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Weight Function Theory for Three-Dimensional Elastic Crack Analysis

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ABSTRACT: Recent developments in elastic crack analysis are discussed based on extensions and applications of weight function theory in the three-dimensional regime. It is shown that the weight function, which gives the stress intensity factor distribution along the crack front for arbitrary distributions of applied force, has a complementary interpretation: It characterizes the variation in displacement field throughout the body associated, to first order, with a variation in crack-front position. These properties, together with the fact that weight functions have now been determined for certain three-dimensional crack geometries, have allowed some new types of investigation. They include study of the three-dimensional elastic interactions between cracks and nearby or emergent dislocation loops, as are important in some approaches to understanding brittle versus ductile response of crystals, and also the interactions between cracks and inclusions which are of interest for transformation toughening. The new developments further allow determination of stress-intensity factors and crack-face displacements for cracks whose fronts are slightly perturbed from some reference geometry (for example, from a straight or circular shape), and those solutions allow study of crack trapping in growth through a medium of locally nonuniform fracture toughness. Finally, the configurational stability of cracking processes can be addressed: For example, when will an initially circular crack, under axisymmetric loading, remain circular during growth?

KEY WORDS: fracture mechanics, elasticity theory, weight functions, stress intensity factors, dislocation emission, crack-defect interactions, configurational stability, crack trapping

Bueckner introduced the concept of "weight functions" for two-dimensional elastic crack analysis in 1970 [*1*]. His weight functions satisfy the equations of linear elastic displacement fields, but they equilibrate zero body and surface forces and have a stronger singularity at the crack tip than would be admissible for an actual displacement field. The worklike product of an arbitrary set of applied forces with the weight function gives the crack-tip stress intensity factor induced by those forces. Bueckner's contribution led to what is now a vast literature on two-dimensional elastic crack analysis. One of the earliest works of that literature was a 1972 paper by the writer [*2*] which showed that weight functions could be determined by differentiating known elastic displacement field solutions with respect to crack length. It was also shown [*2*] that knowledge of a two-dimensional elastic crack solution, as a function of crack length, for any one loading enables one to determine directly the effect of the crack on the elastic solution for the same body under any other loading system.

The subject here is three-dimensional weight-function theory. Foundations of the three-dimensional theory were given independently by the writer, in the Appendix of Ref *2*, based

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on displacement field variations associated to first order with an arbitrary variation in position of the crack front, and in a review by Bueckner [3], based on three-dimensional solutions of the elastic displacement field equations that equilibrate null forces and that have arbitrary distributions of strength of a normally inadmissible singularity along the crack front. Bueckner refers to such fields as "fundamental fields."

Since 1985, there has been a surge of interest in the three-dimensional theory. That recent work, to be discussed, has allowed new types of three-dimensional crack investigations, including crack tip interactions with dislocations and other defects, stress analysis for perturbed crack shapes, crack-front trapping in growth through heterogeneous solids, and the configurational stability of crack shape during growth. However, three-dimensional weight function theory had a rather quiet first 13 years or so. Notable developments in that period include Besuner's [4] 1974 observation that the formulation based on crack-front variation [2] could be applied to determine certain weighted averages of K_1 (tensile mode stress intensity factor) along the front of an arbitrarily loaded elliptical crack, by differentiating a known solution with respect to parameters describing the ellipse (Bueckner [3] had earlier used the same approach to construct some examples of his fundamental singular fields). Also, Parks and Kamenetzky [5] outlined a three-dimensional finite-element procedure for calculating numerically the variation of elastic displacement fields with crack-front position that are necessary to determine the three-dimensional weight function by the procedure of Ref 2. In a 1977 paper [6], Bueckner determined fundamental fields for tensile half-plane and circular cracks for distributions of singularity strength that vary trigometrically with distance along the crack front. He also used that approach to rederive known results for the stress intensity factor distribution induced by a pair of wedge-opening point forces acting on the crack surfaces.

In 1985, the writer [7] pointed out the relation between three-dimensional weight function concepts and the determination of tensile-mode stress intensity factors along crack fronts whose locations are perturbed slightly from some simple reference geometry, and used such results to address the configurational stability of crack front shape during quasi-static crack growth. He also solved directly for the Mode 1 weight function for a half-plane crack in a full space, by determining the three-dimensional elastic field variations to first order for an arbitrary variation of crack front location, and generalized the three-dimensional theory to arbitrary mixed-mode conditions in the manner briefly reviewed in the next section. A related paper [8] pointed out how to use weight function concepts to describe the three-dimensional elastic interaction between crack tips and dislocation loops or zones of shape transformation, and Gao and Rice [9] developed the perturbation approach of Ref 7 to determine also the shear-mode stress intensity factors along the front of a generally loaded half-plane crack when that front is slightly perturbed from a straight line.

In a significant recent paper, Bueckner [10] completed the determination of weight functions for all three modes for the half-plane crack and further determined them for a "penny-shaped" circular crack. Also, Gao and Rice [11,12] applied the crack shape perturbation method of Ref 7 to determine tensile-mode stress intensity factors along crack fronts whose shapes are moderately perturbed from circles, dealing with the respective cases of near-circular cracks in full spaces and near-circular connections (that is, external cracks) bonding elastic half-spaces. They also note [11], and compare their methods to, a much earlier but apparently little known paper by Panasyuk [13], which directly derived a first-order perturbation solution for a near-circular crack (see also the 1981 review by Panasyuk et al. [14]). Gao and Rice [11,12] used their results to determine conditions for configurational stability in the growth of cracks with initially circular fronts under axisymmetric loading, as will be discussed subsequently. By using shear-mode results of Bueckner [10], Gao [15] solved for shear-mode intensity factors along a slightly noncircular shear crack and used the results to determine, to first-order accuracy, the shape of a shear loaded crack having constant energy release rate along its front. Rice [*16*] applied the crack front perturbation analysis to address some elementary problems in crack front trapping by tough obstacles in growth through heterogeneous microstructures. Also, Anderson and Rice [*17*] applied the methods of Ref *8* to evaluate the three-dimensional stress field and energy of a prismatic dislocation loop emerging from a half-plane crack tip, and studies of this type were recently extended by Gao [*18*] and Gao and Rice [*19*] to general shear dislocation loops. Sham [*20*] recently gave a new finite-element procedure for three-dimensional weight-function determination in bounded solids, as an alternative to the virtual crack extension method of Ref *5*.

Since weight functions are interpretable as intensity factors induced by arbitrarily located point forces, they can sometimes also be extracted from the existing literature on three-dimensional elastic crack analysis. That is too extensive to summarize here, but the reader is referred to the review by Panasyuk et al. [*14*] and also to the recent work of Fabrikant [*21,22*], which gives general solutions for arbitrarily loaded circular cracks.

Theory of Three-Dimensional Weight Functions

Background and Notation

For background, Fig. 1*a* shows a local coordinate system along a three-dimensional elastic crack front. Axes of the local system are labelled to agree with mode number designations for stress intensity factors $K_\alpha(\alpha = 1,2,3)$. Thus, at small distance ρ' ahead of the tip, on the prolongation of the crack plane, the stress components σ_{11}, σ_{12}, σ_{13} have the asymptotic

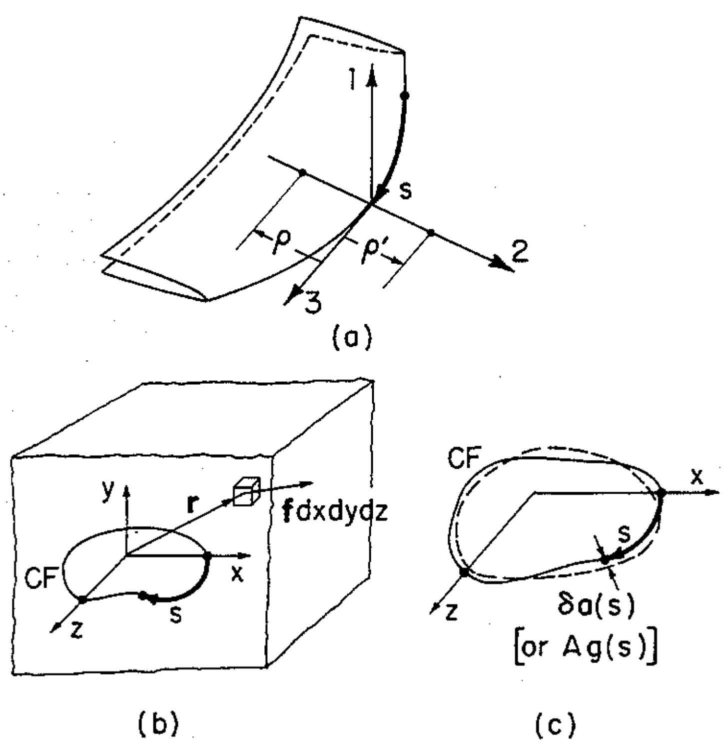

FIG. 1—(a) *Local coordinates along front of three-dimensional crack; numbering of axes corresponds to stress intensity modes.* (b) *Loaded solid with planar crack on* y = *0; CF denotes crack front, arc length* s *parameterizes locations along CF, vector* **r** *denotes position in body.* (c) *Advance of crack front normal to itself by* δa(s); *advance sometimes labelled* Ag(s) *where* A *is amplitude and* g(s) *a fixed function.*

form

$$\sigma_{1\alpha} \sim K_\alpha / \sqrt{2\pi\rho'} \tag{1}$$

Similarly, at small distance ρ behind the tip (that is, along the -2 axis) one has the asymptotic form of displacement discontinuities Δu_1, Δu_2, Δu_3 between the upper and lower crack surfaces of

$$\Delta u_\alpha \sim 8\Lambda_{\alpha\beta} K_\beta \sqrt{\rho/2\pi} \tag{2}$$

Here there is summation on repeated Greek indices over 1, 2, 3, associated with the local coordinate system at the point of interest along the crack front. (Repeated Latin indices, as appear later, are to be summed over directions x, y, z of a fixed-coordinate system as in Fig. 1*b*.) The matrix $\Lambda_{\alpha\beta}$ is given by

$$[\Lambda_{\alpha\beta}] = \frac{1}{2\mu}\begin{bmatrix} 1-\nu & 0 & 0 \\ 0 & 1-\nu & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3}$$

for an isotropic material (μ = shear modulus, ν = Poisson ratio); for the general anisotropic material, $\Lambda_{\alpha\beta}$ remains symmetric but not necessarily diagonal, and is proportional to the inverse of a prelogarithmic energy factor matrix arising in the expression for self energy of a straight dislocation line with same direction as the local crack-front tangent [*23,24*]. Also, the energy G released per unit area of crack advance at the crack-front location considered is

$$G = \Lambda_{\alpha\beta} K_\alpha K_\beta \tag{4}$$

Figure 1*b* shows an elastic solid with a planar crack on $y = 0$. The crack front is denoted as CF, and arc length s parameterizes position along CF. The cracked body is loaded by some distribution of force vector $\mathbf{f} = \mathbf{f}(\mathbf{r})$ per unit volume. Here $\mathbf{r}$ is the position vector relative to the fixed x, y, z system, that is, $\mathbf{r} = (x,y,z)$, and $\mathbf{f}$ has cartesian components denoted by f_j where $j = x, y, z$. Also, in cases involving loading by a distribution of imposed stresses, or tractions, on the surface of the body, it will be convenient to regard those surface tractions as a singular layer of body force. Thus, the work-like product of the entire set of applied loadings with any vector field $\mathbf{u} = \mathbf{u}(\mathbf{r})$ will generally be written as

$$\int_{\text{Body}} \mathbf{f}(\mathbf{r}) \cdot \mathbf{u}(\mathbf{r})\, dxdydz \text{ or } \int_{\text{Body}} f_j(\mathbf{r}) u_j(\mathbf{r})\, dxdydz$$

where the integral of $\mathbf{f} \cdot \mathbf{u}$ over the "Body" is to be interpreted as an integral of $\mathbf{f} \cdot \mathbf{u}$ over the interior of the body (with surface layer excluded) plus an integral of $\mathbf{T} \cdot \mathbf{u}$ over the surface of the body, including crack surfaces, where $\mathbf{T}$ is the vector of imposed surface tractions.

