N}$'s associated with any general [F], so it is not easy to use this strain definition. However, for the special choice identified as $g^M(\lambda) = (\lambda^2 - 1)/2$ above, it may be shown that $2\ E^M_{ij} = \sum_{k=1}^{3} F_{ki}F_{kj} - \delta_{ij} = \partial u_i/\partial X_j + \partial u_j/\partial X_i + \sum_{k=1}^{3} \partial u_k/\partial X_i\ \partial u_k/\partial X_j$, which, like the finite strain generated by any other $g(\lambda)$, reduces to ε_{ij} when linearized in $[\partial u/\partial X]$.

Stress-Strain Relations

Linear elastic isotropic solid. The simplest case is that of the *linear elastic solid*, considered in circumstances for which $|\partial u_i/\partial X_j| << 1$, and for *isotropic* materials, whose mechanical response is independent of the direction of stressing. If a material point sustains a stress state $\sigma_{11} = \sigma$, with all other $\sigma_{ij} = 0$, it is subjected to *uniaxial tensile stress*. This can be realized in a homogeneous bar loaded by an axial force. We may write the resulting strain as $\varepsilon_{11} = \sigma/E$, $\varepsilon_{22} = \varepsilon_{33} = -\nu\varepsilon_{11} = -\nu\sigma/E$, $\varepsilon_{12} = \varepsilon_{23} = \varepsilon_{31} = 0$. Two new parameters have been introduced here, E and ν; E is called *Young's modulus* and it has dimensions of [Force]/[Length]^2 and is measured in units such as Pa (Pa = Pascal = 1 N/m^2), or dyne/cm^2, or psi (pounds per square inch); ν is dimensionless and is called the *Poisson ratio*. If the isotropic solid is subjected only to shear stress, $\sigma_{12} = \sigma_{21} = \tau$, with all other $\sigma_{ij} = 0$, then the response is shearing strain of the same type, $\varepsilon_{12} = \tau/2G$, $\varepsilon_{23} = \varepsilon_{31} = \varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} = 0$. Notice that because $2\varepsilon_{12} = \gamma_{12}$, this is equivalent to $\gamma_{12} = \tau/G$. The constant G introduced is called the *shear modulus*. Frequently, the symbol μ is used instead for it. The shear modulus G is not independent of E and ν, but is related to them by $G = E/2\,(1 + \nu)$, as follows from the tensor nature of stress and strain. The general stress-strain relations are then

$$\varepsilon_{ij} = (1 + \nu)\sigma_{ij}/E - \nu\,\delta_{ij}\,(\sigma_{11} + \sigma_{22} + \sigma_{33})/E \qquad (i,j = 1,2,3)$$

and where δ_{ij} is defined as 1 when its indices agree and 0 otherwise.

These relations can be inverted to read $\sigma_{ij} = \lambda\delta_{ij}\,(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + 2\mu\varepsilon_{ij}$ where here we have used μ rather than G as the notation for the shear modulus, following convention, and where $\lambda = 2\nu\mu/(1 - 2\nu)$. The elastic constants λ and μ are sometimes called the *Lamé constants*. Since ν is typically in the range 1/4 to 1/3 for hard polycrystalline solids, λ falls often in the range between μ and 2μ. (Navier's particle model with central forces leads to $\lambda = \mu$ for an isotropic solid.) Another elastic modulus often cited is the *bulk modulus* K, defined for a linear solid under pressure p ($\sigma_{11} = \sigma_{22} = \sigma_{33} = -p$) such that the fractional decrease in volume is p/K. If we consider a small cube of side length L in the reference state, observe that shearing strain does not change volume, and that the length along, say, the 1 direction changes to $(1 + \varepsilon_{11})L$, we see that the fractional change of volume is $(1 + \varepsilon_{11})(1 + \varepsilon_{22})(1 + \varepsilon_{33}) - 1 = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$, neglecting quadratic and cubic order terms in the ε_{ij} compared to linear, as appropriate when using linear elasticity. Thus $K = E/3(1 - 2\nu) = \lambda + 2\mu/3$.

Thermal strains. Temperature change can also cause strain. In an isotropic material the thermally induced extensional strains are equal in all directions, and there are no shear strains. In the simplest cases, we can treat these *thermal* strains as being linear in the temperature change, $\theta - \theta_0$ (where θ_0 is the temperature of the reference state) writing $\varepsilon_{ij}^{\text{thermal}} = \delta_{ij}\,\alpha\,(\theta - \theta_0)$ for the strain produced by temperature change in the absence of stress. Here α is called the *coefficient of thermal expansion*. Thus, in cases of temperature change, we replace ε_{ij} in the stress-strain relations above with $\varepsilon_{ij} - \varepsilon_{ij}^{\text{thermal}}$, with the thermal part given as a function of temperature. Typically, when temperature changes are modest, we can neglect the small dependence of E and ν on temperature.

