Crack Paralleling an Interface Between Dissimilar Materials

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A crack paralleling a bonded plane interface between two dissimilar isotropic elastic solids is considered. When the distance of the crack from the interface is small compared to the crack length itself and to other length scales characterizing the geometry, a simple universal relation exists between the Mode I and Mode II stress intensity factors and the complex stress intensity factor associated with the corresponding problem for the crack lying on the interface. In other words, if the influence of external loading and geometry on the interface crack is known, then this information can immediately be used to generate the stress intensity factors for the sub-interface crack. Conditions for cracks to propagate near and parallel to, but not along, an interface are derived.

1 Introduction

Bonded interfaces between dissimilar elastic materials often separate by cracking, as would be expected if the toughness of the interface is low compared to that of the abutting materials. In some instances cracking is observed to occur approximately parallel to the interface but with the crack lying entirely within one of the two materials. The aim of this paper is to analyze subinterface cracks which parallel the interface and to examine conditions under which they might be expected.

The mathematical problem which is analyzed is introduced in Fig. 1. Each material is taken to be isotropic and linearly elastic. The interface lies along the x_1 axis with material #1 lying above and #2 below. Plane strain deformations are considered. Attention will be restricted to subinterface cracks which lie below the interface at a distance h which is small compared to the length of the crack L and to all other relevant geometric length quantities in the problem. As indicated in Fig. 1, we will consider the asymptotic problem for the semi-infinite subinterface crack. The remote field in the asymptotic problem is prescribed to be the near-tip field of the interface crack problem (everywhere but in material #2 between the crack and the interface). That is, the solution to the subinterface crack problem at any point a fixed distance from the tip approaches the solution to the corresponding interface crack problem as $h \to 0^+$ with L fixed. Thus, at distances from the tip which are large compared to h and small compared to L, the near-tip field of the interface crack problem pertains. Posing the problem in this manner permits us to develop a universal relation between the Mode I and II stress intensity factors of the subinterface crack and the corresponding "complex" stress intensity factor of the interface crack. This relation is otherwise independent of loading, crack length, and external geometry.

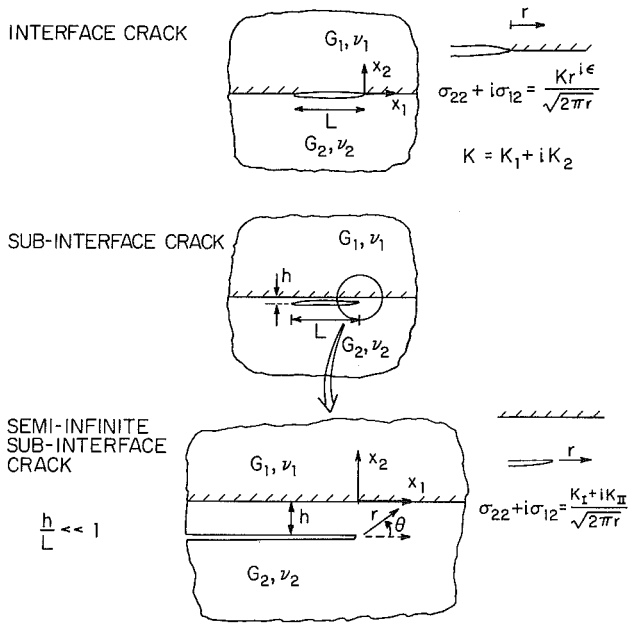

Fig. 1 Relation of asymptotic subinterface crack problem to interface crack problem

With the universal relation in hand, we examine conditions under which propagation of a parallel sub-interface crack should be expected. When conditions do favor such cracks, the analysis predicts the separation distance from the interface.