In addition, it will be assumed in general that the body considered is restrained against displacement over some part of its surface so that it can sustain arbitrary force distributions. This requirement can be disregarded if, in the intended applications, the actual imposed loadings are self-equilibrating.

Weight Functions and Their Properties

The weight functions $\mathbf{h}_1$, $\mathbf{h}_2$, $\mathbf{h}_3$ are three vector functions of position $\mathbf{r}$ in the body and locations s along CF: $\mathbf{h}_\alpha = \mathbf{h}_\alpha(\mathbf{r};s)$. One such vector function is associated with each crack tip stressing mode at location s, the mode being indicated by the value of subscript α on $\mathbf{h}_\alpha$. The vector functions $\mathbf{h}_\alpha$ have cartesian components $h_{\alpha j}(\alpha = 1,2,3;\ j = x,y,z)$, so that altogether there are nine scalar functions involved.

The weight functions have two properties, now outlined, as introduced in the developments via Refs *2* and *7*. Either of the first or the second property may be taken to define the weight function, and then the other property may be derived from that one by basic elasticity and fracture mechanics principles.

The first property is that the stress intensity factors induced at location s along the crack front, by arbitrary loading of the body (Fig. 1*b*) are given by

$$K_\alpha(s) = \int_{\text{Body}} \mathbf{h}_\alpha(\mathbf{r};s) \cdot \mathbf{f}(\mathbf{r})\ dxdydz \tag{5}$$

Thus $h_{\alpha j}(\mathbf{r};s)$ gives the mode α intensity factor induced at location s along CF by a unit point force in the j direction at $\mathbf{r}$.

The second property is that if, under fixed applied loadings, the crack front is advanced normal to itself, in the plane $y = 0$, by an amount $\delta a = \delta a(s)$, variable along CF as in Fig. 1*c*, the associated change in the displacement field $\mathbf{u}(\mathbf{r})$ is

$$\delta\mathbf{u}(\mathbf{r}) = 2\int_{\text{CF}} \Lambda_{\alpha\beta}(s)\mathbf{h}_\alpha(\mathbf{r};s)K_\beta(s)\delta a(s)\ ds \tag{6}$$

to first order in $\delta a(s)$. Thus $h_{\alpha j}(\mathbf{r},s)$, when weighted with $2\Lambda_{\alpha\beta}K_\beta$, gives the increase of displacement component u_j at $\mathbf{r}$ per unit enlargement of crack area near s [note that $\delta a(s)$ ds is an element of area].

To state the second property, Eq 6, more precisely, as well as to aid certain derivations, the following alternative is useful: Let $g(s)$ be an arbitrary but, once chosen, fixed dimensionless function of position along CF. Then a family of crack-front locations, with parameter A, may be defined by advancing the original crack front, CF, normal to itself by amount $A\ g(s)$. That is, the increment labelled $\delta a(s)$ in Fig. 1*c* is now understood as $A\ g(s)$. The loading is regarded as fixed so that the displacement field associated with this family of crack-front locations may be written as $\mathbf{u} = \mathbf{u}(\mathbf{r},A)$. Then the statement that Eq 6 holds to first order in $\delta a(s)$ is equivalent to

$$\left.\frac{\partial\mathbf{u}(\mathbf{r},A)}{\partial A}\right|_{A=0} = 2\int_{\text{CF}} \Lambda_{\alpha\beta}(s)\mathbf{h}_\alpha(\mathbf{r};s)K_\beta(s)g(s)\ ds \tag{7}$$

Since the growth increment is written as $A\ g(s)$, this equation corresponds, of course, to writing $\delta a(s) = \delta A\ g(s)$ in Eq 6, which is then required to hold to first order in δA.

Example: Mode 1 Weight Function for the Half-Plane Crack

The writer solved [7] for the Mode 1 weight function for a half-plane crack, denoted here as $\mathbf{h}_1(\mathbf{r};s)$. As shown in Fig. 2, s now denotes the z-coordinate of the location of interest

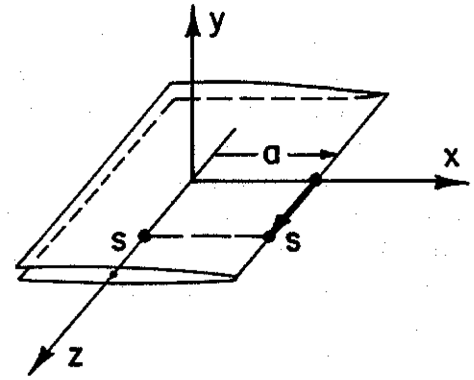

FIG. 2—*Half-plane crack with straight front in infinite solid.*

along the crack front, and the front is at $x = a$ on $y = 0$. The derivation was accomplished using the second property, Eqs 6 or 7, to define $\mathbf{h}_1$. The relevant elasticity equations were solved directly for $\partial \mathbf{u}(\mathbf{r}, A)/\partial A$, at $A = 0$, for arbitrary growth functions $g(s)$ and arbitrary Mode 1 loadings [hence arbitrary $K_1(s)$], and the solution was put in the form of Eq 7 to identify $\mathbf{h}_1$. The results are

$$h_{1y} = H - [1/2(1 - \nu)] y \partial H/\partial y$$

$$\{h_{1x}, h_{1z}\} = -[1/2(1 - \nu)]\{\partial/\partial x, \partial/\partial z\} \left[yH - (1 - 2\nu) \int_y^\infty H\, dy \right] \tag{8}$$

where

$$H = \frac{\mathrm{Im}[(x - a + iy)^{1/2}]}{\pi\sqrt{2\pi}[(x - a)^2 + y^2 + (z - s)^2]} \tag{9}$$

and $i = \sqrt{-1}$, Im means "imaginary part of," and the branch cut for the ½ power term is along the crack.

The solution for $\mathbf{h}_1$ could also have been developed by using Fourier analysis together with some fundamental fields given by Bueckner [*6*], having a cos ωz variation along the crack front, to construct what he has called a fundamental field with a point of concentration. Essentially, his fundamental fields would have to be superposed over all ω by weighting each with the Fourier transform at frequency ω of a Dirac function, centered at $z = s$, and integrating over all ω to obtain $\mathbf{h}_1(\mathbf{r};s)$.

As remarked, Bueckner later derived [*10*] the full set of three weight functions for the half-plane crack and for the circular crack. From his work, the function defined by the integral in Eq 8, with $a = s = 0$, is

$$\frac{1}{\pi\sqrt{2\pi}} \int_y^\infty \frac{\mathrm{Im}(x + iy)^{1/2}\, dy}{x^2 + y^2 + z^2} = \mathrm{Re}\left\{ \frac{1}{2\pi\sqrt{\pi}\zeta} \ln \left[\frac{q + \zeta}{q - \zeta} \right] \right\} \tag{10}$$

where $\zeta = (x + iz)^{1/2}$ and $q = \mathrm{Re}[\sqrt{2}(x + iy)^{1/2}]$, and all the weight functions for the half-plane crack may be expressed in terms of linear differential operation on the complex function whose real part appears on the right in Eq 10.

Derivation of Second Property from First

Assume that the $\mathbf{h}_\alpha(\mathbf{r};s)$ are defined primitively by their first property, Eq 5, that is, $h_{\alpha j}(\mathbf{r};s)$ is defined as the mode α intensity factor at location s on CF due to a unit point force in the j direction at position $\mathbf{r}$. The writer's derivation [2,7] of the second property is outlined, and modestly recast, here. Let U be the strain energy of the cracked solid in Fig. 1*b* and let

$$V = -\int_{\text{Body}} \mathbf{f}(\mathbf{r}) \cdot \mathbf{u}(\mathbf{r})\, dxdydz \tag{11}$$

be the potential energy of the applied forces (regarded as fixed), which induce intensity factors $K_\alpha(s)$ along the crack front.

Consider a family of crack shapes defined by advancing CF normal to itself by $Ag(s)$, where, again, $g(s)$ is arbitrary but fixed once chosen, and the advance process corresponds to Fig. 1*c* with the label $\delta a(s)$ replaced by $A\, g(s)$. Observe that the area elements swept out in incremental change δA in A can be written as

$$\delta(\text{area}) = p(A,s)\, \delta A\, g(s)\, ds \tag{12}$$

where $p(A,s)$ is a function dependent on the curvature of CF at s but need not be written out here since we will, in the end, only need its value at $A = 0$, at which $p(0,s) = 1$.

Also, suppose that in addition to the given load system, an arbitrary point force $\mathbf{F}$ is applied to the body at $\mathbf{r}$, where the displacement is $\mathbf{u}(\mathbf{r})$ or, more fully, $\mathbf{u}(\mathbf{r}, A,\mathbf{F})$. (Formally, $\mathbf{u}$ is unbounded at $\mathbf{r}$ when $\mathbf{F}$ differs from zero, but we shall shortly be setting $\mathbf{F} = 0$. To keep things finite, we may distribute $\mathbf{F}$ uniformly over a small sphere [7] of radius ϵ about $\mathbf{r}$, interpret $\mathbf{u}(\mathbf{r})$ as the average over that same sphere, and later let $\epsilon \rightarrow 0$ after setting $\mathbf{F} = 0$.) Thus the total stress intensity factors $\hat{K}_\alpha = \hat{K}_\alpha(s, A,\mathbf{F})$ have the form, when $A = 0$,

$$\hat{K}_\alpha(s,0,\mathbf{F}) = K_\alpha(s) + \mathbf{h}_\alpha(\mathbf{r},s) \cdot \mathbf{F} \tag{13}$$

Now, by the definition of the elastic energy release rate G, and the relation between increments of work and energy, one must have

$$\delta U(A,\mathbf{F}) = -\delta V(A,\mathbf{F}) + \mathbf{F} \cdot \delta\mathbf{u}(\mathbf{r}, A,\mathbf{F}) - \int_{\text{CF}} G(s, A,\mathbf{F})[p(A,s)\, \delta A\, g(s)\, ds] \tag{14}$$

for arbitrary variations of $\mathbf{F}$ and A, where U is the strain energy of the cracked body. Thus

$$\delta[\mathbf{F} \cdot \mathbf{u} - V - U] = \mathbf{u}(\mathbf{r}, A,\mathbf{F}) \cdot \delta\mathbf{F} + \left\{\int_{\text{CF}} G(s, A,\mathbf{F}) p(A,s) g(s)\, ds\right\} \delta A \tag{15}$$

and, evidently, the right side must be a perfect differential in $\delta\mathbf{F}$ and δA. Thus their coefficients must satisfy the Maxwell-like reciprocal relations

$$\frac{\partial \mathbf{u}(\mathbf{r},A,\mathbf{F})}{\partial A} = \frac{\partial}{\partial \mathbf{F}}\left\{\int_{CF} G(s,A,\mathbf{F})p(A,s)g(s)\,ds\right\} \quad (16)$$

Since, by Eq 4, $G = \Lambda_{\alpha\beta}\hat{K}_\alpha\hat{K}_\beta$, this means that

$$\frac{\partial \mathbf{u}(\mathbf{r},A,\mathbf{F})}{\partial A} = 2\int_{CF} \Lambda_{\alpha\beta}(s)\,\frac{\partial \hat{K}_\alpha(s,A,\mathbf{F})}{\partial \mathbf{F}}\,\hat{K}_\beta(s,A,\mathbf{F})p(A,s)g(s)\,ds \quad (17)$$

One now sets $A = 0$, recalling that then $p(0,s) = 1$ and, from Eq 13

$$\partial\hat{K}_\alpha(s,0,\mathbf{F})/\partial\mathbf{F} = \mathbf{h}_\alpha(\mathbf{r};s) \quad (18)$$

Setting $\mathbf{F} = 0$ also, in which case $\hat{K}_\beta(s,0,0) = K_\beta(s)$, the left and right sides of Eq 17 coincide with those of Eq 7, thus providing the desired proof of the second property enunciated for the weight functions.