Anisotropy. Anisotropic solids are also common in nature and technology. Examples are: single crystals; polycrystals in which the grains are not completely random in their crystallographic orientation but have a "texture", typically due to some plastic or creep flow process which has left a preferred grain orientation; fibrous biological materials like wood or bone; and composite materials which, on a microscale, have the structure of reinforcing fibers in a matrix, with fibers oriented in a single direction or in multiple directions (e.g., to ensure strength along more than a single direction), or may have the structure of a lamination of thin layers of separate materials. In the most general case the application of any of the six components of stress induces all six components of strain, and there is no shortage of elastic constants. There would seem to be 6 × 6 = 36 in the most general case but, as a consequence of the laws of thermodynamics, the maximum number of independent elastic constants is 21 (compared to 2 for isotropic solids). In many cases of practical interest, *symmetry considerations* reduce the number to far below 21. Crystals of cubic symmetry, like rocksalt (NaCl), or face-centered-cubic metals such as aluminum, copper, or gold, or body-centered-cubic metals like iron at low enough temperature or tungsten, or non-metals such as diamond, germanium or silicon, have only 3 independent elastic constants. Also solids with a special direction, and with identical properties along any direction perpendicular to that direction, are called *transversely isotropic*, and have 5 independent elastic constants. Examples are provided by fiber-reinforced composite materials, with fibers that are randomly emplaced but aligned in a single direction in an isotropic, or transversely isotropic, matrix, and by single crystals of hexagonal-close-packing such as zinc.

General linear elastic stress strain relations have the form $\sigma_{ij} = \sum_{k=1}^{3}\sum_{l=1}^{3} C_{ijkl}\,\varepsilon_{kl}$ where, because the ε_{kl} are symmetric, we can write $C_{ijkl} = C_{ijlk}$, and because the σ_{ij} are symmetric, $C_{ijkl} = C_{jikl}$. Hence the 3 × 3 × 3 × 3 = 81 components of C_{ijkl} reduce to the 6 × 6 = 36 mentioned. In cases of temperature change, we replace ε_{ij} above by $\varepsilon_{ij} - \varepsilon_{ij}^{\text{thermal}}$ where $\varepsilon_{ij}^{\text{thermal}} = \alpha_{ij}\,(\theta - \theta_o)$ and α_{ij} is the set of thermal strain coefficients, with $\alpha_{ij} = \alpha_{ji}$. An alternative matrix notation is sometimes employed, especially in the literature on single crystals. That approach introduces 6-element columns of stress and strain $\{\sigma\}$ and $\{\varepsilon\}$, defined so that the columns, when transposed (super-script T) or laid out as rows, are $\{\sigma\}^T = (\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{31})$ and $\{\varepsilon\}^T = (\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, 2\varepsilon_{12}, 2\varepsilon_{23}, 2\varepsilon_{31})$. These forms assure that the scalar $\{\sigma\}^T\{d\varepsilon\} \equiv \mathrm{tr}\,([\sigma][d\varepsilon])$ is an increment of stress working per unit volume. The stress-strain relations are then written $\{\sigma\} = [c]\{\varepsilon\}$ where [c] is the 6 by 6 matrix of elastic moduli. Thus, $c_{13} = C_{1133}$, $c_{15} = C_{1123}$, $c_{44} = C_{1212}$, etc.

Thermodynamic considerations. In thermodynamic terminology, a state of purely elastic material response corresponds to an *equilibrium* state, and a process during which there is purely elastic response corresponds to a sequence of equilibrium states and hence to a *reversible process*. The *second law of thermodynamics* assures that the heat absorbed per unit mass can be written θ ds where θ is *thermodynamic (absolute) temperature* and s is the *entropy* per unit mass. Hence, writing the work per unit volume of reference configuration in a manner appropriate to cases when infinitesimal strain can be used, and letting ρ_o be the density in that configuration, we have from the *first law of thermodynamics* that $\rho_o\theta\, ds + \mathrm{tr}\,([\sigma][d\varepsilon]) = \rho_o de$ where e is the internal energy per unit mass. This relation shows that if we express e as a function of entropy s and strains $[\varepsilon]$, and if we write e so as to depend identically on ε_{ij} and ε_{ji}, then $\sigma_{ij} = \rho_o\, \partial e([\varepsilon], s)/\partial\varepsilon_{ij}$. Alternatively, we may introduce the *Helmholtz free energy f* per unit mass, $f = e - \theta s = f([\varepsilon], \theta)$, and show that $\sigma_{ij} = \rho_o\, \partial f([\varepsilon], \theta)/\partial\varepsilon_{ij}$. The later form corresponds to the variables with which the stress-strain relations were written above. Sometimes $\rho_o f$ is called the *strain energy* for states of isothermal (constant θ) elastic deformation; $\rho_o e$ has the same interpretation for *isentropic* (s = constant) elastic deformation, achieved when the time sale is too short to allow heat transfer to or from a deforming element. Since the mixed partial derivatives must be independent of order, a consequence of the last equation is that $\partial\sigma_{ij}([\varepsilon], \theta)/\partial\varepsilon_{kl} = \partial\sigma_{kl}([\varepsilon], \theta)/\partial\varepsilon_{ij}$, which requires that $C_{ijkl} = C_{klij}$, or equivalently that the matrix [c] be symmetric, $[c] = [c]^T$, reducing the maximum possible number of independent elastic constraints from 36 to 21. The strain energy at constant temperature θ_o is $W([\varepsilon]) \equiv f([\varepsilon], \theta_o) = (1/2)\, \{\varepsilon\}^T[c]\{\varepsilon\}$.

