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APPLICATION OF 3-D WEIGHT FUNCTIONS—II. THE STRESS FIELD AND ENERGY OF A SHEAR DISLOCATION LOOP AT A CRACK TIP

HUAJIAN GAO† and JAMES R. RICE

Division of Applied Sciences, Harvard University, Cambridge, MA 02138, U.S.A.

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ABSTRACT

THE GENERAL weight function expressions given in GAO (*J. Mech. Phys. Solids* **37**, 133, 1989), referred to here as part I, for combined-mode crack-dislocation interaction problems in the three-dimensional regime are applied to solve for the stress field and energy of a shear dislocation loop emerging from the tip of a half-plane crack. The results are compared to the previously proposed approximate estimates for shear loops by ANDERSON and RICE (*J. Mech. Phys. Solids* **35**, 743, 1987), who solved exactly for prismatic opening dislocation loops that are co-planar with the crack and also for the analogous 2-D cases of general crack tip-parallel line dislocations. The energy results are presented in terms of a correction factor m, following Anderson and Rice, to the usual estimate of energy for an emergent crack tip loop as half the energy of a full loop (identified as the emergent loop and its image relative to the crack front) in an uncracked solid. For a full circular shear loop the energy is $U = [(2-\nu)\mu b^2 r/4(1-\nu)]\ \ln\,(8r/e^2 r_0)$, where r_0 denotes the core cut-off parameter and μ, ν are the shear modulus and Poisson ratio. Thus for a semicircular loop emerging from the crack tip, the energy is expressed as $U = [(2-\nu)\mu b^2 r/8(1-\nu)]\ \ln\,(8mr/e^2 r_0)$, where the constant m depends on the orientation angle ψ of the Burgers vector relative to a line normal to the crack tip and the inclination angle ϕ of the dislocated plane relative to the crack plane. The m factors are calculated at selected angles ϕ for rectangular and semicircular loops. This involves multiple numerical integrations based on the weight functions of part I, first to obtain the stress field and then to integrate it over the dislocated area to get the energy, and requires a large amount of computing CPU time. An approximate formula for m is proposed for general inclined dislocation loops, based on known 2-D results for m factors for arbitrary angles ϕ calculated by ANDERSON and RICE (1987) and the 3-D $m(\phi = 0)$ results given here for shear dislocation loops in the crack plane. It compares well to the exact results.

INTRODUCTION

IN PART I (GAO, 1989) we have presented some explicit formulae for calculation of stress intensity factors induced by interaction of transformation strains and dislocations with crack tips. The calculation is based on the three-dimensional weight function solutions by BUECKNER (1987) while the formulation of the problem is based on the analysis of a crack tip interacting with sources of internal stress of RICE (1985) and ANDERSON and RICE (1987).

It remains an interesting topic to study the ductile vs brittle response to cracks in various materials, and this partly includes considerations of whether a solid is intrinsically cleavable. RICE and THOMSON (1974) have proposed that such intrinsic cleavability is determined by the competition between cleavage decohesion and crack

† Current address: Division of Applied Mechanics, Stanford University, Stanford, CA 94305, U.S.A.

tip emission of blunting dislocations. In other words, if the dislocation nucleation conditions are reached before those for decohesion of the interface or crystal plane ahead of the crack, and if the dislocations are sufficiently mobile once nucleated, it is feasible to conclude that there will be an active plastic zone in front of the crack and the fracture will involve a ductile flow mechanism such as microvoid growth to coalescence and/or shear band localization, rather than cleavage. In the dislocation nucleation process, the concentration of the applied stress field at the tip favors the emission to relieve the elevated stresses, but the creation of the dislocation itself and a "ledge" at the crack tip tend to increase the energy and hence prevent the emission. See THOMSON (1966), LIN and THOMSON (1986), ANDERSON and RICE (1986) and RICE (1987) for recent discussions of this topic. Its study requires calculation of the self energy of crack tip dislocation loops. The energy results for 3-D shear dislocation loops emanating from a crack tip have not been accurately determined, although ANDERSON and RICE (1987) have proposed approximate estimations based on their exact results for prismatic loops coplanar with the crack and for general 2-D configurations of line dislocations lying parallel to the crack tip. We solve for the 3-D shear loop energy results in this paper using 3-D weight function methods, as facilitated by results in part I.

For convenience we set up the crack-dislocation system in Cartesian coordinates x, y, z so that the crack lies in the plane $y = 0$ with tip parallel to the z axis at $x = a$ and crack plane on $x < a$; Fig. 1 shows the configuration when $a = 0$. In addressing the problems of dislocation loops inclined to the crack plane by an angle ϕ we also adopt coordinates t, n, z with t lying in the dislocation plane perpendicular to the z axis and n normal to the dislocation plane. The t, n, z system has the same origin as the x, y, z coordinates (Fig. 1). When $\phi = 0$ the two coordinate systems coincide.

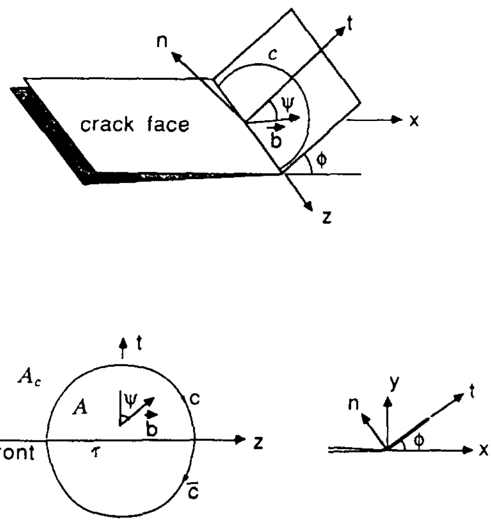

FIG. 1. Geometry of an emergent crack-tip shear dislocation loop on surface inclined to the crack plane by an angle ϕ. Two coordinate systems x, y, z and t, n, z are adopted. The Burgers vector $\bar{b}$ makes angle ψ with the t axis.

The general dislocation is represented by a displacement discontinuity $\Delta\vec{u} = \vec{u}^+ - \vec{u}^-$ on the cut surface A having normal N_i pointing from its $(-)$ side to $(+)$ side. Following RICE (1985), the variation in the strain energy of the above system, in the absence of body forces, is

$$\delta U = -\int_{-\infty}^{\infty} \Lambda_{\alpha\beta} K_\alpha K_\beta \delta a \, \mathrm{d}z - \int_A N_i \sigma_{ij} \delta(\Delta u_j) \, \mathrm{d}A \tag{1}$$

where K_α ($\alpha = 1, 2, 3$) are the stress intensity factors and $\Lambda_{\alpha\beta}$ is a symmetric matrix that appears in the expression for Irwin energy release rate as $\mathscr{G} = K_\alpha \Lambda_{\alpha\beta} K_\beta$ and is diagonal for isotropic material with $\Lambda_{11} = \Lambda_{22} = (1-\nu)/2\mu$, $\Lambda_{33} = 1/2\mu$. Here $i, j, k, l = x, y, z$ while Greek indices $\alpha, \beta = 1, 2, 3$. The term $N_i \sigma_{ij}$ is the energetic force conjugate to the dislocation. If Δu_j is held fixed, one can integrate (1) over the crack position variable a and write $U = U^0 + \int_{-\infty}^{a} (\partial U/\partial a) \, \mathrm{d}a$ where U^0 denotes the strain energy which the dislocation caused in an uncracked body. The first term in (1) is just $(\partial U/\partial a)\delta a$, and hence

$$U = U^0 - \int_{-\infty}^{a} \int_{-\infty}^{\infty} \Lambda_{\alpha\beta} K_\alpha(z; a') K_\beta(z; a') \, \mathrm{d}z \, \mathrm{d}a'. \tag{2}$$

We have emphasized in (2) the dependence of the intensity factors $K_\alpha(z; a)$ on the crack front location a and position z along the front. We have written the following in part I for the intensity factors induced by a dislocation loop:

$$K_\alpha(z'; a) = 2\mu \int_A U^\alpha_{kl}(x-a, y, z-z') N_k \Delta u_l(x, y, z) \, \mathrm{d}A(x, y, z), \tag{3}$$

where the quantities $U^\alpha_{kl} = U^\alpha_{kl}(x-a, y, z-z')$ are related to derivatives of the weight functions. Special attention is needed in calculating K_α for loops emerging from the crack tip, for which case $a = 0$ and the integral in (3) is not convergent since quantities U^α_{kl} are of order $(x^2+y^2)^{-3/4}$ in the vicinity of the crack tip. It was discussed in ANDERSON and RICE (1987) and in part I that this problem can be remedied by removing a null stress state due to a uniformly dislocated half plane A_u coplanar with the dislocation and extending from the crack tip to infinity so that

$$K_\alpha(z'; 0) = 2\mu \int_{A_u} U^\alpha_{kl}(x, y, z-z') N_k [\Delta u_l(x, y, z) - \Delta u_l(0, 0, z')] \, \mathrm{d}A(x, y, z). \tag{4}$$