2 Formulation and Solution

The singular near-tip field of the *interface crack problem*

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Table 1 Values of $\phi(\alpha, \beta)$

β \ α	−.8	−.6	−.4	−.2	−.1	0	.1	.2	.4	.6	.8
−.4	−.1202	−.0801	−.0467	−.0186	−.0060	.0058	.0168	.0273	.0468	.0653	.0810
−.2	−.1814	−.1162	−.0708	−.0351	−.0197	−.0056	.0075	.0197	.0419	.0618	.0798
−.1	−.2057	−.1281	−.0764	−.0368	−.0199	−.0046	.0096	.0227	.0465	.0675	.0865
−.05	−.2167	−.1328	−.0779	−.0363	−.0187	−.0027	.0120	.0256	.0501	.0718	.0912
−.02	−.2229	−.1354	−.0785	−.0356	−.0176	−.0012	.0139	.0277	.0527	.0748	.0946
0	−.2270	−.1369	−.0787	−.0350	−.0167	0	.0153	.0293	.0547	.0770	.0970
.02	−.2309	−.1384	−.0788	−.0343	−.0156	.0013	.0168	.0311	.0567	.0793	.0995
.05	−.2366	−.1403	−.0787	−.0330	−.0138	.0035	.0193	.0339	.0601	.0830	.1035
.1	−.2456	−.1431	−.0780	−.0301	−.0101	.0079	.0243	.0393	.0663	.0900	.1110
.2	−.2620	−.1468	−.0744	−.0219	−.0003	.0191	.0367	.0528	.0815	.1065	.1287
.4	−.2902	−.1449	−.0566	.0055	.0307	.0531	.0733	.0917	.1242	.1522	.1769

(England, 1965; Erdogan, 1965; Rice and Sih, 1965) gives rise to tractions directly ahead of the tip ($\theta = 0$) given by

$$\sigma_{22} + i\sigma_{12} = K(2\pi r)^{-1/2} r^{i\epsilon} \tag{2.1}$$

where $K = K_1 + iK_2$ is the complex stress intensity factor, $i = \sqrt{(-1)}$, and

$$\epsilon = \frac{1}{2\pi} ln\left[\frac{G_1 + G_2(3-4\nu_1)}{G_2 + G_1(3-4\nu_2)}\right] \tag{2.2}$$

where G is the shear modulus and ν is Poisson's ratio. Here $K = (k_1 + ik_2)\sqrt{\pi}\cosh \pi\epsilon$ where $k_1 + ik_2$ is the complex intensity factor as originally introduced by Rice and Sih (1965). The $\sqrt{\pi}$ is standard in converting the lower case k's of that period to K's; we include the factor $\cosh \pi\epsilon$ so that the magnitude of the traction vector on the interface is given by $\sqrt{(\sigma_{22}^2 + \sigma_{12}^2)} = |K|/\sqrt{2\pi r}$, analogously to the homogeneous material case. The associated crack face displacements a distance r behind the tip are given by

$$\delta_2 + i\delta_1 = 2\,\frac{[(1-\nu_1)/G_1 + (1-\nu_2)/G_2]}{(1+2i\epsilon)\cosh \pi\epsilon} K(r/2\pi)^{1/2} r^{i\epsilon} \tag{2.3}$$

where $\delta_\alpha = u_\alpha(-r, 0^+) - u_\alpha(-r, 0^-)$. The fact that equation (2.3) predicts interpenetration in a (usually) small neighborhood of the crack tip is not relevant in the present context. Dimensional considerations dictate that K must be of the form

$$K = (\text{applied stress}) \times (\sqrt{L}\, L^{-i\epsilon}) \times f \tag{2.4}$$

where L is a length quantity such as crack length and f is a nondimensional possibly complex function of dimensionless combinations of the material moduli and the geometric parameters. Two specific examples will be given in Section 3. The energy release-rate (per unit extension along the interface per unit length of crack front) is

$$\mathcal{G} = \left[\frac{(1-\nu_1)/G_1 + (1-\nu_2)/G_2}{4\cosh^2 \pi\epsilon}\right] K\bar{K} \tag{2.5}$$

where $\bar{}$ denotes the complex conjugate.