A New Derivation of the Second Property

Consider an arbitrary location s along CF. Relative to the local coordinate system there, Fig. 1*a*, let us move along the negative 2 axis (that is, perpendicular to CF, into the crack zone) a small distance ρ and, at that site apply a force pair $\mathbf{Q}$ to the upper crack surface and $-\mathbf{Q}$ to the lower. Let us now apply the elastic reciprocal theorem to the load system just described and to another system consisting of a point force $\mathbf{F}$ at $\mathbf{r}$. The latter causes intensity factors $\mathbf{h}_\alpha(\mathbf{r};s)\cdot\mathbf{F}$ at location s and hence, by Eq 2, the relative crack surface displacements $\Delta\mathbf{u}^F$ induced by the force $\mathbf{F}$ at distance ρ, very near to location s along CF, is

$$\Delta u_\alpha^F = 8\sqrt{\rho/2\pi}\Lambda_{\alpha\beta}(s)h_{\beta j}(\mathbf{r},s)F_j \quad (19)$$

By the elastic reciprocal theorem, the work product $Q_\alpha\Delta u_\alpha^F$ equals the product $F_j u_j^Q(\mathbf{r})$, where $u_j^Q(\mathbf{r})$ denotes the j direction displacement induced at $\mathbf{r}$ by the pair of point forces $\mathbf{Q}$ and $-\mathbf{Q}$ at (small) distance ρ from location s along CF.

Thus another characterization of the weight functions which emerges is that

$$u_j^Q(\mathbf{r}) = 8\sqrt{\rho/2\pi}Q_\alpha\Lambda_{\alpha\beta}(s)h_{\beta j}(\mathbf{r},s) \quad (20)$$

is the j direction displacement induced at $\mathbf{r}$ by the force pair near CF. The only sense in which ρ is assumed small in this derivation is that terms of order higher than $\sqrt{\rho}$, in the expression for $\Delta\mathbf{u}^F$ induced at distance ρ from CF by force $\mathbf{F}$ at $\mathbf{r}$, must be negligible by comparison to $\sqrt{\rho}$. Note that the components Q_α in Eq 20 are referred to the local 1, 2, 3 coordinate system at s, just as are those of Δu_α^F in Eq 19.

Now consider the process of crack advance by $\delta a(s)$, as in Fig. 1*c*. The variation $\delta\mathbf{u}(\mathbf{r})$ in displacement at $\mathbf{r}$ can be calculated as the effect of removing the stresses of type

$$\sigma_{1\alpha} = K_\alpha(s)/\sqrt{2\pi\rho'} \quad (21)$$

which acted before enlargment. This manner of addressing the effects of crack enlargement is similar to Panasyuk's [13] approach to the perturbed circular crack. Stress removal is equivalent to placing pairs of infinitesimal forces

$$Q_\alpha = [K_\alpha(s)/\sqrt{2\pi\rho'}]\, d\rho' ds \tag{22}$$

on the crack faces at distance $\rho = \delta a(s) - \rho'$ from the new crack tip. Each such force causes the displacement u_j at $\mathbf{r}$ identified above as $u_j^\rho(\mathbf{r})$, and thus the net displacement variation $\delta u_j(\mathbf{r})$ due to the considered crack advance is

$$\delta u_j(\mathbf{r}) = \int_{CF} \int_0^{\delta a(s)} \left[8\sqrt{\frac{\delta a(s) - \rho'}{2\pi}} \frac{K_\alpha(s)}{\sqrt{2\pi\rho'}} \Lambda_{\alpha\beta}(s) h_{\beta j}(\mathbf{r};s) \right] d\rho' ds \tag{23}$$

The integral on ρ' is elementary, and one readily confirms that this equation agrees with Eq 6, providing the alternate derivation. As stated, this derivation assumes that $\delta a(s)$ is everywhere positive. It is not hard to modify it when $\delta a(s)$ is negative (in those zones one applies infinitesimal forces Q_α to create, rather than remove, the appropriate near-tip stresses $\sigma_{1\alpha}$), and thus to make the derivation fully general.

The reciprocal interpretation of the three-dimensional weight functions in Eq 20 has also been noticed by Bueckner (private communication). It generalizes an interpretation given by Paris et al. [25] in the two-dimensional case.

Variation of Green's Function with Change of Crack Front Position

The Green's function $G_{jk}(\mathbf{r},\mathbf{r}')$ for an elastic body is defined by the property

$$u_j(\mathbf{r}) = \int_{\text{Body}} G_{jk}(\mathbf{r},\mathbf{r}') f_k(\mathbf{r}')\, dx'dy'dz' \tag{24}$$

and, naturally, the Green's function depends on the position of the crack and varies with change of that position. Letting $\delta G_{jk}(\mathbf{r},\mathbf{r}')$ be that variation, it is seen by the second property of the weight functions, when the $K_\beta(s)$ in Eq 6 is expressed by use of the first property, Eq 5, that

$$\delta G_{jk}(\mathbf{r},\mathbf{r}') = 2 \int_{CF} \Lambda_{\alpha\beta}(s) h_{\alpha j}(\mathbf{r},s) h_{\beta k}(\mathbf{r}',s) \delta a(s)\, ds \tag{25}$$

to first order in $\delta a(s)$ when the crack front is advanced, as in Fig. 1*c*.

This emphasizes the remarkable information content of the weight functions. While primitively they have the relatively humble role of describing only the distribution of stress intensity factors induced along the crack front by arbitrary point forces, they turn out to relate to the Green's function and thus to the entire displacement field induced throughout the body by such point forces. In fact, if the weight functions are known for a sequence of crack-front positions, corresponding to introduction of the crack and enlargement to its present size, then $G_{jk}(\mathbf{r},\mathbf{r}')$ can be calculated directly from the weight functions, by integrating $\delta G_{jk}(\mathbf{r},\mathbf{r}')$ of Eq 25, provided that the initial $G_{jk}(\mathbf{r},\mathbf{r}')$ is known for the uncracked solid.

For example, consider a half-plane crack with tip position at a, as in Fig. 2, or a circular crack of radius a (Fig. 3*b*). Then $G_{jk} = G_{jk}(\mathbf{r},\mathbf{r}';a)$ and, by letting $\delta a(s)$ be uniform in s,

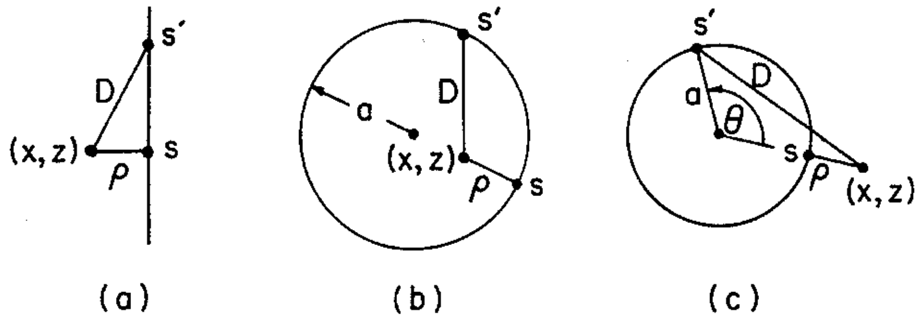

FIG. 3—*Cracks in the plane* y = *0 and notation to describe crack face weight functions for* (a) *half-plane crack,* (b) *circular crack, and* (c) *circular connection* (*externally cracked*).

and dividing both sides of Eq 25 by that uniform value, the left side becomes $\partial G_{jk}(\mathbf{r},\mathbf{r}';a)/\partial a$. Thus, by knowing the classical Kelvin G_{jk} for an uncracked full-space and adding to it the integral of $\partial G_{jk}/\partial a$, from $-\infty$ to a for the half-plane crack or from 0 to a for the circular crack, one obtains G_{jk} for the cracked solid.

Sometimes it is not necessary to know all three weight functions to calculate what will serve as a Green's function for the class of loadings actually experienced by a cracked solid. For example, suppose our concern is exclusively with loading systems that produce pure Mode 1 along the crack front. In that case, when $\delta G_{jk}(\mathbf{r},\mathbf{r}')$, or its integral over some crack introduction sequence, is actually multiplied by $f_k(\mathbf{r}')$ and integrated over all volume elements $dx'dy'dz'$ of the body, the product $h_{\beta k}f_k$ integrates to zero, by Eq 5, for all load systems of the class considered when $\beta = 2$ or 3. It then suffices, for the pure Mode 1 load systems which are considered, to write (for example, for an isotropic solid)

$$\delta G_{jk}(\mathbf{r},\mathbf{r}') = \frac{1-\nu}{\mu}\int_{CF} h_{1j}(\mathbf{r};s)h_{1k}(\mathbf{r}';s)\delta a(s)\,ds \tag{26}$$

The concepts outlined here have been used to derive the Green's function or, related to it, the expression for relative crack surface displacement $\Delta\mathbf{u}$ under general loadings, for half-plane [*7*] and circular [*11,15*] cracks and circular connections [*12*].

Relation to Bueckner's Concept of Fundamental Fields

Bueckner's approach to the three-dimensional theory may be summarized as follows: Let $\mathbf{v}(\mathbf{r})$ be a fundamental field, that is, a solution to the Navier displacement equations of three-dimensional elasticity, equilibrating null applied loading. In general, no such field other than $\mathbf{v} = 0$ (or $\mathbf{v}$ = rigid motion) would exist, for such is clearly a solution of the elasticity equations for null loading and, by the uniqueness theorem, no other type of solution could exist. However, the fundamental fields lie outside the scope of fields covered by the uniqueness theorem, since the fundamental fields, to be useful, must have unbounded strain energy. In fact, such fields have displacements which become infinite as $1/\sqrt{\rho}$ near the crack front, and hence stresses and strains which become infinite as $1/\rho\sqrt{\rho}$. Their strength at location s along the crack front is characterized in terms of the discontinuity $\Delta\mathbf{v}$ between upper and lower crack surfaces by the Bueckner strength function

$$B_\alpha(s) = \lim_{\rho\to 0}\left[\sqrt{\pi\rho/2}\,\Delta v_\alpha(s,\rho)\right] \tag{27}$$

where now the $\Delta \mathbf{v}$ are referred to the local coordinates (Fig. 1*a*) at s and $\Delta v_\alpha(s,\rho)$ means Δv_α at distance ρ along the -2 axis through the location along CF at arc length s.

In terms of these fundamental fields and their strength distributions around CF, Bueckner's basic result is that

$$\int_{CF} B_\alpha(s) K_\alpha(s)\, ds = \int_{\text{Body}} \mathbf{v}(\mathbf{r}) \cdot \mathbf{f}(\mathbf{r})\, dxdydz \tag{28}$$

The proof is as follows: $\mathbf{v}(\mathbf{r})$ can be regarded as an unobjectional elastic displacement field for a cracked solid from which we exclude a small cylindrical tube, say, of radius ρ, along CF. The stresses associated with $\mathbf{v}$ equilibrate zero body and surface force everywhere except along the tube surface, where tractions $\mathbf{T}$ of order $1/\rho\sqrt{\rho}$ must be applied to maintain displacements $\mathbf{v}$.