Since $-\sigma_{kl}(x, y, z)$ is the differential coefficient of δU with respect to $N_k \delta(\Delta u_l)$, it follows from (2) and (3) that

$$\sigma_{kl}(x, y, z,) = \sigma^0_{kl}(x, y, z) + 4\mu \int_{-\infty}^{a} \int_{-\infty}^{\infty} \Lambda_{\alpha\beta} U^\alpha_{kl}(x-a, y, z-z') K_\beta(z'; a) \, \mathrm{d}z' \, \mathrm{d}a, \tag{5}$$

where $\sigma^0_{kl}(x, y, z)$ is the stress field that would be induced by the dislocation $\Delta\vec{u}$ over surface A in an uncracked full space. Using the expressions for U^α_{kl} presented in part I, we are, in principle, able to calculate the stress field.

In this paper we are only interested in planar Volterra-type dislocation loops, i.e. dislocations with uniform Burgers vector ($\Delta u_j = b_j = \text{const}$), emerging from a crack tip. Let us introduce the "full loop" associated with an emergent dislocation at the tip as the dislocation itself plus its mirror image with respect to the crack front, this imagined "full loop" lying in an uncracked body. The energy of the full loop is denoted by $U^{\text{full loop}}$. The formulae for the calculation of $U^{\text{full loop}}$ for arbitrary dislocation loops with arbitrary Burgers vector $\vec{b}$ can be found, for example, in HIRTH and LOTHE (1968), and can be written in the general form

$$U^{\text{full loop}} = \Omega\alpha b^2 \ln (Qr/r_0), \tag{6}$$

where Ω denotes the perimeter of the full loop, b is the magnitude of the Burgers vector, α is the energy constant (proportional to μ through a factor dependent on ν, loop shape and dislocation type), Q is some geometry dependent numerical constant (e.g. $Q = 8/e^2$ for a circular loop), r is a measure of loop size (e.g. a radius), and r_0 is the core cut-off size. Here we do not go into detail in discussing solutions for full loops.

Now consider an emergent dislocation loop inclined to the crack plane by angle ϕ and adopt the t, n, z system. We follow ANDERSON and RICE (1987) in writing the energy of such a loop with Burgers vector pointing in the jth ($j = t, n, z$) coordinate direction as

$$U_j = (\Omega\alpha_j b^2 \ln m_j + U_j^{\text{full loop}})/2 \tag{7}$$

(no summation here on j), where the subscript j emphasizes that the Burgers vector of the loop is in the j direction. Note that the energy constants α_j depend on the material properties and also possibly on the geometry of the loop such that the coefficient $\Omega\alpha_j b^2$ is also the prelogarithmic coefficient for the energy of the full loop with Burgers vector in the j direction. For a semicircular loop in an isotropic solid the constants α_j are

$$\alpha_n = \frac{\mu}{4\pi(1-\nu)}, \quad \alpha_t = \alpha_z = \frac{(2-\nu)\mu}{8\pi(1-\nu)}. \tag{8}$$

The constant m_j, which corrects the energy expression to account for the presence of the crack, then represents a prismatic loop for $j = n$ and represents shear loops with Burgers vector in t and z directions respectively when $j = t, z$. These shear loops are dislocated in edge character relative to the tip when $j = t$ and in screw character when $j = z$.

If the Burgers vector of a dislocation loop is not parallel to a coordinate direction, then (6) applies for the full loop and we define m by writing the energy of the emergent loop as

$$U = (\Omega\alpha b^2 \ln m + U^{\text{full loop}})/2. \tag{9}$$

For a crack tip dislocation loop that is symmetric about the axis t, the full loop has two axes of symmetry and thus

$$\alpha = \alpha_t \cos^2 \psi + \alpha_z \sin^2 \psi, \tag{10}$$

where ψ is the angle between the Burgers vector and the t direction normal to the crack tip. Equation (10) follows because for a doubly symmetric full loop the stresses induced by b_t ($=b \cos \psi$) do not net work on slip b_z ($= b \sin \psi$) and *vice versa*. We will show that the general m of (9) can be expressed in terms of α_j/α and m_j for such symmetric crack tip dislocation loops.

The above reduces the problem of finding the energy of an emergent crack tip loop to that of finding the correction factor m (or set of factors m_j) on the original approximation of RICE and THOMSON (1974) who wrote the energy of a loop emerging from the crack tip as half that of its related full loop, i.e. they effectively set $m = 1$. The assumption of Rice and Thomson is motivated by their exact 2-D result for the force on a near tip dislocation, showing that the force on a crack tip–parallel dislocation line is equal to the force that would be exerted on it by its mirror image dislocation (relative to the crack front) existing in an uncracked full space. In fact, this observation actually leaves an undetermined constant m even for the 2-D energy expression, as remarked by ANDERSON and RICE (1987). They also showed that $m = 2$ for 2-D straight dislocations of any Burgers vector direction on the prolongation of the crack plane ($\phi = 0$) and ranges between 1.1 and 2 in general, and they calculated the value of m for 3-D rectangular and semicircular shaped emerging prismatic loops ahead of the crack on the crack plane. Then based on their exact 2-D results for the dependence of m for dislocations with arbitrary Burgers vector on the orientation angle ϕ of the dislocated surface relative to the tip, using well-known elastic solutions as summarized by LIN and THOMSON (1986), Anderson and Rice gave an approximate estimate for the energy of the shear dislocation loops inclined to the crack plane. They could not consider the dependence of the m factor on the orientation of the Burgers vector for a 3-D shear dislocation loop coplanar with the crack and, guided by their 2-D results, ignored any such dependence. We find here that the orientation angle ψ has a significant effect on the value of m. In fact the difference between the m_t and m_z, which coincide in the 2-D case when $\phi = 0$, is for a semicircular loop biggest when $\phi = 0$, monotonously decreases as ϕ increases toward 45°, and reverses sign around 45° as ϕ continues to increase.

In this paper we present exact calculations for a crack tip shear dislocation loop. We discuss the stress field for an emergent dislocation loop, especially one lying in the crack plane. We carry out the calculation of the m factors for loops inclined to the crack plane. The values are obtained only at a few selected angles due to the CPU computing time requirements. However an approximate formula, with relative error within the bound of 5%, is proposed based on results for the m factors for shear dislocation loops in the crack plane and on the known 2-D results.

FEATURES OF THE STRESS FIELD OF THE CRACK–DISLOCATION SYSTEM

We study the stress field of a crack tip dislocation loop on a surface inclined to the crack plane by an angle ϕ. Let us adopt t, n, z coordinates as in Fig. 1 and observe that on the dislocation plane $n = 0$ the following relations are valid

$$x = t\cos\phi, \quad y = t\sin\phi. \tag{11}$$

Indices p, q, s range over t, z only in the following. We define the quantities $U^{\alpha}_{ns}(t, z-z'; a; \phi)$ on the plane $n = 0$ by

$$U^{\alpha}_{nt} = \frac{U^{\alpha}_{yy} - U^{\alpha}_{xx}}{2}\sin 2\phi + U^{\alpha}_{xy}\cos 2\phi, \quad U^{\alpha}_{nz} = U^{\alpha}_{yz}\cos\phi - U^{\alpha}_{xz}\sin\phi. \tag{12}$$