The elastic moduli for isentropic response are slightly different from those for isothermal response. In the case of the isotropic material, it is most convenient to give results in terms of G and K, the isothermal shear and bulk moduli. We find that the isentropic moduli $\overline{G}$ and $\overline{K}$ are then $\overline{G} = G$ and $\overline{K} = K\ (1 + 9\, \theta_o K \alpha^2 / \rho_o c_\varepsilon)$ where $c_\varepsilon = \theta_o\, \partial s([\varepsilon], \theta)/\partial\theta$, evaluated at $\theta = \theta_o$ and $[\varepsilon] = [0]$, is the *specific heat at constant strain*. The fractional change in bulk modulus, given by the second term in the parenthesis, is very small, typically of the order of 1% or less even for metals and ceramics of relatively high α, of order 10^{-5}/degree Kelvin. The fractional change in absolute temperature during an isentropic deformation is found to involve the same small parameter: $[(\theta - \theta_o)/\theta_o]_{s = \mathrm{const}} = -(9\, \theta_o K \alpha^2 / \rho_o c_\varepsilon)\, [(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})/3\alpha\theta_o]$. Values of α for most solid elements and inorganic compounds are in the range 10^{-6} to 4×10^{-5} / degree Kelvin, and room temperature is around 300 Kelvin, so $3\alpha\theta_o$ is typically in the range 10^{-3} to 4×10^{-2}. Thus, if the fractional change in volume is of the order of 1%, which is quite large for a metal or ceramic deforming in its elastic range, the fractional change in absolute temperature is also of order 1%. For those reasons, it is usually appropriate to neglect the alteration of the temperature field due to elastic deformation, and hence to use purely mechanical formulations of elasticity in which distinctions between isentropic and isothermal response are neglected.

Finite elastic deformations. When we deal with elastic response under arbitrary deformation gradients, because rotations, if not strains, are large or, in a material such as rubber, because the strains are large too, it is necessary to dispense with the infinitesimal strain theory. Instead, the combined first and second laws of thermodynamics has the form $\rho_o \theta \; ds$ + $\det [F] \; tr ([F]^{-1}[\sigma][dF]) = \rho_o \, de$. Here $[F]^{-1}$ is the matrix inverse of [F]. If we have deformed a parcel of material by [F] and then give it some additional rigid rotation, we would insist that the free energy be unchanged in that rotation. In terms of the polar decomposition [F] = [R][U], this is equivalent to saying that *f* is independent of the rotation part [R] of [F], which is then equivalent to saying that *f* is a function of the finite strain measure $[E^M] = (1/2) ([F]^T[F] - [I])$ based on change of metric or, for that matter, on any member of the family of material strain tensors. Thus $\sigma_{ij} = (1 / \det [F]) \sum_{k=1}^{3} \sum_{l=1}^{3} F_{ik} \, F_{jl} \, S_{kl} \, ([E^M], \, \theta)$ where S_{kl} (= S_{lk}) is sometimes called the *second Piola-Kirchhoff* stress, and is given by $S_{kl} = \rho_o \, \partial f([E^M], \, \theta) \, / \, \partial E^M_{kl}$, it being assumed that *f* has been written so as to have identical dependence on E^M_{kl} and E^M_{lk}.

Inelastic response. The above mode of expressing [σ] in terms of [S] is valid for solids showing *viscoelastic* or *plastic* response as well, except that [S] is then to be regarded not only as a function of the present $[E^M]$ and θ, but to depend on the prior history of both. Assuming that such materials show elastic response to sudden stress changes, or to small unloading from a plastically deforming state, we may still express [S] as a derivative of *f*, as above, but the derivative is understood as being with respect to an elastic variation of strain and is to be taken at fixed θ and with fixed prior inelastic deformation and temperature history. Such dependence on history is sometimes represented as a dependence of *f* on *internal state variables* whose laws of evolution are part of the inelastic constitutive description. There are also simpler models of inelastic response and the most commonly employed forms for plasticity and creep in isotropic solids are presented next.

To a good approximation, plastic deformation of crystalline solids causes no change in volume, and *hydrostatic* changes in stress, amounting to equal change of all normal stresses, have no effect on plastic flow, at least for changes that are of the same order or magnitude as the strength of the solid in shear. Thus plastic response can be formulated in terms of *deviatoric stress*, defined by $\tau_{ij} = \sigma_{ij} - \delta_{ij} \, (\sigma_{11} + \sigma_{22} + \sigma_{33})/3$. Following von Mises, in a procedure which is found to agree moderately well with experiment, the plastic flow relation is formulated in terms of the second invariant of deviatoric stress, commonly rewritten as $\bar{\sigma} = \sqrt{(3/2)\ \mathrm{tr}\ ([\tau][\tau])}$ and called the *equivalent tensile stress*. The definition is made so that for a state of uniaxial tension, $\bar{\sigma}$ equals the tensile stress, and the stress-strain relation for general stress states is formulated in terms of data from the tensile test. In particular, a plastic strain $\bar{\varepsilon}^{\,p}$ in a uniaxial tension test is defined from $\bar{\varepsilon}^{\,p} = \bar{\varepsilon} - \bar{\sigma}/E$, where here $\bar{\varepsilon}$ is interpreted as the strain in the tensile test according to the logarithmic definition, $\bar{\varepsilon} = ln\lambda$, and the elastic modulus E is assumed to remain unchanged with deformation; also, in the situations considered $\bar{\sigma}/E << 1$.