We now apply the elastic reciprocal theorem to the pair of fields consisting of the fundamental field $\mathbf{v}(\mathbf{r})$ just described and the actual displacement field $\mathbf{u}(\mathbf{r})$ induced by the applied forces $\mathbf{f}(\mathbf{r})$. Thus, the work of the forces of the $\mathbf{v}$ field (that is, of the tractions $\mathbf{T}$, of order $1/\rho\sqrt{\rho}$ along the tube) on the $\mathbf{u}$ displacements equals the work of the forces $\mathbf{f}$, and of tractions on the tube resulting from the $\mathbf{u}$ field, on the $\mathbf{v}$ displacements. We can let $\rho \rightarrow 0$ in the two work expressions. The work of $\mathbf{f}$ is plainly given by the right side of Eq 28, and it should appear plausible that the limit of the works of the tube tractions is given by the left side, since the $\mathbf{u}$ field near CF is proportional to the K's times $\sqrt{\rho}$. Thus $\mathbf{T} \cdot \mathbf{u}$ is of order $1/\rho$ along the tube, as is the work on $\mathbf{v}$, and the $1/\rho$ gets cancelled out when we integrate over the surface of the tube, so there is a well-defined limit as $\rho \rightarrow 0$. Of course, the strengths $B_\alpha(s)$ have been so defined in Eq 27 that the tube-surface work terms combine to what is written on the left of Eq 28.

Consider a limiting fundamental field which may be said to have a point of concentration at location s', that is, for which the strength distribution is

$$B_\alpha(s) = \delta^K_{\alpha\beta}\delta^D(s - s') \tag{29}$$

where $\delta^K_{\alpha\beta}$ is the Kronecker-δ and $\delta^D(\ldots)$ the Dirac-δ. Then by comparing the result of Eqs 28 to 5, giving the first property of the weight functions, it is evident that the limiting fundamental field described is just

$$\mathbf{v}(\mathbf{r}) = \mathbf{h}_\beta(\mathbf{r};s') \tag{30}$$

Since the field $\mathbf{u}(\mathbf{r})$ created by general applied loadings satisfies the Navier displacement equations of elasticity, so also must $\delta\mathbf{u}(\mathbf{r})$ of Eq 6 [and $\partial\mathbf{u}(\mathbf{r},A)/\partial A$ of Eq 7]. Further, since both $\mathbf{u}(\mathbf{r})$ and $\mathbf{u}(\mathbf{r}) + \delta\mathbf{u}(\mathbf{r})$ equilibrate the same system of loadings, $\delta\mathbf{u}(\mathbf{r})$ and $\partial\mathbf{u}(\mathbf{r},A)/\partial A$ satisfy the displacement equations of elasticity corresponding to null loading. One therefore suspects that not only $\mathbf{h}_\alpha(\mathbf{r};s)$, but also every field of type $\partial\mathbf{u}(\mathbf{r},A)/\partial A$ meets the requirements to be a Bueckner fundamental $\mathbf{v}(\mathbf{r})$ field. It is easy to confirm that $\partial\mathbf{u}(\mathbf{r},A)/\partial A$ has a singularity of the appropriate order, $1/\sqrt{\rho}$, near the crack front so that it is indeed a candidate $\mathbf{v}(\mathbf{r})$. The Bueckner strength distribution $B_\alpha(s)$ associated with $\partial\mathbf{u}(\mathbf{r},A)/\partial A$ is readily determined, either by examining the field near $\rho = 0$ or by substituting $\partial\mathbf{u}(\mathbf{r},A)/\partial A$ as expressed by Eq 7 into Eq 28 for $\mathbf{v}(\mathbf{r})$ and then using Eq 5. Either way, one finds that

$$B_\alpha(s) = 2\Lambda_{\alpha\beta}(s)K_\beta(s)g(s) \tag{31}$$

We have just seen that every field $\partial\mathbf{u}(\mathbf{r},A)/\partial A$, corresponding to variation of crack front location under fixed load, provides a Bueckner fundamental field $\mathbf{v}(\mathbf{r})$. The converse applies too: Every Bueckner fundamental field $\mathbf{v}(\mathbf{r})$ can be identified with a field $\partial\mathbf{u}(\mathbf{r},A)/\partial A$ for a suitably loaded cracked solid with suitable choice of growth function $g(s)$. This is true because $\Lambda_{\alpha\beta}$ is invertible, and hence every distribution of $B_\alpha(s)$ corresponds, by Eq 31, to an equivalent distribution of products $K_\beta(s)g(s)$ in the description of crack growth under fixed load. Since by suitable choice of body-force distribution it is possible to make the $K_\beta(s)$ vary in any desired manner around the crack front, this confirms that every $\mathbf{v}(\mathbf{r})$ has a $\partial\mathbf{u}(\mathbf{r},A)/\partial A$ representation. Thus Bueckner's class of fundamental fields and the writer's class of fields generated by incremental crack growth are identical.

Crack-Face Weight Functions, Some Examples, and Symmetry Properties

For applications to perturbations of crack shape, and for some other purposes, there is no need to know the full-field weight functions $\mathbf{h}_\alpha(\mathbf{r};s)$ for positions $\mathbf{r}$ throughout the entire body. Rather, it suffices to know their jumps across the crack plane, that is, to know the crack-face weight functions defined by

$$\mathbf{k}_\alpha(x,z;s') = \mathbf{h}_\alpha(x,0^+,z;s') - \mathbf{h}_\alpha(x,0^-,z;s') \tag{32}$$

for all positions (x,z) within the crack zone and locations s' along CF. The properties of the $\mathbf{k}_\alpha$ are analogous to those enunciated for the $\mathbf{h}_\alpha$ earlier. First, if the loading consists of tractions $\mathbf{T} = \mathbf{T}(x,z)$ per unit area on the upper crack face, and $-\mathbf{T}$ on the lower, then the intensity factors at location s' along the crack front are

$$K_\alpha(s') = \int_{\text{crack}} \mathbf{k}_\alpha(x,z;s') \cdot \mathbf{T}(x,z)\, dxdz \tag{33}$$

Of course, general loadings can always be reduced to this case by the well-known superposition procedure. Second, if the crack front location is altered by $\delta a(s)$, as in Fig. 1*c*, under fixed loading conditions then the variation of the crack-surface displacement discontinuity is

$$\delta[\Delta\mathbf{u}(x,z)] = 2\int_{CF} \Lambda_{\alpha\beta}(s')\mathbf{k}_\alpha(x,z;s')K_\beta(s')\delta a(s')\, ds' \tag{34}$$

to first order in $\delta a(s)$. These are the specific forms of Eqs 5 and 6 for crack-face loading and crack-face displacement discontinuities.

Some examples of crack-face weight functions are cited now for cracks in unbounded isotropic solids. First one may observe that symmetry requires

$$k_{1x} = k_{1z} = k_{2y} = k_{3y} = 0 \tag{35}$$

in all such cases. Thus, for tensile (Mode 1) loadings, one needs only $k_{1y}(x,z;s')$. The half-plane crack, circular crack of radius a, and circular connection of radius a (external annular crack of infinite outer radius) are shown in Figs. 3*a*, *b*, and *c*, respectively. Choose a point (x,z) within the crack space, let $\rho = \rho(x,z)$ be the shortest distance to the crack front, and $D = D(x,z;s')$ be the distance from (x,z) to location s' along the crack front. Then for the half-plane crack (for examples, see Refs *6* and *7*)

$$k_{1y}(x,z;s') = \sqrt{2\rho}/\pi\sqrt{\pi}\, D^2 \tag{36}$$

and for the circular crack [*6,11,13*]

$$k_{1y}(x,z;s') = \sqrt{\rho(2\alpha - \rho)}/\pi\sqrt{\pi a}\, D^2 \tag{37}$$

When the two half-spaces joined by the circular connection are restrained against displacement at infinity [*12*]

$$k_{1y}(x,z;s') = \sqrt{\rho(2a + \rho)}/\pi\sqrt{\pi a}\, D^2 \tag{38}$$

When the restraints at infinity are removed so as to allow free translation in the y direction, free rotation, or both (that is, completely unrestrained), various terms must be added to the expression for k_{1y} just given. For example, in the completely unrestrained case, the terms [*12*]

$$(\pi a)^{-3/2}\{\cos^{-1}[1/(1 + \rho/a)][1 + 3(1 + \rho/a)\cos\theta] + [\sqrt{\rho(2a + \rho)}/(a + \rho)]\cos\theta\}$$

must be added, where the angle $\theta = \theta(x,z;s')$ is identified in Fig. 3*c*.

Shear mode crack face weight functions, k_{2x}, k_{2z}, k_{3x}, and k_{3z} are given in Refs *7* and *9* for the half-plane crack and in Ref *15* for the circular crack.

Let us limit attention to pure Mode 1 conditions in homogeneous isotropic solids for the rest of this section. Consider a general crack shape as in Fig. 4, choose two locations s and s' along the crack front, locate a point (x,z) by moving into the crack zone a small perpendicular distance ρ from s, and a point (x',z') by moving a small distance ρ' from s'.

Guided by the above examples, we observe that $k_{1y}(x,z;s')/\sqrt{\rho(x,z)}$ has a well-defined limit as $\rho \to 0$, that is, as (x,z) approaches the location s along the crack front. Let us therefore introduce the general representation

$$k_{1y}(x,z;s') = \frac{\sqrt{2\rho(x,z)}\,W(x,z;s')}{\pi\sqrt{\pi}\, D^2(x,z;s')} \tag{40}$$

where $W(x,z;s')$ has a well-defined limit, denoted by

$$W(s,s') = \lim_{\rho(x,z)\to 0}[W(x,z;s')] \tag{41}$$

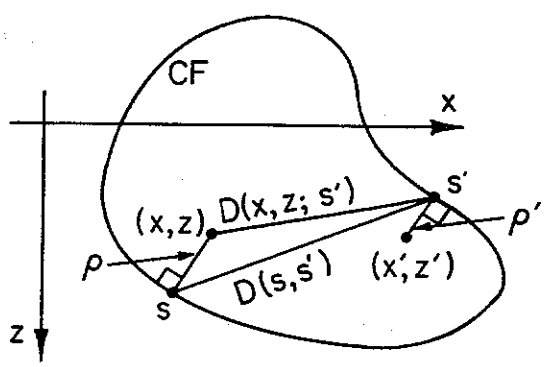

FIG. 4—*General crack shape, for discussion of crack face weight function and symmetry properties.*

as (x,z) approaches location s. We use $D(s,s')$ to denote the corresponding limit of $D(x,z;s')$; it is the distance from location s to location s' (Fig. 4). For example

$$W(s,s') = 1 \tag{42}$$

for the cases of the half-plane crack and the circular crack, and also for the circular connection when points at infinity are restrained against displacement, whereas

$$W(s,s') = 1 + D^2(s,s')[1 + 6 \cos \theta(s,s')]/a^2 \tag{43}$$

for the circular connection with unrestrained displacements at infinity.

Consider now the opening displacement $\Delta u_y{}^Q(x',z')$ induced by force Q_y in the y (or 1) direction on the upper crack face, and $-Q_y$ on the lower crack face, at (x,z). By Eqs 20 and 32

$$\Delta u_y^Q(x',z') = 8\sqrt{\rho(x,z)/2\pi}[(1 - \nu)/2\mu]k_{1y}(x',z';s)Q_y \tag{44}$$

and by the reciprocal theorem this must be the same as the opening displacement induced at (x,z) when the same force pair is applied at (x',z'). Thus

$$\sqrt{\rho(x,z)}\; k_{1y}(x',z';s) = \sqrt{\rho'(x',z')}\; k_{1y}(x,z;s') \tag{45}$$

and dividing this equation by $\sqrt{\rho\rho'}$ and then letting $\rho \rightarrow 0$ and $\rho' \rightarrow 0$ (recall that Eq 20 is exact in this limit), we obtain the symmetry property

$$W(s',s) = W(s,s') \tag{46}$$

Some Applications of Three-Dimensional Weight-Function Theory

Guided by the two-dimensional theory, it might be thought that the main application of three-dimensional weight-function theory is to determine stress intensity factors for a given crack geometry under a variety of different loading conditions. There is evidently much to be done on such applications, but here the focus will be on some new types of investigation allowed by the theory.