The quantity U^{α}_{ns} depends on a through $x-a = t\cos\phi - a$ in $U^{\alpha}_{kl}(x-a, y, z-z')$. In order to calculate the self energy of a dislocation loop, it suffices to compute the shear stresses $\sigma_{ns}(t, z)$ on the plane $n = 0$, i.e. on the dislocation plane. Here we understand $\sigma_{ns}(t, z)$ as $\sigma_{ns}(t, n, z)|_{n=0}$. We may use Eqs (3, 5, 12) to write that

$$\sigma_{ns}(t,z) = \sigma^{0}_{ns}(t,z) + \sigma'_{ns}(t,z) \tag{13}$$

with

$$\sigma'_{ns}(t,z) = 8\mu^2 \int_A \int_{-\infty}^{0} \int_{-\infty}^{\infty} \Lambda_{\alpha\beta} U^{\alpha}_{ns}t, z-z'; a; \phi) \times U^{\beta}_{nq}(\tilde{t}, \tilde{z}-z'; a; \phi)\Delta u_q(\tilde{t}, \tilde{z})\, \mathrm{d}z'\, \mathrm{d}a\, \mathrm{d}\tilde{t}\, \mathrm{d}\tilde{z}. \tag{14}$$

If we let t, z be also denoted by t_t, t_z in subscripted notation, the stress field $\sigma^{0}_{ns}(t, z)$ of the emergent dislocation loop in the absence of the crack is related to an integral representation given by BUI (1977) as

$$\sigma^{0}_{ns}(t,z) = \frac{\mu}{4\pi(1-\nu)} \int_A \frac{(1-\nu)\, D_p \delta_{sq} + \nu D_s \delta_{pq}}{D^3} \frac{\partial}{\partial \tilde{t}_p} \Delta u_q(\tilde{t}, \tilde{z})\, \mathrm{d}\tilde{t}\, \mathrm{d}\tilde{z}, \tag{15}$$

where $D^2 = (t-\tilde{t})^2 + (z-\tilde{z})^2$ and $D_p = t_p - \tilde{t}_p$; $\partial\Delta u_q/\partial t_p$ is understood to be Dirac singular around the bounding contour of a uniformly dislocated area A. A similar set of equations, in slightly different form, was also presented by WEAVER (1977). GAO (1988) showed the consistency of Weaver's equations with those of BUI (1977) shown in (15) and presented an alternative way of getting this set of equations by a Papkovitch–Neuber potential function method. We note that the associated full loop is defined as a loop lying in an uncracked full space, constructed as the loop itself plus its image relative to the crack front. Equation (15) also gives the stress field for a full loop when the integration area A is extended to the full loop area. The loop is enclosed by contours c (the boundary of the crack tip loop excluding the part τ on the crack front) and $\bar{c}$ (mirror image of c) (Fig. 1).

As observed by ANDERSON and RICE (1987) concerning tensile stress for a prismatic dislocation loop in the crack plane, it is also true for shear loops that both σ'_{ns} and σ^{0}_{ns} ($s = t$, z) contain a $1/t$ singularity with the same magnitude but opposite sense, so that the total stress field is of only square root singularity $1/\sqrt{t}$ near the tip. Following Anderson and Rice we adopt the same method of eliminating the $1/t$ singularity of σ'_{ns} by subtracting the null stress state of a uniformly dislocated half-plane emanating from the tip with Burgers vector equal to $\Delta u_s(0, z)$; this null stress state also eliminates the singularity in σ^{0}_{ns} at the tip. Thus we write

$$\sigma'_{ns}(t,z) = 8\mu^2 \int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{0}\int_{-\infty}^{\infty} \Lambda_{\alpha\beta}\, U^{\alpha}_{ns}(t, z-\dot{z}'; a; \phi)$$

$$\times\, U^{\beta}_{nq}(\tilde{t}, \tilde{z}-z'; a; \phi)\,[\Delta u_q(\tilde{t}, \tilde{z}) - \Delta u_q(0, z)]\, \mathrm{d}z'\,\mathrm{d}a\,\mathrm{d}\tilde{t}\,\mathrm{d}\tilde{z} \quad (16)$$

and

$$\sigma^0_{ns}(t,z) = \frac{\mu}{4\pi(1-\nu)}\int_{-\infty}^{\infty}\int_{0-}^{\infty} \frac{(1-\nu)D_p\delta_{sq} + \nu D_s\delta_{pq}}{D^3}$$

$$\times\, \frac{\partial}{\partial \tilde{t}_p}(\Delta u_q(\tilde{t}, \tilde{z}) - H(\tilde{t})\Delta u_q(0, z))\, \mathrm{d}\tilde{t}\,\mathrm{d}\tilde{z}, \quad (17)$$

where $H(\tilde{t})$ denotes the Heaviside unit step function, i.e. $H(\tilde{t})$ equals one for non-negative $\tilde{t}$ and equals zero otherwise. In writing Eqs (16, 17) we have understood $\Delta u_q(\tilde{t}, \tilde{z})$ to be zero outside the dislocation loop area A. We further observe that in (17) a line integral along the loop contour $c+\tau$ (Fig. 1) is implied since $\partial[\Delta\mu_q(\tilde{t}, \tilde{z})]/\partial\tilde{t}_p$ is Dirac singular along $c+\tau$. Moreover, $\partial H(\tilde{t})/\partial\tilde{t}$ is Dirac singular along the z axis, which reduces the corresponding part of the area integral in (17) also to a line integral.

Now we consider an emergent Volterra type dislocation loop at a crack tip, on a plane inclined to the crack plane by an angle ϕ. We will calculate the self energy of the loop by integrating the shear stresses times their work conjugate, i.e. the Burgers vector $\vec{b}$ over surface A, cutting off at r_0 the singular terms which diverge inversely with distance from the dislocation line. For constant Burgers vector loops, the energy is obtained by directly integrating the shear stresses over A. Since we are only interested in calculating the difference between the energy of the crack tip loop and half that of the full loop, we need only to study the difference in stress, $\sigma_{ns}(t,z) - \sigma^{\text{full loop}}_{ns}(t, z)$. The difference in energy can be calculated by integrating this stress difference times the work conjugate b_s (the component of Burgers vector in s direction) over the area of the loop A. Note that in calculating the energy for the full loop, the presence of a singularity of the inverse distance from the perimeter of the loop requires an elastic core cut-off for the dislocation so as to keep the energy bounded. Here the choice of a core cut-off is avoided by calculating the bounded integral of the difference in stress $\sigma_{ns}(t, z) - \sigma^{\text{full loop}}_{ns}(t, z)$, between the exact elastic result and that for the full loop in an infinite body. This procedure is not strictly consistent with other core cut-off procedures such as that by GAVAZZA and BARNETT (1976) of excluding the energy of a tube along the dislocation front.

For convenience we assume that the crack tip loop is symmetric about the t axis; such is a feature of semicircular and rectangular loops. Consider a symmetric loop of edge character, i.e. $b_s = b\delta_{st}$ (Kronecker delta), the associated full loop being doubly symmetric. By symmetry we observe that the shear stress component $\sigma_{nz}(t, z)$ of the crack tip loop must be an odd function of z and also that $\sigma^{\text{full loop}}_{nz}(t, z)$ is an odd function of z. Therefore the z component of the stress difference $\sigma_{nz}(t, z) - \sigma^{\text{full loop}}_{nz}(t, z)$ integrates to zero over A and hence does no net work on the slip b_t. By a similar symmetry argument the t component of the stress difference $\sigma_{nt}(t, z) - \sigma^{\text{full loop}}_{nt}(t, z)$ does no net work on the slip b_z.

Therefore the coupling terms containing $b_t b_z$ in the final expression for the self energy of a symmetric loop vanish. Hence the self energy of an emergent shear loop with Burgers vector $\vec{b}$ simply equals the energy of a loop with Burgers vector $b_t = b \cos \psi$ in the t direction plus that of a loop with Burgers vector $b_z = b \sin \psi$ in z direction. Thus the general energy problem is decoupled into two independent problems, one with $\psi = 0$ and the other with $\psi = \pi/2$. Therefore it suffices to study the symmetric loops with Burgers vector pointing in a coordinate direction. Again, semicircular and rectangular loops fall into this category.

ENERGY CORRECTION FACTOR

The energy correction factor m is in general a function of the inclination angle ϕ and the orientation angle ψ of the Burgers vector relative to the t axis. Using Eqs (13, 16, 17) and the U^{α}_{kl} presented in part I, the stress field of an emergent crack tip loop can be numerically evaluated and integrated to calculate the value of $m(\phi, \psi)$.