Thus in the *rate-independent plasticity* version of the theory, tensile data (or compressive, with appropriate sign reversals) from a monotonic load test.is assumed to define a function $\bar{\varepsilon}^{\,p}(\bar{\sigma})$. In the *viscoplastic* or *high-temperature creep* versions of the theory, tensile data is interpreted to define $d\bar{\varepsilon}^{\,p}/dt$ as a function of $\bar{\sigma}$ in the simplest case representing, for example, *secondary creep*, and as a function of $\bar{\sigma}$ and $\bar{\varepsilon}^{\,p}$ in theories intended to represent transient creep effects or rate-sensitive response at lower temperatures. Consider first the *rigid-plastic* material model in which elastic deformability is ignored altogether, as sometimes appropriate for problems of large plastic flow as in metal forming or long term creep in the Earth's mantle, or for analysis of plastic collapse loads on structures. The *rate of deformation* tensor D_{ij} is defined by $2\ D_{ij} = \partial v_i/\partial x_j + \partial v_j/\partial x_i$, and in the rigid-plastic case [D] can be equated to what we will shortly consider to be its *plastic* part $[D^p]$, given as $D^p_{ij} = 3\ (d\bar{\varepsilon}^{\,p}/dt)\ \tau_{ij} / 2\ \bar{\sigma}$. The numerical factors secure agreement between D^p_{11} and $d\bar{\varepsilon}^{\,p}/dt$ for uniaxial tension in the 1 direction. Also, the equation implies that $D^p_{11} + D^p_{22} + D^p_{33} = 0$ and that $d\bar{\varepsilon}^{\,p}/dt = \sqrt{(2/3)\ \mathrm{tr}\ ([D^p][D^p])}$, which must be integrated over previous history to get $\bar{\varepsilon}^{\,p}$ as required for viscoplastic models in which $d\bar{\varepsilon}^{\,p}/dt$ is a function of $\bar{\sigma}$ and $\bar{\varepsilon}^{\,p}$. In the rate-independent version, we define $[D^p]$ as *zero* whenever $\bar{\sigma}$ is less than the highest value that it has attained in the previous history, or when the current value of $\bar{\sigma}$ is the highest value but $d\bar{\sigma}/dt < 0$. (In the *elastic-plastic* context, this means that "unloading" involves only elastic response.) For the *ideally-plastic solid*, which is idealized to be able to flow without increase of stress when $\bar{\sigma}$ equals the yield strength level, we regard $d\bar{\varepsilon}^{\,p}/dt$ as an undetermined but necessarily non-negative parameter, which can be determined (sometimes not uniquely) only through the complete solution of a solid mechanics boundary value problem.

The *elastic-plastic* material model is then formulated by writing $D_{ij} = D^e_{ij} + D^p_{ij}$ where D^p_{ij} is given in terms of stress and possibly stress rate as above, and where the elastic deformation rates $[D^e]$ are related to stresses by the usual linear elastic expression, $D^e_{ij} = (1 + \nu)\ \dot{\sigma}^*_{ij}/E - \nu\ \delta_{ij}\ (\dot{\sigma}^*_{11} + \dot{\sigma}^*_{22} + \dot{\sigma}^*_{33})/E$. Here the stress rates are expressed as the Jaumann *co-rotational* rates

$\dot{\sigma}^{*}_{ij} = \dot{\sigma}_{ij} + \sum_{k=1}^{3} (\sigma_{ik} \Omega_{kj} - \Omega_{ik} \sigma_{kj})$ where $\dot{\sigma}_{ij} = d\sigma_{ij}/dt$ is a derivative following the motion of a material point, and where the *spin* Ω_{ij} is defined by $2\,\Omega_{ij} = \partial v_i/\partial x_j - \partial v_j/\partial x_i$. The co-rotational stress rates are those calculated by an observer who spins with the average angular velocity of a material element. The elastic part of the stress strain relation should be consistent with the existence of a free energy *f* as discussed above. This is not strictly satisfied by the form just given, but the differences between it and one which is consistent in that way involves additional terms which are of order $\bar{\sigma}/E^2$ times the $\dot{\sigma}^{*}_{kl}$, and negligible in typical cases in which the theory is used, since $\bar{\sigma}/E$ is usually an extremely small fraction of unity, say, 10^{-4} to 10^{-2}. A *small-strain* version of the theory is in common use for purposes of elastic-plastic stress analysis. In that one replaces [D] with $\partial[\varepsilon(\mathbf{X}, t)]/\partial t$, where $[\varepsilon]$ is the small strain tensor, $\partial/\partial x$ with $\partial/\partial X$ in all equations, and $[\dot{\sigma}^{*}]$ with $\partial[\sigma(\mathbf{X}, t)]/\partial t$. The last two steps cannot always be justified even in cases of very small strain when, for example, in a rate-independent material, $d\bar{\sigma}/d\bar{\varepsilon}^{\,p}$ is not large compared to $\bar{\sigma}$, or, just as is a concern for buckling problems in purely elastic solids, when rates of rotation of material fibers can become much larger than rates of stretching.

SOME PROBLEMS INVOLVING ELASTIC RESPONSE

Equations of linear elasticity, mechanical theory. The final equations of the purely mechanical theory of linear elasticity (i.e., when we neglect coupling with the temperature field) and assume isothermal response are obtained as follows. We use the stress strain relations, write the strains in terms of displacement gradients, and insert the final expressions for stress into the equations of motion, replacing $\partial/\partial x$ with $\partial/\partial X$ in those equations. In the case of an isotropic and homogeneous solid, these reduce to

$$(\lambda + \mu)\, \nabla(\nabla \cdot \mathbf{u}) + \mu\, \nabla^2 \mathbf{u} + \mathbf{f} = \rho\, \partial^2 \mathbf{u}/\partial t^2 ,$$

known as the Navier equations (here $\nabla = \mathbf{e}_1\, \partial/\partial X_1 + \mathbf{e}_2\, \partial/\partial X_2 + \mathbf{e}_3\, \partial/\partial X_3$ and ∇^2 is the Laplacian operator defined by $\nabla \cdot \nabla$). Such equations hold in the region V occupied by the solid; on the surface S one prescribes each component of **u**, or each component of **T** (expressed in terms of $[\partial u/\partial X]$), or sometimes mixtures of components or relations between them. For example, along a freely slipping planar interface with a rigid solid, the normal component of **u** and the two tangential components of **T** would be prescribed, all as zero.