Crack Tip Interaction with Transformation Strains and Dislocations

This class of applications is of interest for the micromechanics of fracture. The three-dimensional weight-function concepts enable one to develop useful relations for study of the effects of particles of different elastic moduli, or of particles that undergo a stress-induced shape transformation (as in transformation toughening), or of smaller-scale micro-cracks, on the stress intensity factors and stress fields near a crack tip. Similarly, they allow one to address various three-dimensional dislocation interactions with crack tips, including development of models for three-dimensional loop emission from stressed crack tips.

The starting point is with the Eshelby [*26*] concept of a distribution of stress-free transformation strain $\epsilon_{pq}^T(\mathbf{r})$. If we consider a solid of elastic moduli $C_{jkpq}(\mathbf{r})$, then the stress field in the presence of this strain distribution is assumed to be given by

$$\sigma_{jk}(\mathbf{r}) = C_{jkpq}(\mathbf{r})[u_{p,q}(\mathbf{r}) - \epsilon_{pq}^T(\mathbf{r})] \tag{47}$$

Here $C_{jkpq} = C_{pqjk} = C_{kjpq} = C_{jkqp}$, and if $L = L(\mathbf{r})$, then L,j for $j = (x,y,z)$ denotes $\partial L/\partial(x,y,z)$. Equilibrium in the absence of any actual body force requires that $\sigma_{jk,j} = 0$ and this equation, using Eq 42, is the same as the equilibrium equation in terms of $\mathbf{u}(\mathbf{r})$ for a solid having zero transformation strain but being subject to a distribution of "effective" body force

$$f_k^{\text{eff}}(\mathbf{r}) = -[C_{jkpq}(\mathbf{r})\epsilon^T_{pq}(\mathbf{r})],j \tag{48}$$

(and also effective surface tractions when ϵ^T_{pq} is nonzero along the crack faces or external surfaces).

Thus the writer [*8*] showed that the intensity factors induced at location s along CF by a distribution of $\epsilon^T_{pq}(\mathbf{r})$ in some region V are

$$K_\alpha(s) = \int_V h_{\alpha k,j}(\mathbf{r};s)C_{jkpq}(\mathbf{r})\epsilon^T_{pq}(\mathbf{r})\ dxdydz \tag{49}$$

General Somigliana dislocations, that is, surfaces S on which a displacement jump $\Delta\mathbf{u}(\mathbf{r})$ is prescribed for $\mathbf{r}$ on S, can also be considered as a limiting case. Thus [*8*]

$$K_\alpha(s) = \int_S h_{\alpha k,j}(\mathbf{r};s)C_{jkpq}(\mathbf{r})N_p(\mathbf{r})\Delta u_q(\mathbf{r})\ dS(\mathbf{r}) \tag{50}$$

where $\mathbf{N}$ is the local normal to S, chosen so that $\Delta\mathbf{u}$ is the difference between $\mathbf{u}$ on the side of S towards which $\mathbf{N}$ points and $\mathbf{u}$ on the other side.

This formula requires a modification, outlined in Refs *17* to *19* for the case of the half-plane crack, to by-pass formally divergent integrals for locations s along any segment of CF which happens to lie in the dislocated surface S or along a part of its border. The latter is particularly an issue for a microcrack [representable as some distribution, albeit unknown, of $\Delta\mathbf{u}(\mathbf{r})$] or dislocation loop emanating from the crack tip. In the simplest case, when the crack can be regarded as a half-plane crack with straight front, as in Fig. 2, and when the surface S is a planar zone emanating from the crack tip, the fix-up is simple. We replace $\Delta\mathbf{u}(x,y,z)$ in Eq 50 by $\Delta\mathbf{u}(x,y,z) - \Delta\mathbf{u}(0,0,s)$, where the location of interest along the crack front is at $z = s$ and the crack tip is assumed to be at $a = 0$ (Fig. 2), and we extend the integral over the entire half-plane emanating from the tip that contains S [*17*]. This is equivalent to subtracting the effect of a constant displacement, $\Delta\mathbf{u}(0,0,s)$, over that entire half-plane, which has no effect on the stress field.

Stress Field—In the types of micromechanical applications mentioned, one generally wants to know the stress field induced in V by the transformation strain, or on S by the dislocation. For example, if we use $\epsilon^T_{pq}(\mathbf{r})$ as an artifact, to represent the effect of an inclusion of different moduli than its surroundings (the Eshelby [*26*] procedure), then the ϵ^T_{pq} distribution over the inclusion region V should, in principle, be chosen so that the local values of σ_{jk}, due to ϵ^T_{pq} and to the external loading, relate to the local values of $u_{j,k}$ within V in a manner compatible with the actual constitutive relation (not necessarily linear) for the inclusion material. This clearly poses a formidable problem, but one which would be tractible in a more approximate form, for example, if ϵ^T_{pq} were taken as locally uniform in V, or in some subregions of V, and the constitutive relation were required to be satisfied only at the centroid of V, or at centroids of its subregions. Similarly, when S represents a microcrack one would, in principle, choose $\Delta\mathbf{u}(\mathbf{r})$ on S so that the traction stresses $N_j\sigma_{jk}$, due to that $\Delta\mathbf{u}$ distribution and to the external loading, vanish on S. Equally, for calculating the overall energy and alteration of Peach-Koehler (configurational) forces along a near-tip (Volerra, constant $\Delta\mathbf{u}$) dislocation loop, the stresses are required.

The stress field due to transformation strain or dislocation can be represented as follows for the half-plane crack with tip along the z axis (that is, with $a = 0$ in Fig. 2). Let $K_\alpha(s,a)$ denote the intensity factor distribution calculated from Ref *49* or *50* when the tip is at $x = a$. Also, let $\mathbf{h}_\alpha(x - a,y,z - s)$ denote the weight functions $\mathbf{h}_\alpha(\mathbf{r};s)$; this is the form they take for the half-plane crack in a homogeneous material (for example, see Eqs 8 and 9). Then [*8*]

$$\sigma_{pq}(\mathbf{r}) = \sigma^0_{pq}(\mathbf{r}) + 2\Lambda_{\alpha\beta}C_{jkpq}\int_{-\infty}^{0}\int_{-\infty}^{+\infty} h_{\alpha k,j}(x - a,y,z - s)K_\beta(s,a)\, ds da \tag{51}$$

Here $\sigma^0_{pq}(\mathbf{r})$ is the stress field which that same distribution of transformation strain or dislocation would induce in an infinite, uncracked solid. When the crack tip contains segments which lie in or on the border of a dislocated surface S, certain subtraction procedures, analogous to subtracting $\Delta\mathbf{u}(0,0,s)$ as explained earlier, have to be used to avoid formally divergent terms. For example, as discussed in Refs *17* to *19*, when S is a dislocated surface emanating from the crack tip, both σ^0_{pq} and the integral in Eq 51 contribute singular stress terms, of order $1/\rho'$ near the tip (Fig. 1*a*), which cancel one another to leave the proper $1/\sqrt{\rho'}$ singularity.

Although the methods outlined in this section have great potential for the three-dimensional micromechanics of phenomena at crack tips, it should be cautioned that the integrations involved are formidable. Equation 51 for a dislocation loop effectively involves integration over four variables, two to get the K_α from Eq 50 and two more displayed explicitly in Eq 51. Also, the subject is young, the Mode 1 weight function being published only in 1985 [*7*] and the Mode 2 and 3 functions only in 1987 [*10*], and only rather simple applications have been made thus far. Some of these are now summarized, all involving isotropic solids.

Dilatant Shape Transformation—Suppose that a transformation strain distribution corresponding to pure dilatation, $\epsilon^T_{pq}(\mathbf{r}) = \delta_{pq}\theta(\mathbf{r})/3$, occurs in V. Then from Eq 49 and Eqs 8 and 9, K_1 along the crack front is given by [*8*]

$$K_1(s) = \frac{2\mu(1 + \nu)}{3(2\pi)^{3/2}(1 - \nu)}\int_V \frac{\cos(\phi/2)}{\rho^{1/2}D^2}\left[1 - \frac{8\rho^2}{D^2}\sin^2\left(\frac{\phi}{2}\right)\right]\theta(\mathbf{r})\, dxdydz \tag{52}$$

where (notation of Fig. 2) $\tan\phi = y/(x - a)$, $\rho^2 = (x - a)^2 + y^2$, and $D^2 = \rho^2 + (z - s)^2$. This reproduces known results from two-dimensional models [*27,28*] developed in the literature on transformation toughening of ceramics, when $\theta(x,y,z)$ reduces to $\theta(x,y)$.

Microcrack Ahead of Half-Plane Crack—Let the microcrack surface S lie on $y = 0$ and assume that S lies entirely in the region $x > 0$. Here the (main) crack tip is at $x = 0$ (that is, $a = 0$ in Fig. 2) so the microcrack is separated from the main crack. The problem is assumed to have Mode 1 symmetry so that the y (and only) component of displacement discontinuity on S is $\Delta u_y(x,z)$.

Then the intensity $K_1(s)$ induced along the half-plane crack tip by the opening distribution can be evaluated from Eq 50 and has the representation [8]

$$K_1(s) = \frac{\mu}{(2\pi)^{3/2}(1 - \nu)} \int_S \frac{\Delta u_y(x,z)\, dxdz}{\sqrt{x}[x^2 + (z - s)^2]} \tag{53}$$

When $\Delta u_y \geq 0$, $K_1(s)$ is everywhere positive, so microcrack opening directly ahead of the main crack increases K_1 everywhere along the crack front. Of course, to this K_1 must be added the K_1 due to the externally applied loading (in the absence of the microcrack), and the actual presently unknown Δu_y distribution on S will be directly proportional to the intensity of that loading, so that Eq 53, like the equation for σ_{yy} below, is only a result that is useful as part of the process of constructing a fuller solution.

The stress integral of Eq 51 gives in this case [8], for $x > 0$

$$\sigma_{yy}(x,0,z) = \sigma^o_{yy}(x,0,z) + \frac{\mu}{4\pi^2(1 - \nu)\sqrt{x}} \int_S \frac{Q(x,x',z - z')\Delta u_y(x',z')\, dx'dz'}{\sqrt{x'}[(x - x')^2 + (z - z')^2]} \tag{54}$$

where

$$Q = 1 - P \tan^{-1}(1/P) \quad \text{and} \quad P = 2\sqrt{xx'}/\sqrt{(x - x')^2 + (z - z')^2} \tag{55}$$

[so that the integrand is nonsingular when (x',z') coincides with (x,z)] and [29]

$$\sigma^o_{yy}(x,0,z) = \frac{\mu}{4\pi(1 - \nu)} \int_S \frac{(x' - x)\Delta u_{y,x}(x',z') + (z' - z)\Delta u_{y,z}(x',z')}{[(x - x')^2 + (z - z')^2]^{3/2}}\, dx'dz' \tag{56}$$

gives the stress induced by the same (unknown) opening on S in an uncracked full space. Anderson and Rice [17] show how to modify these expressions when a border of S coincides with a segment of the crack tip. Again, the stresses induced by applied loadings, in the absence of the microcrack, must be added to those given here.