We have shown in the last section that we need only study the energy for special cases of dislocations with Burgers vector pointing in a coordinate direction. The correction constant $m = m(\phi, \psi)$ for these special cases will be denoted by m_j, i.e. $m_t(\phi) = m(\phi, 0)$ and $m_z(\phi) = m(\phi, 90°)$. By the definition of m_j as in (7), we have the following general formulae

$$\tfrac{1}{2}\Omega\alpha_j \ln m_j = (U_j - \tfrac{1}{2}U_j^{\text{full loop}})/b^2 = -\tfrac{1}{2}\int_A \Gamma_{nj}(t, z)\, \mathrm{d}A \tag{18}$$

(no summation on j), with

$$\Gamma_{nj} = (\sigma_{nj} - \sigma_{nj}^{\text{full loop}})/b$$

for a loop with Burgers vector b in the j direction. The quantity Γ_{nj}, and hence m_j, depends on the direction j in which the Burgers vector points while it is independent of the magnitude of the Burgers vector. If we consider a point t, z within A in (16), $\Delta u_q(\tilde{t}, \tilde{z}) - \Delta u_q(0, z)$ equals zero when $\tilde{t}, \tilde{z}$ lies inside A and equals $-b_q$ when $\tilde{t}, \tilde{z}$ lies outside A in the complementary area A_c (Fig. 1). For convenience we present here the quantities Γ_{nt} and Γ_{nz}:

$$\begin{aligned}
\Gamma_{nt}(t, z) = {} & \frac{\mu}{4\pi(1-\nu)}\left\{\left(\int_{-\infty}^{z_1} + \int_{z_2}^{+\infty}\right)\frac{t\,\mathrm{d}\tilde{z}}{[t^2 + (\tilde{z} - z)^2]^{3/2}}\right. \\
& \left. + \int_{\mathscr{C}} \frac{(\tilde{t} - t)\,\mathrm{d}\tilde{z} - (1-\nu)(\tilde{z} - z)\,\mathrm{d}\tilde{t}}{D^3}\right\} \\
& - 8\mu^2 \int_{A_c}\int_{-\infty}^{0}\int_{-\infty}^{\infty} \Lambda_{\alpha\beta}\, U^{\alpha}_{nt}(t, z - z'; a; \phi) \\
& \times U^{\beta}_{nt}(\tilde{t}, \tilde{z} - z'; a; \phi)\,\mathrm{d}z'\,\mathrm{d}a\,\mathrm{d}\tilde{t}\,\mathrm{d}\tilde{z},
\end{aligned} \tag{19}$$

$$\Gamma_{nz}(t,z) = \frac{\mu}{4\pi(1-\nu)}\left\{\left(\int_{-\infty}^{z_1} + \int_{z_2}^{+\infty}\right)\frac{(1-\nu)t\,d\tilde{z}}{[t^2+(\tilde{z}-z)^2]^{3/2}}\right.$$
$$\left. + \int_{\bar{c}} \frac{(1-\nu)(\tilde{t}-t)\,d\tilde{z}-(\tilde{z}-z)\,d\tilde{t}}{D^3}\right\}$$
$$-8\mu^2 \int_{A_c}\int_{-\infty}^{0}\int_{-\infty}^{\infty} \Lambda_{\alpha\beta}\, U^{\alpha}_{nz}(t,z-z';a;\phi)$$
$$\times U^{\beta}_{nz}(\tilde{t},\tilde{z}-z';a;\phi)\,dz'\,da\,d\tilde{t}\,d\tilde{z}. \tag{20}$$

The above results and (18) enable us to calculate the value of m_t and m_z when the shape of the dislocation loop is known. With these results we can readily calculate the self energy of a dislocation loop by (7). If the Burgers vector of a loop is orientated in an angle ψ relative to the t axis normal to the crack tip, i.e. $b_t = b\cos\psi$, $b_z = b\sin\psi$ and the loop is symmetric about the t axis, we have

$$U = (\Omega\alpha b^2 \ln m + U^{\text{full loop}})/2$$
$$= [\Omega b^2(\alpha_t \cos^2\psi \ln m_t + \alpha_z \sin^2\psi \ln m_z) + U^{\text{full loop}}]/2, \tag{21}$$

where we see that the general m is defined as

$$\ln m(\phi,\psi) = \frac{\alpha_t \cos^2\psi}{\alpha}\ln m_t(\phi) + \frac{\alpha_z \sin^2\psi}{\alpha}\ln m_z(\phi) \tag{22}$$

where (Eq. 10) $\alpha = \alpha_t\cos^2\psi + \alpha_z\sin^2\psi$. Hence the energy correction factor m for shear dislocation loops with arbitrarily orientated Burgers vector is solely determined by m_t and m_z.

The numerical calculation of m involves six-fold integration. We transform the integrals over an infinite interval to a finite interval by the following relation

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-1}^{1} [f(x)+f(1/x)/x^2]\,dx \tag{23}$$

and the inverse square root singularities are removed by making standard square transformations. Finally we break the six-fold integration into three double integrals and for each double integral we carefully choose a number of Gaussian points. A numerical routine using standard Fortran 77 code was developed for the numerical integrations. We specialize the results to a semicircular loop of radius r and a rectangular loop of dimension r perpendicular to, and dimension $2w$ parallel to, the crack tip (Fig. 2). The parameters are (HIRTH and LOTHE, 1968)

$$\Omega = 2\pi r, \quad \alpha = \alpha_t = \alpha_z = \frac{(2-\nu)\mu}{8\pi(1-\nu)} \tag{24}$$

for a semicircular loop and

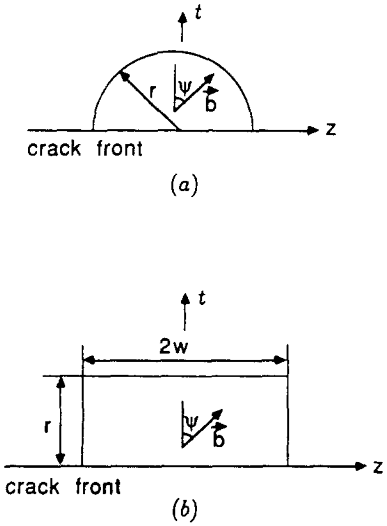

FIG. 2. (a) Geometry of a crack-tip circular loop and (b) a rectangular loop on $n = 0$ plane.

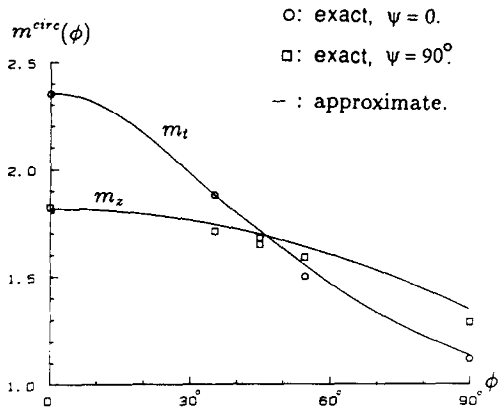

FIG. 3. Values of energy correction factor m_t and m_z as function of inclination angle ϕ of semicircular dislocation loops. The numerical results calculated from the exact formulation are presented as circles for m_t and as squares for m_z. The solid lines represent the proposed approximate formula for m_t.

$$\alpha_t = \frac{1+(r/w)(1-\nu)}{8\pi(1-\nu)(1+r/w)}\mu, \quad \alpha_z = \frac{r/w+(1-\nu)}{8\pi(1-\nu)(1+r/w)}\mu, \tag{25}$$
$$\Omega = 4(w+r), \quad \alpha = \alpha_t \cos^2\psi + \alpha_z \sin^2\psi$$

for a rectangular loop.

The results for m_t and m_z for semicircular loops are calculated at selected angles $\phi = 35.264°$, $45°$ $54.736°$, $90°$, and are presented in Fig. 3 as circles (m_t) and squares (m_z). We take the Poisson ratio $\nu = 0.3$. Similar calculations are also carried out for rectangular loops and are plotted in Fig. 4. Calculations accurate to a reasonable number (e.g. two or three) of significant digits involve a large number of integration

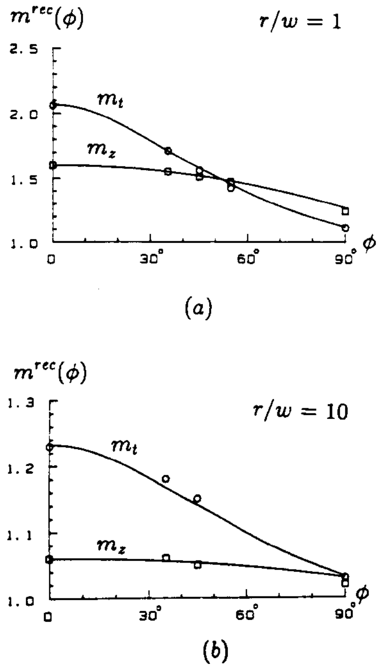

FIG. 4. Values of energy correction factor m_t and m_z as function of inclination angle ϕ of rectangular dislocation loops of aspect ratio (a) $r/w = 1$ and (b) $r/w = 10$. The numerical results calculated from the exact formulation are presented as circles for m_t and as squares for m_z. The solid lines represent the proposed approximate formula for m_j.

points (increases with p^6 for p Gaussian points in one dimension) and require substantial computing CPU time, e.g. 2 h on the Vax 8650 to compute the m value for a chosen angle ϕ and an aspect ratio r/w, the latter being relevant for rectangular loops.