Body wave solutions. By looking for *body wave* solutions in the form $\mathbf{u}(\mathbf{X}, t) = \mathbf{p} f(\mathbf{n} \cdot \mathbf{X} - ct)$, where unit vector **n** is the propagation direction, **p** is the *polarization*, or direction of particle motion, and c is the wave speed, one may show for the isotropic material that solutions exist for arbitrary functions *f*(..) if either $c = c_d \equiv \sqrt{(\lambda + 2\mu)/\rho}$ and $\mathbf{p} = \mathbf{n}$, or $c = c_s = \sqrt{\mu/\rho}$ and $\mathbf{p} \cdot \mathbf{n} = 0$. The first case, with particle displacements in the propagation direction, describes *longitudinal* or *dilatational* waves and the latter case, which corresponds to two linearly independent displacement directions, both transverse to the propagation direction, describes *transverse* or *shear* waves.

Linear elastic beam. The case of a beam treated as a linear elastic line may also be considered. Let the line lie along the 1 axis, Figure 8, have properties that are uniform along its length, and have sufficient symmetry that bending it by applying a torque about the 3 direction causes the line to deform into an arc lying in the 1,2 plane. Make an imaginary cut through the line and let the forces and torque acting at that section on the part lying in the direction of decreasing X_1 be denoted as a shear force V in the positive 2 direction, an axial force P in the positive 1 direction, and a torque M, commonly called a *bending moment*, about the positive 3 direction. The linear and angular momentum principles then require that the actions at that section on the part of the line lying in the direction of increasing X_1 are of equal magnitude but opposite sign. Now let the line be loaded by transverse force F per unit length, directed in the 2 direction, and make assumptions on the smallness of deformation consistent with those of linear elasticity. Let ρA be the mass per unit length (so that A could be interpreted as the cross section area of a homogeneous beam of density ρ) and let u the transverse displacement in the 2 direction. Then, writing X for X_1, the linear and angular momentum principles require that $\partial V/\partial X + F = \rho A\, \partial^2 u/\partial t^2$ and $\partial M/\partial X + V = 0$, where rotatory inertia has been neglected in the second equation, as is appropriate for long-wavelength disturbances compared to cross section dimensions. The curvature κ of the elastic line can be approximated by $\kappa = \partial^2 u/\partial X^2$ for the small deformation situation considered, and the equivalent of the stress-strain relation is to assume that κ is a function of M at each point along the line. The function can be derived by the analysis of stress and strain in pure bending and is $M = EI\kappa$ with $I = \int_A (X_2)^2\, dA$ for uniform elastic properties over all the cross section, and with the 1 axis passing through the section centroid. Hence the equation relating transverse load and displacement of a linear elastic beam is $-\partial^2 (EI\ \partial^2 u/\partial X^2) / \partial X^2 + F = \rho A\, \partial^2 u/\partial t^2$ and this is to be solved subject to two boundary conditions at each end of the elastic line. Examples are $u = \partial u/\partial X = 0$ at a completely restrained ("built in") end, $u = M = 0$ at an end that is restrained against displacement but not rotation, and $V = M = 0$ at a completely unrestrained (free) end. The beam will be reconsidered later in an analysis of response with initial stress present.

The preceding derivation was presented in the spirit of the model of a beam as the *elastic line* of Euler. We may obtain the same equations of motion by the following five steps: (1) Integrate the three-dimensional equations of motion over a section, writing $V = \int_A \sigma_{12}\, dA$; (2) Integrate X_2 times those equations over a section, writing $M = -\int_A X_2\, \sigma_{11}\, dA$; (3) Assume that planes initially perpendicular to fibers lying along the 1 axis remain perpendicular during deformation, so that $\varepsilon_{11} = \varepsilon_o(X, t) - X_2\, \kappa(X, t)$ where $X \equiv X_1$, $\varepsilon_o(X, t)$ is the strain of the fiber along the 1 axis and $\kappa(X, t) = \partial^2 u/\partial X^2$ where $u(X, t)$ is u_2 for the fiber initially along the 1 axis;

(4) Assume that the stress σ_{11} relates to strain as if each point was under uniaxial tension, so that $\sigma_{11} = E\,\varepsilon_{11}$; and (5) Neglect terms of order h^2/L^2 compared to unity, where h is a typical cross section dimension and L is a scale length for variations along the 1 direction. In step (1) the average of u_2 over area A enters but may be interpreted as the displacement u of step (3) to the order retained in (5). The kinematic assumption (3) together with (5), if implemented under conditions that there are no loadings to generate a net axial force P, requires that $\varepsilon_o(X, t) = 0$ and that $\kappa(X, t) = M(X, t)/EI$ when the 1 axis has been chosen to pass through the centroid of the cross section. Hence $\sigma_{11}. = -X_2\, M(X, t) / I = -X_2\, E\, \partial^2 u(X, t)/\partial X^2$ according to these approximations. The expression for σ_{11} is exact for static equilibrium under *pure bending*, since assumptions (3) and (4) are exact and (5) is irrelevant then. That is, of course, what motivates the use of assumptions (3) and (4) in a situation which does not correspond to pure bending.