The present formulation leads to a singular integral equation for $\Delta u_y(x,z)$ on S, analogous to those developed as a starting point in numerical treatments of three-dimensional crack problems in simpler full or half-plane geometries [29–31] and similar methods of discretization can be employed for numerical solution.

Dislocation Loop Emanating from Tip—When $\Delta u_y(x,z) = b$ (a constant; Volterra dislocation) over a region S (on $y = 0$) with border along the crack tip, we have the problem of a prismatic dislocation loop emanating from the crack tip. The two-dimensional elastic interaction between a crack tip and a parallel dislocation line has been worked out in great generality [32,33]. Such interactions are of interest in studies of intrinsic cleavability of solids, where an attempt is made to estimate the critical combination of intensity factors at which a dislocation nucleates from the crack tip [32,34]. Thus far, the three-dimensional aspects of such dislocation emission have been treated only approximately by assuming that the elastic self-energy of an emergent loop is just half of that for the corresponding "full loop" (emergent loop plus its mirror image relative to the crack tip). The problem is of interest for shear dislocations [18,19] emerging from the tip on slip planes that are generally inclined to the crack plane; the prismatic version [17], emerging in that plane, provides a more readily addressed first case. We review here only the half-circular emergent loop, over $x^2 + z^2 \leq R^2$, $x \geq 0$.

The self-energy of a full circular loop (in an uncracked body) is [35]

$$U^{\text{full loop}} = 2\pi R A_0 \ln(8R/e^2R_0) \quad (57)$$

where

$A_0 = \mu b^2/4\pi(1 - \nu)$ for a prismatic dislocation,
R_0 = core cut-off, and
e = natural logarithm base.

The self-energy U of the emergent loop therefore may be calculated as

$$U - \frac{1}{2} U^{\text{full loop}} = -(b/2) \int_S [\sigma_{yy}(x,0,z) - \sigma_{yy}{}^{\text{full loop}}(x,0,z)]\, dxdz \quad (58)$$

which Anderson and Rice [17] show to have the form $\pi R A_0$ times a constant, which they calculate by using the previous formulae for σ_{yy}, and write as $\ln(m)$. Thus

$$U = \pi R A_0 \ln(8mR/e^2R_0) \quad (59)$$

for the emergent loop. It is found [17] that $m = 2.21$ for the half-circular loop. Calculations for a rectangular emergent loop show that the analogously defined (with πR replaced by loop outer perimeter) value of m is 1.92 when the loop and its image form a square full loop, that m increases towards a maximum of 2.27 when the rectangle is elongated from that shape by a factor of about 4 in the direction parallel to the crack tip, and that with greater elongation m diminishes towards its limit $m = 2$ for the infinitely elongated rectangular loop (that is, for the two-dimensional, or line, dislocation [17]).

More recent work [18,19] has used Bueckner's [10] shear-mode weight functions to evaluate an analogously defined m, based on $A_0 = (2 - \nu)\mu b^2/8\pi(1 - \nu)$, for a half-circular shear dislocation loop emerging from the tip. When it is on the same plane as the crack, this gives $m = 2.35$ when $\Delta\mathbf{u}$ is in the x direction, and $m = 1.82$ when $\Delta\mathbf{u}$ is in the z direction. For the highly elongated loop (two-dimensional limit), $m = 2$ independently of the direction of $\Delta\mathbf{u}$ when the slip plane is the crack plane. However, m decreases, in different ways for different directions of $\Delta\mathbf{u}$ in the slip plane, when the slip plane is rotated (about the crack tip) relative to the crack plane [17,19].

The energy of an emergent loop is an important quantity in the theory of dislocation emission and its competition with cleavage decohesion at a crack tip [32,34]. The fact that $m > 1$ for typical cases means that the approximation of estimating self-energy as half that for a full loop has tended to underestimate (by $1/\sqrt{m}$) the K needed for dislocation nucleation.

Variation of Crack Shape

A general crack-front shape is shown in Fig. 5, and advance normal to itself by a distribution $\delta a(s)$ is indicated, as well as the distance function $D(s_1,s)$. The focus here is on setting up the formalism for calculating variations in the K_α along a crack front, both to first-order accuracy in the function $\delta a(s)$ and, in principle, exactly by integrating results for a sequence of infinitesimal advances $\delta a(s)$. For simplicity, attention is limited to Mode 1 conditions in isotropic solids.

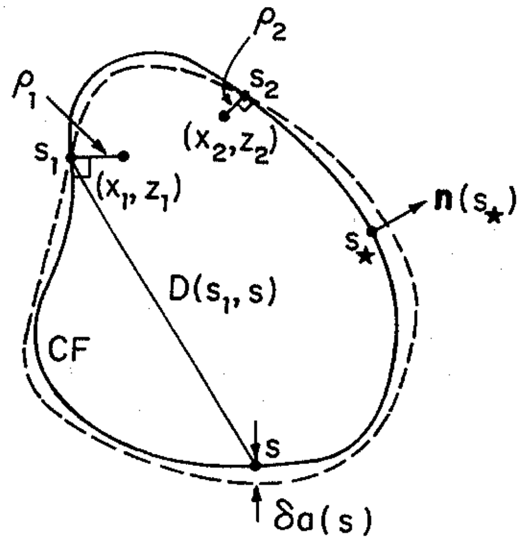

FIG. 5—*Variation of crack front location, for discussion of corresponding variations in* $K_I(s_I)$ *and* $W(s_I,s_2)$.

Note that $\delta a(s) = 0$ when $s = s_1$ or s_2 in Fig. 5. These are important locations because it is possible to work out the first-order variations $\delta K_1(s_1)$, $\delta K_1(s_2)$ and $\delta W(s_1,s_2)$, where W is the function of Eqs 40 to 43 and Eq 46. At least this is so when $d[\delta a(s)]/ds$ exists at s_1 and s_2. Naturally, one is also interested in the δK_1 and δW associated with locations where $\delta a(s) \neq 0$, and a procedure will be explained shortly for calculating those.

Consider a point (x_1, z_1) at small distance ρ_1 from CF along its perpendicular at s_1. By Eqs 2 and 3

$$\Delta u_y(x_1, z_1) \sim [4(1 - \nu)/\mu] K_1(s_1)\sqrt{\rho_1/2\pi} \tag{60}$$

Hence the variation in Δu_y is

$$\delta[\Delta u_y(x_1, z_1)] \sim [4(1 - \nu)/\mu] \delta K_1(s_1)\sqrt{\rho_1/2\pi} \tag{61}$$

to first order in $\delta a(s)$, where $\delta K_1(s_1)$ is the corresponding first-order variation at location s_1 along CF. The perpendicular to the new crack front (dashed line, Fig. 5) at location s_1 no longer passes through (x_1, z_1), but misses it by a distance $\rho_1\delta\theta_1$ measured parallel to CF, where $\delta\theta_1 = d[\delta a(s)]/ds$ at $s = s_1$. This effect may be included in the analysis, recognizing that $\delta[\Delta u_y]$ in Eq 61 should, strictly, be replaced by its value at (x_1, z_1) plus $\rho_1\delta\theta_1$ times the gradient of Δu_y in the direction parallel to CF. However, that modification gives a term of order $\rho_1\sqrt{\rho_1}\, \delta\theta_1$, the same order in ρ_1 as those already deleted on the right in Eqs 60 and 61, and all these will disappear shortly when we divide Eq 61 by $\sqrt{\rho_1}$ and let $\rho_1 \to 0$.

From Eqs 3, 34, and 40, one now observes by the second property of the weight functions that

$$\delta[\Delta u_y(x_1, z_1)] = \frac{1 - \nu}{\mu} \frac{\sqrt{2\rho_1}}{\pi\sqrt{\pi}} \int_{CF} \frac{W(x_1, z_1; s)}{D^2(x_1, z_1; s)} K_1(s)\delta a(s)\, ds \tag{62}$$

Now Eq 61 is used and both sides of Eq 62 are divided by $\sqrt{\rho_1}$, after which one lets $\rho_1 \to 0$ [that is, $(x_1 z_1)$ approaches location s_1 along CF]. Since $W(x_1, z_1; s)$ smoothly approaches

$W(s_1,s)$, Eq 41, in this limit, and since $\delta a(s)$ vanishes at s_1, the limit of the integral exists as a principal value (PV) integral. Thus

$$\delta K_1(s_1) = \frac{1}{2\pi} PV \int_{CF} \frac{W(s_1,s)}{D^2(s_1,s)} K_1(s)\delta a(s)\, ds \tag{63}$$

to first order in $\delta a(s)$ when $\delta a(s_1) = 0$. This generalizes to arbitrary crack front shapes results given in Refs *7*, *11*, and *12* for specific shapes.

Note that $W(s_1,s)$ is known for simple crack shapes, Fig. 3 and Eqs 42 and 43. Thus, Eq 63 is ready for use, provided that $\delta a(s_1) = 0$.

How does one calculate $\delta K_1(s_*)$ at a location like s_* in Fig. 5, where $\delta a(s_*) \neq 0$? In general, the following procedure solves the problem: We represent the given $\delta a(s)$ as the sum of two functions

$$\delta a(s) = \delta_* a(s) + [\delta a(s) - \delta_* a(s)] \tag{64}$$

where $\delta_* a(s) = \delta a(s)$ at $s = s_*$ so that the second function, in brackets, vanishes as required for validity of Eq 63, and where $\delta_* a(s)$ is a simple motion of CF for which it is straightforward to calculate independently $\delta_* K_1(s_*)$, the variation of K_1 corresponding to $\delta_* a(s)$. Thus

$$\delta K_1(s_*) = \delta_* K_1(s_*) + \frac{1}{2\pi} PV \int_{CF} \frac{W(s_*,s)}{D^2(s_*,s)} K_1(s)[\delta a(s) - \delta_* a(s)]\, ds \tag{65}$$

As examples, we may take for $\delta_* a(s)$ a rigid translation of CF of amount $\mathbf{n}(s_*)\delta a(s_*)$, so that

$$\delta_* a(s) = \mathbf{n}(s) \cdot \mathbf{n}(s_*)\delta a(s_*) \tag{66}$$

where $\mathbf{n}(s)$ is the unit outer normal (in the x,z plane) to CF at location s. For a finite-size crack in an unbounded solid under remotely uniform tension, such translation gives $\delta_* K_1(s_*) = 0$. Alternately, as in Refs *11* and *12* for circular CF, one may sometimes take a self-similar scaling of the crack shape by expansion relative to any convenient location of the x,z cordinate origin, which does not lie along the tangent line to CF at s_*. In that case, $\delta_* a(s)$ has the form $\mathbf{r}(s) \cdot \mathbf{n}(s)\delta\lambda$, where here $\mathbf{r}$ is the position vector along CF, and we choose $\delta\lambda$ to make $\delta_* a(s_*) = \delta a(s_*)$. Thus

$$\delta_* a(s) = \mathbf{r}(s) \cdot \mathbf{n}(s)\delta a(s_*)/\mathbf{r}(s_*) \cdot \mathbf{n}(s_*) \tag{67}$$

Yet a third alternative for choosing $\delta_* a(s)$ is provided by a rigid rotation of CF about any point which does not lie on the perpendicular to CF at s_*.

This discussion has focused on growth from the solid line CF in Fig. 5 to the dashed line. However, often the given reality in a problem is the actual (dashed line) crack front. Then the solid line CF is just an artifact that we introduce and are free to locate arbitrarily, subject to the restrictions that we must know how to determine K_1 along CF and that we want it to be close enough to the actual shape to justify use of the first order perturbation formulae.

Thus, if we want to know K_1 at the location marked s_1 along the dashed crack front in Fig. 5, we simply position the (arbitrarily chosen) solid-line CF to pass through that location, as has been done for the figure. Examples follow shortly.