Fortunately there exists an alternative way of getting the m values to a good approximation. It is worth noting that the weight functions become much simplified for dislocation loops coplanar with the crack plane ($\phi = 0$) (see Appendix A). This simplification leads to simplified expressions for the stresses and more effective numerical calculations (e.g. for rectangular loops the CPU computing time for one value of m ($\phi = 0$) drops to several mins). Starting with the accurate values of m for dislocation loops in the crack plane and the exact ψ dependence of m, we show a good approximation for m factors for inclined shear loops based on the 2-D line dislocation results. This motivates further investigation of the coplanar dislocation loops in the following section.

ENERGY CORRECTION FACTOR OF A SHEAR LOOP COPLANAR WITH THE CRACK

In Appendix A we present the derivation of the shear stresses for an emergent dislocation loop coplanar with the crack plane, based on the then simplified weight function expressions. Here we calculate the m factor for loops of this type by integrating the shear stresses over A. In this case (18) reduces to

$$\tfrac{1}{2}\Omega\alpha_j \ln m_j(0) = -\tfrac{1}{2}\int_A \Gamma_{yj}(x,z)\,\mathrm{d}A, \tag{26}$$

where $\Gamma_{yj} = (\sigma_{yj} - \sigma_{yj}^{\text{full loop}})/b$. We use the stress formulae derived in Appendix A in the special case of uniform Burgers vectors to write the quantities Γ_{yx} and Γ_{yz} as

$$\Gamma_{yx}(x,z) = \frac{\mu}{4\pi(1-\nu)}\left\{\left(\int_{-\infty}^{z_1} + \int_{z_2}^{+\infty}\right)\frac{x\,\mathrm{d}\tilde{z}}{[x^2+(\tilde{z}-z)^2]^{3/2}} + \int_{\tilde{C}} \frac{(\tilde{x}-x)\,\mathrm{d}\tilde{z} - (1-\nu)(\tilde{z}-z)\,\mathrm{d}\tilde{x}}{D^3} - \frac{2}{\pi}\int_{A_c}\left[\left(1+\frac{2\nu}{2-\nu}\frac{\partial}{\partial z}(z-\tilde{z})\right)F + \frac{\nu^2}{2-\nu}\frac{\partial^2 G}{\partial \tilde{z}\,\partial z}\right]\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{z}\right\}, \tag{27}$$

$$\Gamma_{yz}(x,z) = \frac{\mu}{4\pi(1-\nu)}\left\{\left(\int_{-\infty}^{z_1} + \int_{z_2}^{+\infty}\right)\frac{(1-\nu)x\,\mathrm{d}\tilde{z}}{[x^2+(\tilde{z}-z)^2]^{3/2}} + \int_{\tilde{C}} \frac{(1-\nu)(\tilde{x}-x)\,\mathrm{d}\tilde{z} - (\tilde{z}-z)\,\mathrm{d}\tilde{x}}{D^3} - \frac{2}{\pi}\int_{A_c}\left[(1-\nu)\left(1-\frac{2\nu}{2-\nu}\frac{\partial}{\partial z}(z-\tilde{z})\right)F + \frac{\nu^2}{2-\nu}\frac{\partial^2 G}{\partial \tilde{z}\,\partial z}\right]\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{z}\right\}, \tag{28}$$

where functions F and G are defined in Appendix A. A scheme to simplify further the integrations over the complementary area A_c is discussed in Appendix B, which is especially useful for rectangular loops, where the m values for many chosen aspect ratios r/w need to be calculated.

(*a*) *Coplanar semicircular loops*

The numerical calculations show that $m_t \simeq 2.35$ and $m_z \simeq 1.82$ for $\nu = 0.3$ [ANDERSON and RICE (1987) quoted an earlier set of values from the present work, as 2.67 and 1.99 respectively, but these reflected an error in the original version of our program]. The m_t is higher than $m_n = 2.21$ for a prismatic dislocation loop as presented by ANDERSON and RICE (1987), while the m_z is lower (by 20%) than the prismatic value. The m factor for a dislocation loop with Burgers vector orientated at angle ψ with the x-axis is

$$\ln m^{\text{circ}}(\phi = 0) = \cos^2\psi \ln 2.35 + \sin^2\psi \ln 1.82. \tag{29}$$

The results for $m^{\text{circ}}(\phi = 0)$ are plotted in Fig. 5. The value of m_z is lower than that

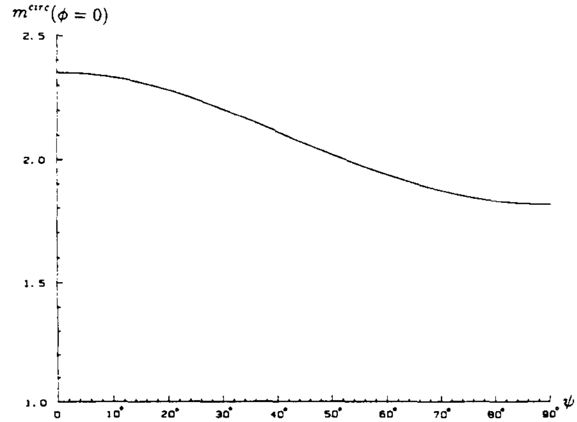

FIG. 5. Values of energy correction factor m vs the angle ψ of Burgers vector relative to the t axis for semicircular loops coplanar with the crack.

of a line dislocation of 2-D geometry, given as $m^{2D}(\phi = 0) = 2$ by ANDERSON and RICE (1987), while m_t is higher than the 2-D value.

(*b*) *Coplanar rectangular loops*

In the case of coplanar rectangular loops, the results for m_t and m_z are plotted in Fig. 6 vs the aspect ratio r/w and w/r. At limiting cases when $w/r \to \infty$, $m_j \to 2$ and hence $m \to 2$. When $r/w \to \infty$, $m_j \to 1$, hence $m \to 1$. The former limit corresponds to the 2-D geometry of a straight dislocation line parallel to the crack front, and the value of m matches the exact 2-D results given by ANDERSON and RICE (1987). It is found that only in the latter limit when $r/w \to \infty$, corresponding to a pair of straight dislocations stretched out from the crack tip, $m = 1$ so that then the elastic energy is given exactly by half that of the full loop. When $r/w = 1$, it is found that $m_t = 2.03$ and $m_z = 1.59$, compared to the prismatic loop value $m_n = 1.92$. In the crack plane

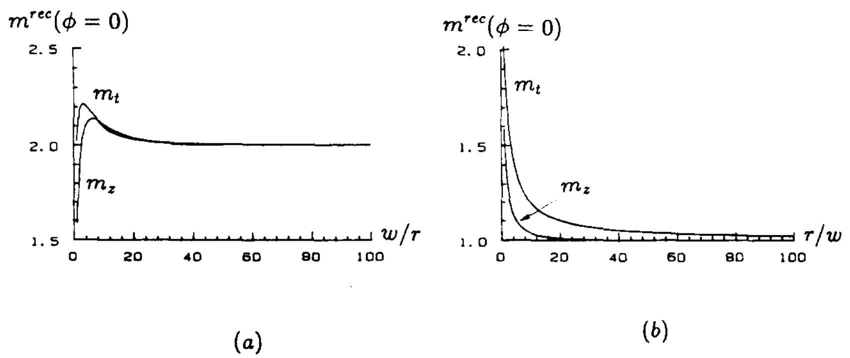

FIG. 6. Values of energy correction factor m_t and m_z vs the aspect ratio (a) r/w and (b) w/r for rectangular loops coplanar with the crack.