Sometimes we wish to deal with solids which are already under stress in what we choose as the reference configuration, from which we measure strain. As a simple example, suppose that the beam just discussed is under an initial uniform tensile stress $\sigma_{11} = \sigma^o$; that is, the axial force P = $\sigma^o A$. If σ^o is negative and of significant magnitude, one generally refers to the beam as a column; if it is large and positive, the beam might respond more like a taut string. The initial stress σ^o contributes a term to the equations of small transverse motion, which now becomes

$$-\partial^2 (EI\ \partial^2 u/\partial X^2) / \partial X^2 + \sigma^o A\, \partial^2 u/\partial X^2 + F = \rho A\, \partial^2 u/\partial t^2 .$$

Free vibrations. Suppose that the beam is of length L, is of uniform properties, and is hinge-supported at its ends at X = 0 and X = L so that u = M = 0 there. Then free transverse motions of the beam, solving the above equation with F = 0, are described by any linear combination of the real part of solutions of the form $u = C_n\, exp\, (i\, \omega_n\, t)\, sin\, (n\, \pi\, X / L)$ where n is any positive integer, C_n is an arbitrary complex constant, and where

$$\rho\, A\, \omega_n^2 = (n\, \pi / L)^4\, E\, I\, [1 + (\sigma^o / E)\, (A\, L^2 / n^2\, \pi^2\, I)]$$

This expression is arranged so that the bracket shows the correction, from unity, of what would be the expression giving the frequencies of free vibration for a beam when there is no σ^o. The correction from unity can be quite significant, even though σ^o/E is always much smaller than unity (for interesting cases, a few times 10^{-6} to, say, 10^{-3} would be a representative range; few materials in bulk form would remain elastic or resist fracture at higher σ^o/E, although good piano wire could reach about 10^{-2}). That is because σ^o/E is multiplied by a term which can become enormous for a beam which is long compared to its thickness; for a square section of side length h, that term (at its largest, when n = 1) is $A\, L^2 / \pi^2\, I \approx 1.2\, L^2 / h^2$, which can combine with a small σ^o/E to produce a correction term within the brackets which is quite non-negligible compared to unity. When $\sigma^o > 0$ and L is large enough to make the bracketed expression much larger than unity, the EI term cancels out and the beam simply responds like a stretched string (here *string* denotes an object which is unable to support a bending moment), although at large enough vibration mode number n, the string-like effects become negligible and beam like response takes over; at sufficiently high n that L/n is reduced to the same order as h, the simple beam theory becomes inaccurate and should be replaced by three-dimensional elasticity or, at least, an improved beam theory which takes account rotary inertia and shear deformability. (While the option of using three-dimensional elasticity for such a problem posed an insurmountable obstacle over most of the history of the subject, by 1990 the availability of computing power and easily used software reduced it to a routine problem which could be studied by an undergraduate engineer or physicist using the finite element method or some other computational mechanics technique.)

Buckling. An important case of that of compressive loading, $\sigma^o < 0$, which can lead to buckling. Indeed, we see that if $\sigma^o A < -\pi^2 EI/L^2$, then the ω_n^2 for at least for n = 1 is negative, which means that the corresponding ω_n is of the form $\pm$ i b, where b is a positive real number, so that the *exp* (i ω_n t) term has a time dependence of type which no longer involves oscillation but, rather, exponential growth, *exp*(b t). The critical compressive force, $\pi^2 EI/L^2$, which causes this type of behavior is called the Euler buckling load; different numerical factors are obtained for different end conditions. We may see that the acceleration associated with the n = 1 mode becomes small in the vicinity of the critical load, and vanishes at that load. Thus solutions are possible, at the buckling load, for which the column takes a deformed shape without acceleration; for that reason, an approach to buckling problems which is equivalent for what, in dynamical terminology, are called conservative systems is to seek the first load at which an alternate equilibrium solution u = u(X), other than u = 0, may exist.

Instability by *divergence*, that is, with growth of displacement in the form *exp*(b t), is representative of conservative systems. Columns under non-conservative loadings, for example, by a follower force which has the property that its line of action rotates so as to always be tangent to the beam centerline at its place of application, can exhibit a *flutter* instability in which the dynamic response is proportional to the real or imaginary part of a term like *exp*(i a t) *exp*(b t) with a real, that is, an oscillation with exponentially growing amplitude. Such instabilities also arise in the coupling between fluid flow and elastic structural response, as in the subfield called *aeroelasticity*, and the prototype is the fluter of an airplane wing. That is a torsional oscillation of the wing, of growing amplitude, which is driven by the coupling between rotation of the wing and the development of aerodynamic forces related to the angle of attack; the coupling feeds more energy into the structure with each cycle. Of course, instability models based on linearized theories, and predicting exponential growth in time, really tell us no more than that the system is deforming out of the range for which our mathematical model applies. Proper nonlinear theories, that take account of the finiteness of rotation and, sometimes, the large and possibly non-elastic strain of material fibers are necessary to really understand the phenomena. An important subclass of such nonlinear analyses for conservative systems involves the static post-buckling response of a perfect structure, such as a perfectly straight column or perfectly spherical shell. That post-buckling analysis allows one to determine if increasing force is required for very large displacement to develop during the buckle, or whether the buckling is of a more highly unstable type for which the load must diminish with buckling amplitude in order to still satisfy the equilibrium equations. The latter type of behavior describes a structure which shows strong sensitivity of the maximum load that it can support to very small imperfections of material or geometry, as do many shell structures.