Variation of W—Equation 63, or its relative, Eq 65, provides a solution for $\delta K_1(s_*)$ provided that one knows $W(s_*,s)$. W is known for simple crack shapes, Eqs 42 and 43. However, one cannot consider yet Eq 65 as providing an infinitesimal δK_1 distribution, associated with infinitesimal $\delta a(s)$, which can be integrated over a sequence of successive crack shapes, starting at a simple shape, to give K_1 for a general crack shape, because a procedure for calculating the evolution of $W(s_*,s)$ has not been given.

Such a procedure can be obtained by applying Eq 63 to loading by a pair of unit y-direction point forces on the crack faces at (x,z). Thus the $K_1(s)$ appearing in Eq 63 becomes $k_{1y}(x,z;s)$ of Eq 32, given for simple crack shapes by Eqs 36 to 38 and, in general form, by Eq 40. We choose a $\delta a(s)$ which, as in Fig. 5, vanishes at two points s_1 and s_2, for which we wish to know $W(s_1,s_2)$. [If the given $\delta a(s)$ does not already so vanish, it can be modified to do so by a procedure like the one in Eq 64, combining any appropriate pair of translations, rotation, and scaling.] We choose (x,z) as the point (x_2,z_2) at distance ρ_2 along the perpendicular to CF at location s_2 in Fig. 5. Then

$$\delta k_{1y}(x_2,z_2;s_1) = \frac{1}{2\pi} PV \int_{CF} \frac{W(s_1,s)}{D^2(s_1,s)} k_{1y}(x_2,z_2;s)\delta a(s)\, ds \tag{68}$$

to first order in $\delta a(s)$ where, from Eq 40,

$$k_{1y}(x_2,z_2;s) = \frac{\sqrt{2\rho_2}\,W(x_2,z_2;s)}{D^2(x_2,z_2;s)} \tag{69}$$

Thus by dividing by $\sqrt{\rho_2}$ and then letting $\rho_2 \rightarrow 0$, we obtain the new result

$$\delta W(s_2,s_1) = \frac{D^2(s_2,s_1)}{2\pi} PV \int_{CF} \frac{W(s_1,s)W(s_2,s)}{D^2(s_1,s)D^2(s_2,s)} \delta a(s)\, ds \tag{70}$$

to first order in $\delta a(s)$, when $\delta a(s_1) = \delta a(s_2) = 0$.

In principle, this equation lets us begin with any simple crack shape for which W is known (for example, as $W = 1$ along a circular crack) and to sum by integration a sequence of δW associated with infinitesimal $\delta a(s)$ to calculate W for an arbitrary crack shape. To make the procedure work, one must find simple functions of type $\delta_* a(s)$ in Eq 64 above, for which the associated $\delta_* W$ is readily computed, and which have two disposable degrees of freedom to make $\delta a(s) = \delta_* a(s)$ at locations corresponding to all possible pairs of arguments of $W(s_*,s_*')$. One then rewrites Eq 70 analogously to how Eq 63 is rewritten as Eq 65. Fortunately, the simple $\delta_* a(s)$ functions can be provided for cracks in unbounded solids by combining translations, rotation, or scaling, all of which then cause zero change $\delta_* W$ in the function W associated with a fixed pair of phase features along CF.

While the idea is simple, the algebra necessary to write out fully the $\delta_* a(s)$ is complex and is not pursued here, except in the simple variants of Eqs 72 and 76 to follow. The important point is that $W(s,s')$ can be determined, in principle, for complex crack shapes by integration, and once $W(s,s')$ is known through a sequence of crack shapes, $K_1(s)$ due to arbitrary loadings is likewise determined, in principle, by integration. That is, Eqs 63 or 65 and 70 can be regarded as a pair of equations which allow us to solve for K along a complex crack geometry by integrating over a sequence of crack shapes, beginning with a simple one. While that is of great theoretical interest, it remains to be seen if the formulation outlined, when implemented numerically, offers any advantage over more customary methods of three-dimensional crack analysis.

First-Order Expressions for K_I *Along Perturbed Crack Fronts*

Half-Plane Crack—Consider a half-plane crack with front lying along the curve $x = b(z)$, Fig. 6*a*, where $b(z)$ differs slightly from constancy. To calculate $K_I(z_*)$, that is, the K_I at the point along the crack front whose coordinates (x,z) are $[b(z_*),z_*]$, we choose for CF the straight crack front, as in Fig. 2, with a identified as $b(z_*)$. Let $K_I^0[z;a]$ denote the K_I distribution that the applied loads would induce along a straight crack with tip at $x = a$, as in Fig. 2. We regard $K_I^0[z,a]$ as a known function of its two arguments. Thus, with the choice of CF made above, we identify $\delta K(z)$ as $K_I(z) - K_I^0[z;b(z_*)]$, and thus have [7]

$$K_I(z_*) = K_I^0[z_*;b(z_*)] + \frac{1}{2\pi}\,\mathrm{PV}\int_{-\infty}^{+\infty} \frac{K_I^0[z;b(z_*)]}{(z - z_*)^2}\,[b(z) - b(z_*)]\,dz \qquad (71)$$

to first order in $b(z) - b(z_*)$. An analogous expression, accurate to first order, based on Eq 34 was also given [7] for the crack-face opening $\Delta u_y(x,z)$.

The shear modes are addressed in a similar manner, but involve somewhat more com-

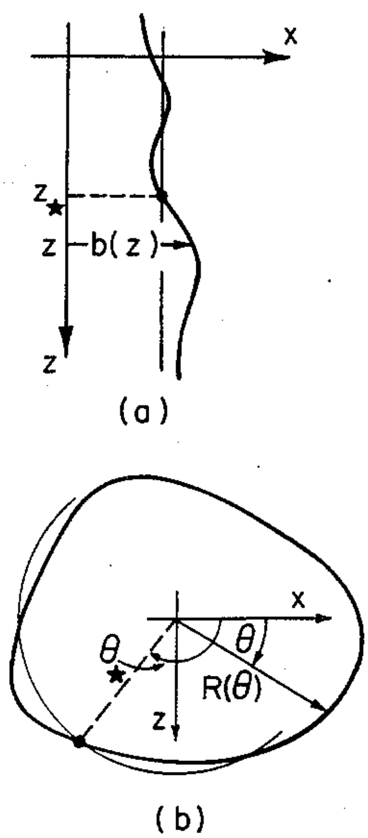

FIG. 6—(a) *Half-plane crack with perturbed front.* (b) *Perturbed circular crack or circular connection.*

plicated expressions because they couple together. The expressions analogous to Eq 71 for K_2 and K_3 are given in Ref *9*.

We may also record a first-order expression, derived from Eq 70 with $W = 1$, for the half-plane crack with tip at $x = b(z)$. Translation and rotation have been used to make vanish the terms corresponding to $\delta a(z)$ at z_1 and z_2. Thus

$$W(z_1,z_2) = 1 + \frac{(z_1 - z_2)^2}{2\pi} \mathrm{PV} \int_{-\infty}^{+\infty} \frac{1}{(z - z_1)^2(z - z_2)^2} \times \left[b(z) - \frac{b(z_2) - b(z_1)}{2(z_2 - z_1)}(2z - z_1 - z_2) - \frac{b(z_2) + b(z_1)}{2}\right] dz \quad (72)$$

to first order in the deviation of $b(z)$ from constancy. A little analysis shows that knowing W to first order enables one to determine $K(z)$ to second order, although this is not pursued further here.

As an example, suppose that

$$b(z) = a_0 + B\cos(2\pi z/\lambda) \quad (73)$$

where $\lambda > 0$ and, for validity of first-order expressions, we assume $2\pi B/\lambda \ll 1$. Then, when the external loadings cause $K_1{}^o[z;a]$ to be independent of z, and then denoted by $K_1{}^o[a]$, we obtain

$$K_1(z) = K_1{}^o[a_0 + B\cos(2\pi z/\lambda)][1 - (\pi B/\lambda)\cos(2\pi z/\lambda)] \quad (74)$$

Expanded consistently to first order in B, this is

$$K_1(z) = K_1{}^o[a_0] + \left\{\frac{dK_1{}^o[a_0]}{da_0} - \frac{\pi}{\lambda}K_1{}^o[a_0]\right\} B\cos(2\pi z/\lambda) \quad (75)$$

The function $W(z_1,z_2)$ associated to first order with this choice of $b(z)$ is

$$W(z_1,z_2) = 1 + \frac{2\pi B}{\lambda}\left[\frac{\sin\eta_2 - \sin\eta_1}{\eta_2 - \eta_1} - \frac{\cos\eta_2 + \cos\eta_1}{2}\right] \quad (76)$$

where $\eta = 2\pi z/\lambda$ and, as remarked, this would actually allow expansion of $K_1(z)$ to order B^2.

Slightly Noncircular Crack—Figure 6*b* shows a slightly noncircular crack, corresponding to radius $R = R(\theta)$ as measured from the adopted coordinate system. Thus, let $K_1{}^o[\theta;a]$ be the intensity factor which the given applied loadings would induce around a circular crack of radius, a (as in Fig. 3*b*). To calculate $K_1(\theta_*)$, we take CF to be the circle of radius $a = R(\theta_*)$. Thus, Eq 63 with $\delta K_1(\theta_*)$ identified as $K_1(\theta_*) - K_1{}^o[\theta_*, R(\theta_*)]$ leads to

$$K_1(\theta_*) = K_1{}^o[\theta_*;R(\theta_*)] + \frac{1}{8\pi}\mathrm{PV}\int_0^{2\pi} \frac{K_1{}^o[\theta;R(\theta_*)][R(\theta)/R(\theta_*) - 1]}{\sin^2[(\theta - \theta_*)/2]} d\theta \quad (77)$$

to first order in $R(\theta) - R(\theta_*)$, as in Ref *11*.

If we consider axisymmetric loading so that $K_I^o[\theta;a]$ is independent of θ, hence denoted $K_I^o[a]$, and consider the perturbation

$$R(\theta) = a_0 + B\cos n\theta \tag{78}$$

(where n is a positive integer and $nB/a_0 \ll 1$), then, when consistently expanded to first order in B, Eq 77 gives a result identical to Eq 75 when we replace z with $a_0\theta$ and identify λ as $2\pi a_0/n$.

Gao and Rice [*11*] tested the range of validity of Eq 77 by using it to calculate K_I along an elliptical crack and found good agreement with the exact solution up to 2:1 aspect ratios. Comparable accuracy was found for displacements Δu_y estimated by the first order formula based on Eq 34. The estimate of K_I was found to remain good up to aspect ratios approaching 5:1 when the reference circular CF was given a radius equal to that of the minor semi-axis of the ellipse (maximum inscribed circle), with the center of the reference circle then being shifted, in general, from that of the ellipse to make $\delta a = 0$ at a location of interest.

Corresponding results for shear loading of cracks perturbed from a circle are given by Gao [*15*]. An alternate, approximate approach to planar tensile cracks of arbitrary noncircular and nonelliptical shape has been given by Fabrikant [*36*].

Slightly Noncircular Connection—Now let a connection, joining two half-spaces, occupy the region shown in Fig. 6*b*, with boundary $R = R(\theta)$. Let $K_I^o[\theta;a]$ be the intensity factor which the given loadings induce when the connection is circular, of radius a, as in Fig. 3*c*. As already suggested by Eqs 42 and 43, in this case the results depend on the conditions of restraint (or its lack) that one assumes for the remote displacements in going from the circular reference shape to the actual shape. Thus [*12*]

$$K_I(\theta_*) = K_I^o[\theta_*;R(\theta_*)] + \frac{1}{8\pi}\,\mathrm{PV}\int_0^{2\pi} \frac{W(\theta_*,\theta)K_I^o[\theta;R(\theta_*)][1 - R(\theta)/R(\theta_*)]}{\sin^2[(\theta - \theta_*)/2]}\,d\theta \tag{79}$$

to first order in $R(\theta) - R(\theta_*)$, where

$$W(\theta_*,\theta) = 1 + 4\sin^2[(\theta - \theta_*)/2][1 + 6\cos(\theta - \theta_*)] \tag{80}$$

from Eq 43 when remote points are unrestrained, where the $6\cos(\theta - \theta_*)$ in the final bracket is dropped when remote points cannot rotate but can move in the y direction, where the 1 in the final bracket is dropped when remote points on a y axis through the center of the reference circle cannot displace in the y direction but can rotate, and from Eq 42 the entire final bracket is dropped when remote points are constrained against any motion.