$\phi = 0$, we find that $m_t > m_n > m_z$ when the aspect ratio $r/w < 2$. When $w/r > 20$ we can treat the rectangular loops as the limiting 2-D case with $m = 2$.

APPROXIMATE FORMULA FOR CALCULATING m FACTORS OF 3-D SHEAR DISLOCATION LOOPS

An approximate formula for $m_t(\phi)$ and $m_z(\phi)$ when $\phi \neq 0$ is proposed based on the numerically calculated data for m. We find that

$$\ln m_j(\phi) = \frac{\ln m_j(\phi = 0)}{\ln m_j^{2D}(\phi = 0)} \ln m_j^{2D}(\phi) \quad (m_j^{2D}(\phi = 0) = 2) \tag{30}$$

is valid within 5% error and serves as a very good approximation for the m factor. Here $m_j^{2D}(\phi)$ are the energy correction factors for 2-D straight dislocations with Burgers vector in the j direction and are calculated by ANDERSON and RICE (1987). Both m_t^{2D} and m_z^{2D} were plotted against the angle ϕ for $0^\circ < \phi < 90^\circ$ by ANDERSON and RICE (1987). In fact, $m_j^{2D}(\phi)$ can be written in the following closed form:

$$m_t^{2D} = 2\cos(\phi/2)\exp\left(-\frac{1}{2}\sin\frac{\phi}{2}\sin\frac{3\phi}{2}\right),$$

$$m_z^{2D} = 2\cos(\phi/2),$$

$$m_n^{2D} = 2\cos(\phi/2)\exp\left(-\frac{1}{2}\sin\frac{\phi}{2}\left(2\sin\frac{\phi}{2} - \sin\frac{3\phi}{2}\right)\right).$$

For a semicircular loop the above equation suggests

$$\ln m_t(\phi) \simeq 1.23 \ln m_t^{2D}(\phi), \quad \ln m_z(\phi) \simeq 0.86 \ln [2\cos(\phi/2)]. \tag{31}$$

The m_j calculated by (31) are plotted in Fig. 3 as solid lines and may be compared to the previous numerical points for m_j. The $m_j(\phi = 0)$ are easily calculated with the help of the simplified analytical results for the stress field in (26–28). As described before, the direct calculation of the numerical values of m when $\phi \neq 0$ involves many hours of computing CPU time so that the above approximation is valuable.

Then the complete result for three-dimensional m factor for general semicircular shear dislocation loops is, from (22) with use of (31),

$$\ln m(\phi, \psi) \simeq 1.23 \cos^2\psi \ln m_t^{2D}(\phi) + 0.86 \sin^2\psi \ln [2\cos(\phi/2)]. \tag{32}$$

In contrast to (32), ANDERSON and RICE (1987) proposed the following m formula for semicircular shear loops ($\nu = 0.3$)

$$\ln m^{A-R}(\phi, \psi) \simeq 1.14 \left(\frac{\cos^2\psi \ln m_t^{2D}(\phi) + 0.7\sin^2\psi \ln [2\cos(\phi/2)]}{\cos^2\psi + 0.7\sin^2\psi}\right). \tag{33}$$

We plot the values of m and m^{A-R} from Eqs (32, 33) in Fig. 7 for $\psi = 0, 30^\circ, 60^\circ, 90^\circ$. It is observed that there is a relatively strong dependence of m on the angle ψ at $\phi = 0$, and that dependence becomes weaker as ϕ increases toward 45° (Fig. 7). When

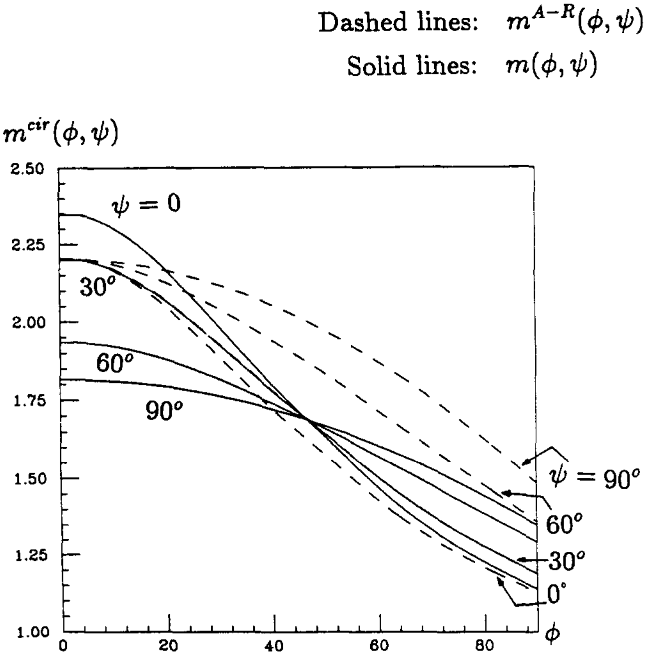

FIG. 7. Proposed approximate values of energy correction factor m as function of the inclination angle ϕ for $\psi = 0, 30°, 60°, 90°$. The solid curves represent the approximate m values calculated from Eq. (32) in the text. The dashed curves are the approximation m^{A-R} by ANDERSON and RICE (1987).

ϕ passes 45°, the difference $m_t - m_z$ reverses sign. These features are not shown in the m results by ANDERSON and RICE (1987) because, based on their exact 2-D results, they omitted any ψ dependence of m for coplanar shear loops.

Approximate relations rooted in (30) and (26) can also be written for rectangular loops. In fact, when the approximate results are compared to the directly calculated m_t and m_z for aspect ratio $r/w = 1, 10$, they show a surprisingly good match with the numerically calculated results, as shown in Fig. 4. In the two limiting cases that $r/w \rightarrow 0, \infty$, obviously Eq. (30) becomes exact. We conclude that Eq. (30) provides an alternative way of calculating the m factor within 5% error for general symmetric dislocation loops emerging from a crack tip.

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APPENDIX A

STRESSES OF A GENERAL DISLOCATION COPLANAR WITH A CRACK

From the analysis of part I, we observe that the weight function simplifies for the special case of dislocations lying within the crack plane $y = 0$. Here we start with Eqs (13, 16, 17) and simplify the formulae for the shear stresses for an arbitrary emergent crack-tip loop coplanar with the crack. In this case $\phi = 0$ and x, y coincide with t, n. Equation (13) becomes

$$\sigma_{ys} = \sigma^0_{ys} + \sigma'_{ys} \tag{A1}$$

($s = x$, z), where σ^0_{ys} are the components of the original stress field that would be induced by the loop in an uncracked body, as presented in (17) in the text. The stress component σ'_{ys} represents the additional stress in the presence of the crack, which will depend on both Δu_x and Δu_z so that

$$\sigma'_{ys} = (\sigma'_{ys})^x + (\sigma'_{ys})^z, \tag{A2}$$

where $(\sigma'_{ys})^x$ is the stress generated on the plane $y = 0$ by the x component of the displacement discontinuity while the second term, similarly, is due to the z component of the displacement discontinuity.

The weight function expressions U^α_{mn} were presented in part I. On the crack plane $y = 0$, they can be expressed as

$$U^1_{yy} = \frac{\sqrt{2/\pi^3}}{8(1-\nu)\eta^{1/2}}\left(\frac{1}{R^2}\right),$$

$$U^2_{yx} = \frac{\sqrt{2/\pi^3}}{8(1-\nu)\eta^{1/2}}\,U\left(\frac{1}{R^2}\right), \quad U^2_{yz} = \frac{\sqrt{2/\pi^3}}{8(1-\nu)\eta^{1/2}}\,V\left(\frac{1}{R^2}\right),$$

$$U^3_{yx} = \frac{\sqrt{2/\pi^3}}{8\eta^{1/2}}\,V\left(\frac{1}{R^2}\right), \qquad U^3_{yz} = \frac{\sqrt{2/\pi^3}}{8\eta^{1/2}}\,W\left(\frac{1}{R^2}\right), \tag{A3}$$

where $R^2 = (x-a)^2 + (z-z')^2$, $\eta = x-a$ and we have denoted differential operators by

$$U\left(\frac{1}{R^2}\right)=\left(1+\frac{2\nu}{2-\nu}\frac{(x-a)^2-(z-z')^2}{R^2}\right)\frac{1}{R^2}=\left(1+\frac{2\nu}{2-\nu}\frac{\partial}{\partial z}(z-z')\right)\frac{1}{R^2},$$

$$V\left(\frac{1}{R^2}\right)=\frac{2\nu}{2-\nu}\frac{2(x-a)(z-z')}{R^4}=-\frac{2\nu}{2-\nu}\eta\frac{\partial}{\partial z}\frac{1}{R^2},$$

$$W\left(\frac{1}{R^2}\right)=\left(1-\frac{2\nu}{2-\nu}\frac{(x-a)^2-(z-z')^2}{R^2}\right)\frac{1}{R^2}=\left(1-\frac{2\nu}{2-\nu}\frac{\partial}{\partial z}(z-z')\right)\frac{1}{R^2}. \quad \text{(A4)}$$