This is a very small sample which, at most, barely touches on most of the issues raised in the *Introduction* and *Historical Sketch*. Nevertheless, it may suggest how considerations from the section on *Basic Principles* are mustered for analysis of specific problems in solid mechanics.

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FIGURE CAPTIONS

1 Coordinate system; position (**x**) and velocity (**v**) vectors; body force **f** dV acting on element dV of volume, and surface force **T** dS acting on element dS of surface.

2 Stress components; first index denotes plane, second denotes direction.

3 Linear momentum principle relates **T** for an arbitrarily inclined face to the σ_{ij}.

4 Principal stresses.

5 Extensional strain; element in reference configuration shown with dashed lines.

6 Simple shear strain; element in reference configuration shown with dashed lines.

7 Diagram to show relation of strains to gradients of displacement.

8 Transverse motion of an initially straight beam, shown at left as an elastic line, and at right as a solid of finite section.

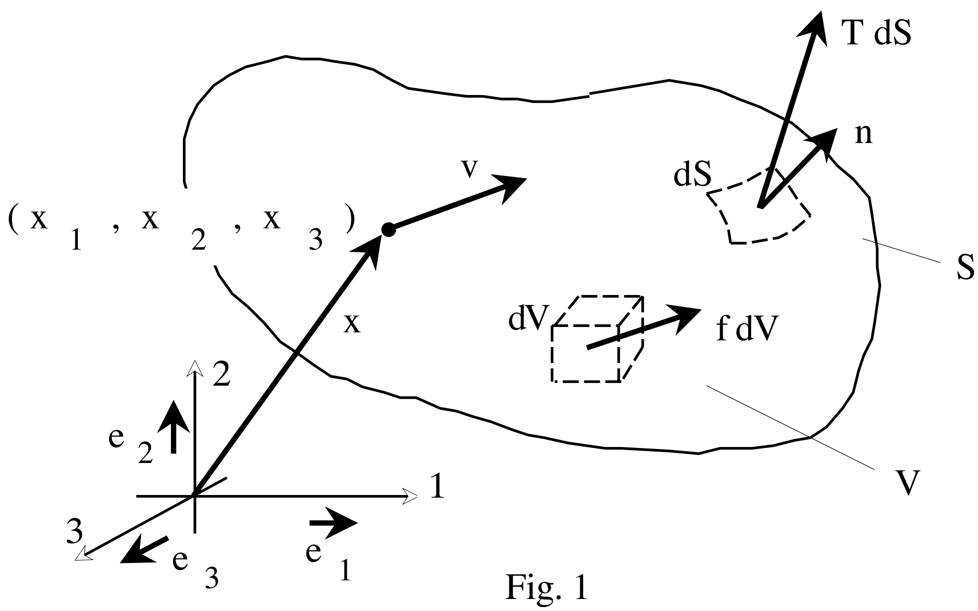


Fig. 1

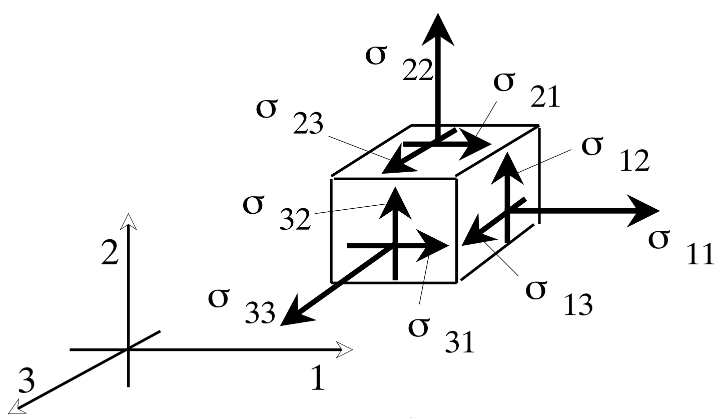
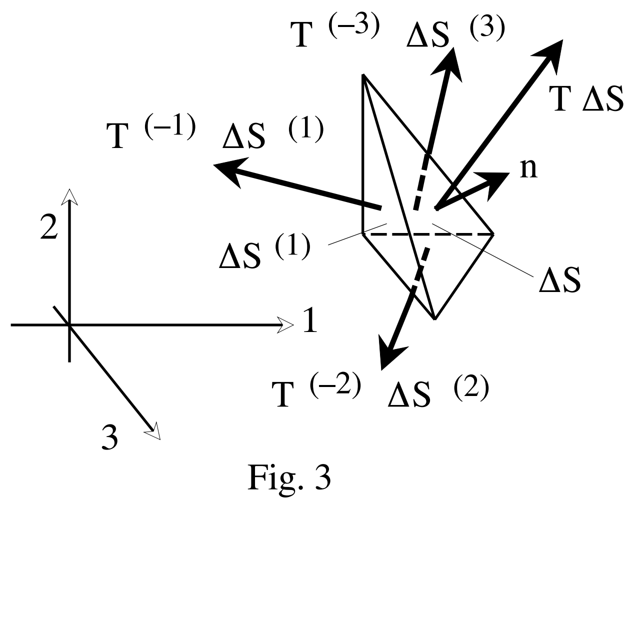