For axisymmetric loading, in the sense that $K_I^o[\theta;a]$ is independent of θ, and denoted $K_I^o[a]$, the perturbation

$$R(\theta) = a_0 - B\cos n\theta \tag{81}$$

(minus sign, compared to Eq 78, so that positive B corresponds to crack growth) leads to

$$K_I(\theta) = K_I^o[a_0] - \left\{\frac{dK_I^o[a_0]}{da_0} + \frac{n_I}{2a_0}K_I^o[a_0]\right\} B\cos n\theta \tag{82}$$

Here, in dealing with remote displacement restraint cases in the order given after Eq 80, one has the following: unrestrained, $n_1 = -3$ when $n = 1$ and $n_1 = n + 2$ otherwise; restrained against rotation only, $n_1 = n + 2$; restrained against y direction displacement along axis only, $n_1 = -5$ when $n = 1$ and $n_1 = n$ otherwise; totally restrained, $n_1 = n$.

Finite-Element Analog—Following deKoning and Lof [37], in three-dimensional finite-element studies of cracked solids, the crack-front position is specified by corner-node positions for the string of elements along CF, and a stress intensity factor $K_1^{(1)}, K_1^{(2)}, \ldots, K_1^{(m)}$ may be associated with each such node, the latter regarded as being defined [37] as in Eq 60 from the calculated near-tip opening, Δu_y, along a perpendicular to CF at the associated corner node. Parameters, $a_1, a_2, \ldots, a_m$ characterize crack front position; they vanish along CF and correspond to outward shifts of the corner-node positions along perpendiculars to CF. The linearized form

$$K_1^{(i)} = [K_1^{(i)}]^{(0)} + [\partial K_1^{(i)}/\partial a_j]^{(0)} a_j \tag{83}$$

is now considered, where the superscript (0) means that the quantity is evaluated with all $a_i = 0$, that is, for the crack front along CF.

The $\partial K_1^{(i)}/\partial a_j$ are calculated from $\partial(\Delta u_y)/\partial a_j$ at fixed distances from the moving crack tip nodes, and these displacement derivatives at $a_i = 0$ are calculated directly from the inverse of the stiffness matrix for the unperturbed crack geometry and from certain quantities that are calculated in the stiffness derivative procedure [37]. In fact, displacement derivatives calculated in this way form the discretized weight functions in the Parks and Kamenetzky [5] finite element formulation, so that the deKoning and Lof procedure seems to be a discretized version of the steps presented here for calculating K_1 to first order along perturbed crack fronts.

Application to Crack Trapping—Crack trapping arises in brittle crack advance through solids of locally heterogeneous fracture toughness. The front advances nonuniformly and has segments which are trapped, at least temporarily, by contact with tough obstacles whose K_{1c} exceeds the local K_1.

In an idealized model of the process the crack may be considered planar, as here, but with a wavy front as in Figs. 6*a* and 6*b*. Important problems are to predict, in a statistical sense, the configuration of the crack front and the far-field K_1 or load level which just enables growth through the heterogeneous fracture resistance. An elementary approach [16] can be based on the first order perturbation results of Eqs 71, 77, 79, and 83. This will not suffice for all problems. For example, with sufficiently tough local obstacles, crack-front segments will tend to bow out between obstacles and ultimately join with neighboring bows, to advance forward as a single crack whose surfaces remain bridged by uncracked obstacles left behind. Description of such processes lie outside the range of the first-order perturbation and will require the integration of perturbation effects along a sequence of crack shapes as discussed following Eq 70.

For cases with sufficiently tame crack fronts, amenable to the first-order perturbation approach, Eqs 71, 77, and 79 become singular integral equations for crack-front locations $b(z)$ or $R(\theta)$. Each point along the crack front is either active (K_1 = local K_{1c}) or trapped ($K_1 <$ local K_{1c}). In general, if we start with an initial half-plane crack front and increase the load, $K_1(z)$ is known as a function of $b(z)$ (on which the local K_{1c} depends) in active segments which are taking part in growth; $b(z)$ is unknown there. Also, $b(z)$ is known where the crack front remains trapped, while $K_1(z)$ is unknown there. Thus, Eq 71 can be regarded as an integral equation whose solution gives crack-front shape $b(z)$. Complications are that the active and trapped zone locations are not known *a priori,* that a previously active zone may later become trapped, and that the problem will not always have a solution (for example, when conditions are met for part of the crack front to jump forward dynamically, either to a new array of trapping obstacles or as the start of final fracture).

The simplest case is for the half-plane crack with $K_I^o[z,a]$ independent of z and, for the size of excursion $b(z)$ considered, of a, too. Hence it is written simply as K_I^o, effectively dependent on applied load only, and then Eq 71 becomes, after integration by parts [9]

$$K_I(z_*)/K_I^o = 1 + \frac{1}{2\pi}\,\mathrm{PV}\int_{-\infty}^{+\infty} \frac{db(z)/dz}{z - z_*}\,dz \tag{84}$$

This is of the same form as for plane stress of a thin elastic sheet occupying the x,z plane, where loading of the sheet is by remotely uniform stress σ_{xx}^o and its entire z axis can open up by an x-direction displacement gap of distribution $\Delta(z)$, hence creating nonuniform stress $\sigma_{xx}(z)$ at points along the z axis. In that case, $\sigma_{xx}(z)/\sigma_{xx}^o$ corresponds to $K_I(z)/K_I^o$ in Eq 84, and $(1 + \nu)\mu\Delta(z)/\sigma_{xx}^o$ to $b(z)$. Thus, prescribing $K_I(z)$ along active portions of the crack front is equivalent to prescribing $\sigma_{xx}(z)$ in the plane stress problem over the same segment of z axis, which is then regarded as a cracked segment. Prescribing $b(z)$ along the trapped portion of crack front corresponds to prescribing the opening gap $\Delta(z)$ over that segment of the z axis.

As a simple example [16], suppose the segment $-H < z < +H$ is active, with $K_I = K_{Ic}$, but that segment $H < z < 2L - H$ is completely trapped with $b(z) = 0$. Let these active and trapped zones alternate periodically, with period $2L$, and symmetry relative to $z = 0$ (middle of an active zone) and $z = L$ (middle of a trap). The problem is analogous to that of a periodic infinite array of collinear cracks in plane stress [38]. Thus, the solution, for example, for the average b_{pen} of crack penetration $b(z)$ over an active zone like $-H < z < +H$, is

$$b_{pen} = (4L^2/\pi H)(1 - K_{Ic}/K_I)\ln[1/\cos(\pi H/2L)] \tag{85}$$

When $K_{Ic} = 1.5K_{Ic}$, this equation predicts $b_{pen}/2H = 0.49$ when $(L - H)/L$, the line fraction of traps, is 0.1 and $b_{pen}/2H = 0.29$ when the line fraction is 0.5. The maximum penetration $b(0)$ for these two cases is only about 25% above b_{pen}.

Configurational Stability in Quasi-Static Crack Growth

Suppose that a tensile crack grows quasistatistically (say, by $R = 0$ fatigue or sustained load stress corrosion) under elastic fracture-mechanics conditions, such that the rate of crack growth is an increasing function of K_I. Further, suppose that the nature of the loading is such that K_I is uniform along the crack front initially, and that a solution exists to the combined equations of elasticity and quasistatic crack growth such that crack can grow while maintaining a uniform K_I along its front.

Such would be the case for homogeneous materials containing half-plane cracks with initially straight fronts, loaded under plane-strain conditions (K_I independent of z), and also for homogeneous materials with circular cracks or connections under axisymmetric loading (K_I independent of θ). In these respective cases, a solution exists such that the crack continues to grow with a straight or circular front.

Here we review conditions for such crack fronts to be configurationally stable. That is, if the crack front starts off slightly deviated from that ideal shape, will those deviations grow or diminish as the crack enlarges? Considering trigometric deviations as in Eqs 73, 78, and 81, stability requires that K_I be smallest where the crack has advanced most and largest where the crack has advanced least, for then the corresponding variations in the growth rate will tend to diminish the amplitude of the crack front fluctuations. Conversely, if K_I is greatest where the crack has advanced most, and smallest where least, then the variation in growth rates causes the fluctuations in crack front position to grow in amplitude.

Thus, since B times the cosine term measures advance of the crack front in Eqs 73, 78, and 81, for configurational stability we require that the coefficient of B times the cosine term in the resulting expressions for K_I be negative. Thus, assuming $K_I^o[a_0] > 0$ and letting

$$Q(a_0) = \{dK_I^o[a_0]/da_0\}/K_I^o[a_0] \tag{86}$$

one has from Eqs 75 and 82 that for configurational stability

$$Q(a_0) < \pi/\lambda \tag{87}$$

for a half-plane crack subject to perturbations of wavelength $\lambda(>0)$ [7],

$$2a_0 Q(a_0) < n \tag{88}$$

for a circular crack with perturbation having $n(>0)$ wavelengths on its circumference ($\lambda = 2\pi a_0/n$) [11], and

$$-2a_0 Q(a_0) < n_1 \tag{89}$$

for a circular connection with perturbations having n wavelengths on the circumference [12], where n_1 is the function of n, dependent on conditions of remote displacement restraint, given after Eq 82. When the inequalities are reversed, Eqs 87, 88, and 89 become conditions for configurational instability.

These expressions show that growth with negative Q for the half-plane and circular crack is always configurationally stable. Also, there is inevitably stability against perturbations of short wavelength λ or high n. Thus, if there is to be configurational instability, it will tend to develop at long wavelengths or low n.

For an edge crack of depth a_0 in a remotely stressed half space, $Q = 1/2a_0$, and thus Eq 87 predicts configurational stability unless λ is so great as to exceed $2\pi a_0$. This could be important for initially short cracks, but the value of λ is so great compared to crack length that no reliance can then be put on predictions based on the half-plane model.

For circular cracks under remote tension, $2a_0Q = 1$ so that by Eq 88 there is neutral stability when $n = 1$ (corresponding to translational shift of the crack front) and stability for all higher n. For a crack loaded by pressure [11], $p = p(R) > 0$ dependent on distance R from the center, there is stability to all n if $dp/dR < 0$. If $dp/dR > 0$, the translational shift mode ($n = 1$) is unstable, and if p increases rapidly enough with R (as happens for p = constant × R), higher n become unstable too, such as $n = 2$ corresponding to an elliptical-like growth mode Note that these unstable growth modes represent effects, presumed to be small at least initially, superposed on the basic axisymmetric growth mode.

Similarly, for the circular connection under remote axially symmetric tensile loading [12], there is stability when $n > 1$. However, the $n = 1$ translational shift mode is neutrally stable [12] when the remote points are constrained against rotation (this means that the crack grows as a circle but does not necessarily remain of fixed center). For loading only by a remotely imposed force, centered on the circular connection, with no restraint against remote rotation, the translational shift mode is unstable. The crack initially tends to grow in a circle but of shifting center, and this induces an unfavorable moment relative to the shifting center which aggravates the effect and ultimately leads to strongly noncircular growth [*12*].

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