The following integrals will be encountered when substituting (A3) into (16) of the text to calculate the stresses:

$$\int_{-\infty}^{\infty}\frac{dz'}{R^2\tilde{R}^2}=\pi\left(\frac{1}{\tilde{\eta}}+\frac{1}{\eta}\right)/S^2,\quad \int_{-\infty}^{\infty}\frac{(z-z')(\tilde{z}-z')}{R^2\tilde{R}^2}dz'=\pi(\tilde{\eta}+\eta)/S^2, \quad \text{(A5)}$$

where $S^2=(z-\tilde{z})^2+(\eta+\tilde{\eta})^2$ and $\tilde{R}^2$, $\tilde{\eta}$ correspond to replacing x, z by $\tilde{x}$, $\tilde{z}$ in R^2, η. Denoting $[\![\Delta u_s]\!]=\Delta u_s(\tilde{x},\tilde{z})-\Delta u_s(0,z)$ and substituting (A3) into (16) leads to

$$(\sigma'_{yx})^x=\frac{\mu}{8\pi^3(1-\nu)}\int_{-\infty}^{0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{[\![\Delta u_x]\!]}{\eta^{1/2}\tilde{\eta}^{1/2}}$$
$$\times\left\{U\left(\frac{1}{R^2}\right)\tilde{U}\left(\frac{1}{\tilde{R}^2}\right)+(1-\nu)V\left(\frac{1}{R^2}\right)\tilde{V}\left(\frac{1}{\tilde{R}^2}\right)\right\}d\tilde{x}\,d\tilde{z}\,dz'\,da. \quad \text{(A6)}$$

The following manipulation is helpful

$$\int_{-\infty}^{\infty}\left[\frac{\partial}{\partial z}(z-z')+\frac{\partial}{\partial\tilde{z}}(\tilde{z}-z')\right]\frac{1}{\tilde{R}^2R^2}dz'=\frac{\partial}{\partial\tilde{z}}(\tilde{z}-z)\int_{-\infty}^{\infty}\frac{1}{\tilde{R}^2R^2}dz'. \quad \text{(A7)}$$

Using (A4, A5, A7) one may show that

$$\int_{-\infty}^{\infty}\left\{U\left(\frac{1}{R^2}\right)\tilde{U}\left(\frac{1}{\tilde{R}^2}\right)+(1-\nu)V\left(\frac{1}{R^2}\right)\tilde{V}\left(\frac{1}{\tilde{R}^2}\right)\right\}dz'$$
$$=\pi\left\{M_1\left[\left(\frac{1}{\eta}+\frac{1}{\tilde{\eta}}\right)\Big/S^2\right]+M_2[(\eta+\tilde{\eta})/S^2]\right\}, \quad \text{(A8)}$$

where M_1 and M_2 are operators:

$$M_1=1+\frac{2\nu}{2-\nu}\frac{\partial}{\partial\tilde{z}}(\tilde{z}-z),\quad M_2=\frac{4\nu^2}{2-\nu}\frac{\partial^2}{\partial\tilde{z}\,\partial z}. \quad \text{(A9)}$$

The above integration on z' appears in (A6). Further noticing that neither M_1 nor M_2 has dependence on the crack tip position variable a, we can carry out the integral in (A6) on a. Following RICE (1985) in handling similar integrals, we use the transformation $t=2(\eta\tilde{\eta})^{1/2}$ and derive

$$\int_{-\infty}^{0}\frac{\eta+\tilde{\eta}}{\eta^{3/2}\tilde{\eta}^{3/2}S^2}da=4F,\quad \int_{-\infty}^{0}\frac{\eta+\tilde{\eta}}{\eta^{1/2}\tilde{\eta}^{1/2}S^2}da=G, \quad \text{(A10)}$$

where for conciseness we have introduced

$$F(x,\tilde{x},z-\tilde{z}) = \frac{1}{D^3}\left(\frac{D}{2(x\tilde{x})^{1/2}} - \arctan\frac{D}{2(x\tilde{x})^{1/2}}\right), \quad G(x,\tilde{x},z-\tilde{z}) = \frac{1}{D}\arctan\frac{D}{2(x\tilde{x})^{1/2}}, \tag{A11}$$

where $D^2 = (x-\tilde{x})^2+(z-\tilde{z})^2$. Finally we have

$$(\sigma'_{yx})^x = \frac{\mu}{2\pi^2(1-\nu)}\int_{-\infty}^{\infty}\int_0^{\infty}\left\{\left(1+\frac{2\nu}{2-\nu}\frac{\partial}{\partial z}(z-\tilde{z})\right)F+\frac{\nu^2}{2-\nu}\frac{\partial^2 G}{\partial\tilde{z}\,\partial z}\right\}[\![\Delta u_x]\!]\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{z}. \tag{A12}$$

Equation (A12) is much simplified compared to the original stress expression in (16) of the text. Following steps similar to those leading to (A12) we can also derive the part of σ'_{xy} generated by $[\![\Delta u_z]\!]$. Also repeating the process for σ'_{yz}, finally we derive

$$(\sigma'_{yx})^z = -\frac{2\nu}{2-\nu}\frac{\mu}{8\pi^2(1-\nu)}\frac{\partial}{\partial z}\left\{\int_{-\infty}^{\infty}\int_0^{\infty}\left[\left(4(x-\tilde{x})F+2\nu\frac{\partial G}{\partial x}\right)-\nu L\right][\![\Delta u_z]\!]\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{z}\right\}, \tag{A13}$$

$$(\sigma'_{yz})^x = -\frac{2\nu}{2-\nu}\frac{\mu}{8\pi^2(1-\nu)}\frac{\partial}{\partial z}\left\{\int_{-\infty}^{\infty}\int_0^{\infty}\left[\left(4(1-\nu)(x-\tilde{x})F-2\nu\frac{\partial G}{\partial x}\right)+\nu L\right][\![\Delta u_x]\!]\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{z}\right\}, \tag{A14}$$

$$(\sigma'_{yz})^z = \frac{1}{2\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{0}\left\{\left(1-\frac{2\nu}{2-\nu}\frac{\partial}{\partial z}(z-\tilde{z})\right)F+\frac{\nu^2}{(2-\nu)(1-\nu)}\frac{\partial^2 G}{\partial\tilde{z}\,\partial z}\right\}[\![\Delta u_z]\!]\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{z}, \tag{A15}$$

where the new function L is defined as

$$L = \int_{-\infty}^{0}\frac{(\eta+\tilde{\eta})}{\eta^{1/2}\tilde{\eta}^{3/2}S^2}\,\mathrm{d}a = \frac{2}{D^2}\left[\left(\frac{x}{\tilde{x}}\right)^{1/2}-1-(x-\tilde{x})G\right] + \frac{z-\tilde{z}}{D^3}\ln\left(\frac{[D(x+\tilde{x})/2\sqrt{x\tilde{x}}-(z-\tilde{z})]\,[D+(z-\tilde{z})]}{[D(x+\tilde{x})/2\sqrt{x\tilde{x}}+(z-\tilde{z})]\,[D-(z-\tilde{z})]}\right). \tag{A16}$$