Fig. 2



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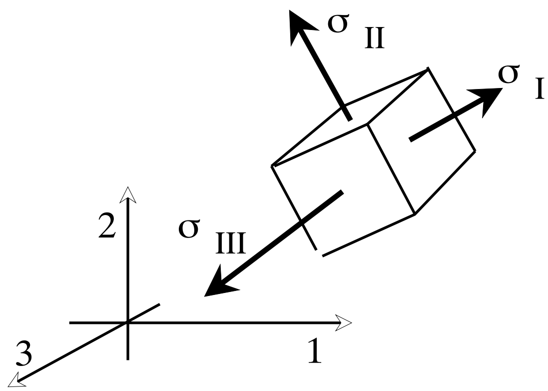

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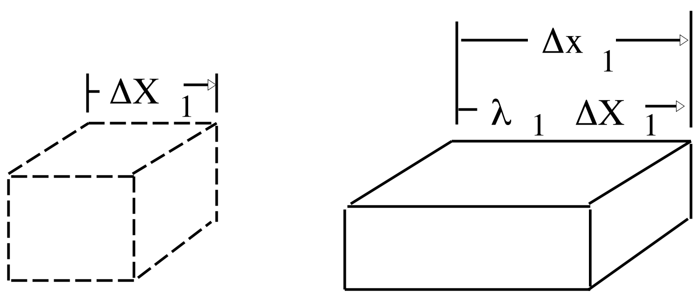

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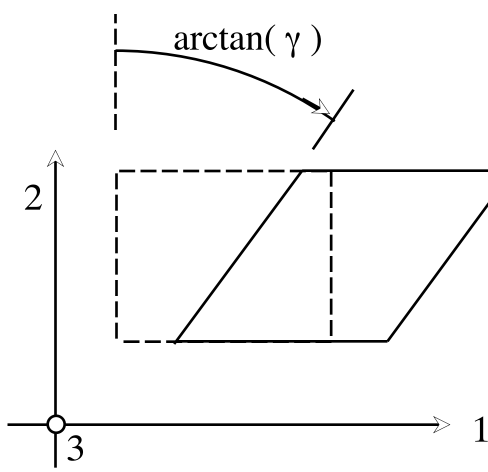

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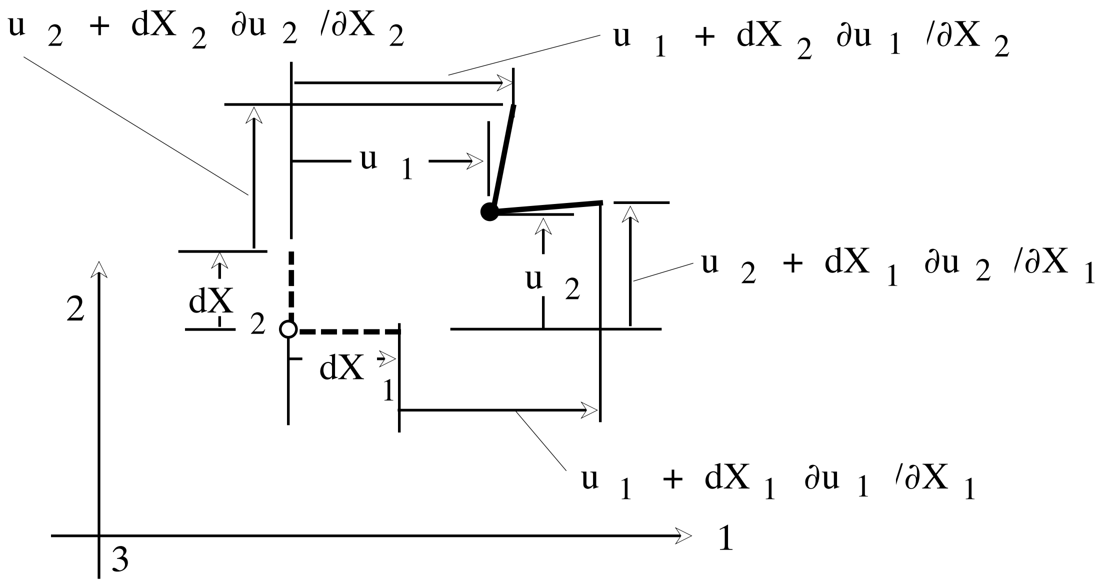
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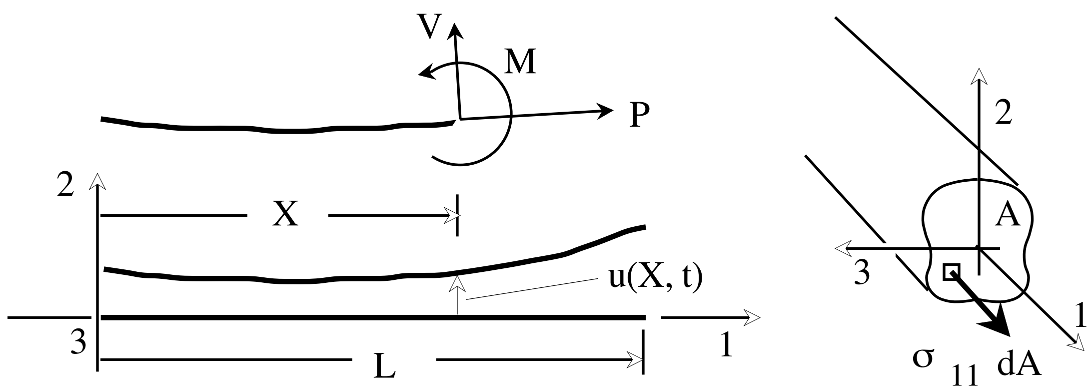
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Fig. 8