The original stress field σ^0_{ns} can be directly written from (17) for the present case $\phi = 0$. We give the following final expression for the total stress field σ_{yx} of (A1) at $(x, 0, z)$ due to the respective x and z components of the shear dislocation:

$$\begin{aligned}(\sigma_{yx})^x = &-\frac{\mu x}{4\pi(1-\nu)}\int_{-\infty}^{\infty}\frac{\Delta u_x(x,\tilde{z})-\Delta u_x(0^+,z)}{[x^2+(z-\tilde{z})^2]^{3/2}}\,\mathrm{d}\tilde{z}+\frac{\mu}{2\pi^2(1-\nu)}\\ &\times\int_{-\infty}^{\infty}\int_0^{\infty}\left\{\left[1+\frac{2\nu}{2-\nu}\frac{\partial}{\partial\tilde{z}}(\tilde{z}-z)\right]F\right.\\ &\left.+\frac{\nu^2}{2-\nu}\frac{\partial^2 G}{\partial\tilde{z}\,\partial z}\right\}[\Delta u_x(\tilde{x},\tilde{z})-\Delta u_x(0^+,z)]\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{z}\\ &+\frac{\mu}{4\pi(1-\nu)}\int_{-\infty}^{\infty}\int_{0^+}^{\infty}\left(\frac{\tilde{x}-x}{D^3}\frac{\partial}{\partial\tilde{x}}+(1-\nu)\frac{\tilde{z}-z}{D^3}\frac{\partial}{\partial\tilde{z}}\right)\Delta u_x(\tilde{x},\tilde{z})\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{z}\end{aligned} \tag{A17}$$

and

$$\begin{aligned}(\sigma_{yx})^z = \frac{\mu\nu}{4\pi(1-\nu)}&\left\{\int_{-\infty}^{\infty}\int_{0^+}^{\infty}\frac{\tilde{x}-x}{D^3}\frac{\partial\Delta u_z(\tilde{x},\tilde{z})}{\partial\tilde{z}}\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{z}-\frac{1}{\pi(2-\nu)}\frac{\partial}{\partial z}\right.\\ &\left.\times\left[\int_{-\infty}^{\infty}\int_0^{\infty}\left(\left(4(x-\tilde{x})F+2\nu\frac{\partial G}{\partial x}\right)-\nu L\right)\Delta u_z(\tilde{x},\tilde{z})\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{z}\right]\right\}.\end{aligned} \tag{A18}$$

The total stress field σ_{yz} can be written out analogously, and we leave this to the interested reader. By definition

$$K_\beta(z) = \lim_{x\to 0^+} \sqrt{2\pi x}\,\sigma_{y\beta}(x, 0, z).$$

Equation (4) of the text for stress intensity factors is verified by observing that the $1/\sqrt{x}$ singular terms in $\sigma_{yx}(x, z)$ and $\sigma_{yz}(x, z)$ are those containing $D/2(x\tilde{x})^{1/2}$ and carrying out the limit. Equations (A17) and (A18) are valid for calculations of the stress field for general Somigliana type dislocations, i.e. arbitrary displacement discontinuities coplanar with the crack.

APPENDIX B

SOME INTEGRAL CALCULATIONS

In Eqs (27, 28) of the text, the double integrals over the complementary area of the dislocation loop on the crack plane give rise to difficulties in formulating an efficient numerical integration scheme. A large number of Gaussian integration points is needed, which results in long computing time. It is then desired to find an efficient way to calculate these complementary area integrals. Without loss of generality assume the maximum height of the loop perpendicular to the crack front is unity. Let us write

$$\int_{A_c} (\cdots)\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{z} = \int_{-\infty}^{+\infty}\int_{1}^{+\infty} (\cdots)\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{z} + \int_{A_r} (\cdots)\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{z}, \tag{B1}$$

where A_c denotes the complementary area (Fig. 1) and A_r denotes an infinite strip area with unit width ($0 < x < 1$, $-\infty < z < \infty$) excluding the crack tip loop area A. We have split the complementary integral into two parts. The first part denotes an integral I_1 over a half-plane emanating from the line $x = 1$. The second part I_2 is an integral over the rest of the area A_r. The second integral is usually easier for numerical calculation than the first one in that it has a finite dimension in the x direction. However, the first part I_1 can actually be worked out analytically. Both complementary integrals shown in (27, 28) can be split in this way. Let us note

$$\int_{-\infty}^{+\infty}\int_{1}^{+\infty} \frac{\partial}{\partial \tilde{z}}(z-\tilde{z})F\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{z} = 0, \quad \int_{-\infty}^{+\infty}\int_{1}^{+\infty} \frac{\partial^2 G}{\partial \tilde{z}\,\partial z}\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{z} = 0. \tag{B2}$$

Hence the I_1 integrals arising in (27, 28) reduce to calculating

$$I_1 = \int_{-\infty}^{+\infty}\int_{1}^{+\infty} F\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{z} = \frac{\pi}{\sqrt{x}(1+\sqrt{x})}. \tag{B3}$$

With the above result the integration over the complementary area is simplified to one over a complementary strip area A_r with finite width. We find that this both speeds up the numerical calculation and increases the precision.

Calculation of the m *factors for a coplanar rectangular loop*

As a demonstration we consider a rectangular crack-tip loop within the crack plane with dimension $2w$ by r ($=1$) as shown in Fig. 2. Defining $\hat{\Gamma}_{yj} = \Gamma_{yj} \times [4\pi(1-\nu)/\mu]$ for conciseness, (27) now specializes to

$$\hat{\Gamma}_{yx} = \hat{\Gamma}^L_{yx} - \frac{2}{\pi}\int_{A_c} \left[\left(1 + \frac{2\nu}{2-\nu}\frac{\partial}{\partial z}(z-\tilde{z})\right)F + \frac{\nu^2}{2-\nu}\frac{\partial^2 G}{\partial \tilde{z}\,\partial z}\right]\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{z}, \tag{B4}$$

where $\hat{\Gamma}^L_{yx}$ represents the remaining part excluding the complementary area integral, and it leads to

$$J_x = \int_A \hat{\Gamma}^L_{yx}(x,0,z)\,\mathrm{d}x\,\mathrm{d}z$$

$$= 2w\left\{\int_0^1\left[\frac{2-\sqrt{(x/w)^2+4}}{x}+\frac{\sqrt{((x+1)/\omega)^2+4}}{x+1}\right]\mathrm{d}x\right.$$

$$\left.+\int_{-1}^1\frac{1-v}{z+1}\left[\sqrt{4/w^2+(z+1)^2}-2\sqrt{1/w^2+(z+1)^2}+(z+1)\right]\mathrm{d}z\right\}. \quad \text{(B5)}$$

A similarly defined quantity J_z associated with $\hat{\Gamma}^L_{yz}$ is

$$J_z = \int_A \hat{\Gamma}^L_{yz}(x,0,z)\,\mathrm{d}x\,\mathrm{d}z$$

$$= 2w\left\{\int_0^1\left[\frac{2-\sqrt{(x/w)^2+4}}{x}+\frac{(1-v)\sqrt{((x+1)/w)^2+4}}{x+1}+\frac{2v}{w}\right]\mathrm{d}x\right.$$

$$\left.+\int_{-1}^1\frac{1}{z+1}\left[\sqrt{4/w^2+(z+1)^2}-2\sqrt{1/w^2+(z+1)^2}+(z+1)\right]\mathrm{d}z\right\}. \quad \text{(B6)}$$

It is readily shown that for a rectangular loop

$$\int_A I_1\,\mathrm{d}x\,\mathrm{d}z = 4w\pi\ln 2 \quad \text{(B7)}$$

so that we can calculate the m factors according to

$$4(1-v-w)\ln m_t = 8w\ln 2 - J_x + \frac{2}{\pi}\int_A I_2^x\,\mathrm{d}x\,\mathrm{d}z,$$

$$4[1+w(1-v)]\ln m_z = 8(1-v)w\ln 2 - J_z + \frac{2}{\pi}\int_A I_2^z\,\mathrm{d}x\,\mathrm{d}z, \quad \text{(B8)}$$

where I_2^x, I_2^z are the corresponding I_2 integrals over the complementary strip area A_r. In the limiting case, $w \to \infty$, the rectangular loop approaches a 2-D straight dislocation; in that case

$$J_x = 4w\ln 2,\quad J_z = 4(1-v)w\ln 2,\quad I_2^x = I_2^z = 0. \quad \text{(B9)}$$

Therefore by Eqs (B8, B9) we see that $m_t^{2D}\ (\phi = 0) = m_z^{2D}\ (\phi = 0) = 2$ as derived by ANDERSON and RICE (1987).

Similar simplifications can be carried out for the case of a semicircular shear loop, although it is not as important as for rectangular loops, for which many calculations for m values at chosen aspect ratios r/w needed to be done. Hence we do not discuss it here. The prismatic opening dislocation loop involves the same kind of complementary integrals (see ANDERSON and RICE, 1987). In fact the prismatic m value corresponds to the shear values when $v = 0$. Hence by setting $v = 0$ in our calculations we can obtain the m values for prismatic loops too.