The tractions on the line directly ahead of the *subinterface crack* tip satisfy

$$\sigma_{22} + i\sigma_{12} = (K_{\mathrm{I}} + iK_{\mathrm{II}})(2\pi r)^{-1/2} \tag{2.6}$$

where K_{I} and K_{II} are the standard Mode I and Mode II stress intensity factors. The energy release-rate is

$$\mathcal{G} = \left[\frac{1-\nu_2}{2G_2}\right](K_{\mathrm{I}}^2 + K_{\mathrm{II}}^2) \tag{2.7}$$

As discussed earlier, the remote stresses in the *semi-infinite subsurface crack problem* are required to approach (for all θ but $\theta = \pi$) the characteristic Williams singular field of the interface crack, which can be written as

$$\sigma_{\alpha\beta} = \mathrm{Re}[K(2\pi r)^{-1/2} r^{i\epsilon} \tilde{\sigma}_{\alpha\beta}(\theta)] \tag{2.8}$$

with universal (complex) angular dependence $\tilde{\sigma}_{\alpha\beta}(\theta)$ for a given material pair. The remote crack face displacements approach equation (2.3). The only length quantity in the semi-infinite sub-interface crack problem is h. From dimensional considerations and by linearity it follows that

$$K_{\mathrm{I}} + iK_{\mathrm{II}} = cKh^{i\epsilon} + d\bar{K}h^{-i\epsilon} \tag{2.9}$$

where c and d are dimensionless complex constants depending only on dimensionless combinations of the moduli of the materials. The depth of the crack below the interface must appear as the factor $h^{i\epsilon}$ to combine with $L^{-i\epsilon}$ in equation (2.4) as the dimensionless term $(h/L)^{i\epsilon}$.

By considering a unit advance of the semi-infinite crack, one concludes by an energy argument, or equivalently by application of the J integral, that the energy release-rate given by equation (2.7) must be equal to that given in equation (2.5). That is

$$K_{\mathrm{I}}^2 + K_{\mathrm{II}}^2 = q^2 K\bar{K} \tag{2.10}$$

where

$$q = \left\{\frac{1}{2\cosh^2 \pi\epsilon}\left[\frac{G_2(1-\nu_1)}{G_1(1-\nu_2)} + 1\right]\right\}^{1/2} \tag{2.11}$$

Using an argument similar to that of Thouless et al. (1987), one can show that $d = 0$, and then substitution of equation (2.9) into (2.10) gives $c\bar{c} = q^2$ and thus

$$c = qe^{i\phi} \tag{2.12}$$

so that the relation (2.9) is fully determined apart from the single dimensionless function ϕ of the elastic moduli. A further simplification is achieved when use is made of Dundurs' (1969) observation that for problems of this class the moduli dependence can be expressed in terms of just two (rather than three) special nondimensional combinations. In plane strain, Dundurs' parameters are

$$\alpha = \frac{G_1(1-\nu_2) - G_2(1-\nu_1)}{G_1(1-\nu_2) + G_2(1-\nu_1)} \tag{2.13}$$

and

$$\beta = \frac{1}{2}\,\frac{G_1(1-2\nu_2) - G_2(1-2\nu_1)}{G_1(1-\nu_2) + G_2(1-\nu_1)} \tag{2.14}$$

where the roles of 1 and 2 are switched from Dundurs' definitions. These parameters vanish for identical materials across the interface and they change sign when the materials are switched. The quantities ϵ and q can be reexpressed as

$$\epsilon = \frac{1}{2\pi} ln\left[\frac{1-\beta}{1+\beta}\right] \tag{2.15}$$

and

$$q = \left[\frac{1-\beta^2}{1+\alpha}\right]^{1/2} \tag{2.16}$$

Table 2 Some special systems:

SYSTEM ①/②	α	β	ε	φ
Al_2O_3/Cu	.51	.089	−.028	.078
Cu/Al_2O_3	−.51	−.089	.028	−.104
MgO/Au	.51	.011	−.0036	.069
Au/MgO	−.51	−.011	.0036	−.108
Si/Cu	.094	−.033	.0105	.012
Cu/Si	−.094	.033	−.0105	−.014
MgO/Ni	.14	−.015	.0049	.020
Ni/MgO	−.14	.015	−.0049	−.023
Al_2O_3/Ti	.56	.12	−.039	.089
Ti/Al_2O_3	−.56	−.12	.039	−.114
Al_2O_3/Nb	.57	.060	−.019	.081
Nb/Al_2O_3	−.57	−.060	.019	−.122

	G	ν		G	ν
Au:	0.293×10^{11} N/M^2	0.417	Cu:	0.478	0.345
Ti:	0.434	0.322	Al_2O_3:	1.792	0.207
Ni:	0.808	0.314	Nb:	0.377	0.392
MgO:	1.283	0.175	Si:	0.688	0.220

An integral equation formulation of the semi-infinite subinterface crack problem is given in the Appendix. Numerical solution of the integral equation for various combinations of α and β has been carried out, the numerical values for $\phi(\alpha, \beta)$ are presented in Table 1. As discussed in the Appendix, the accuracy of these numerical results is believed to be within a small fraction of a percent. For sufficiently small α and β the linear approximation (obtained by a numerical fit in the range of small α and β)

$$\phi = 0.1584\,\alpha + 0.0630\,\beta \tag{2.17}$$

provides an adequate estimate of ϕ. For example, with $\alpha = 0.05$ and $\beta = 0.005$ the error of this formula is only 1.3 percent while with $\alpha = 0.14$ and $\beta = -0.015$ it is 6.7 percent.

Combining equations (2.12) and (2.9) gives the basic result for the stress intensity factors of the subinterface crack in terms of the complex stress intensity factor of the corresponding interface crack for conditions when $(h/L) \ll 1$:

$$K_{\mathrm{I}} + iK_{\mathrm{II}} = qe^{i\phi}Kh^{i\epsilon} \tag{2.18}$$

Note that $K_{\mathrm{I}} = K_1$ and $K_{\mathrm{II}} = K_2$ when α and β both vanish.

3 Applications and Implications

Moduli and values of α, β, ϵ, and ϕ are presented for six representative material combinations in Table 2. The shear modulus and Poisson's ratio listed for each material are polycrystalline values derived from Simmons and Wang (1971). The values for the cubic materials are the average of the Hashin-Shtrikman bounds and the values for the noncubic materials are the average of the Voigt and Reuss bounds. The largest values of ϵ are attained by the Cu/Al_2O_3 systems.

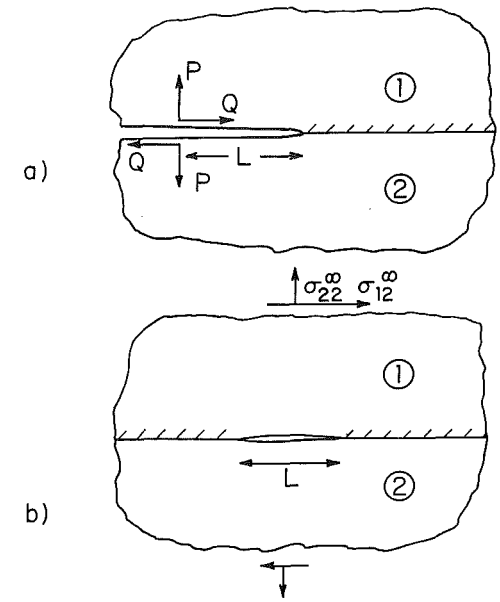

Fig. 2 Two basic interface crack problems

Solutions to a number of different interface crack problems exist in the literature. See Atkinson (1979) and Park and Earmme (1986) for recent discussions. The complex intensity factors for the two basic problems (Erdogan, 1965; Rice and Sih, 1965) shown in Fig. 2 will serve for discussion purposes here. For the semi-infinite crack along the interface between two elastic half-spaces and loaded by concentrated loads (per unit thickness) a distance L behind the tip,

$$K = (P + iQ)\,(\pi L/2)^{-1/2} L^{-i\epsilon} \cosh\, \pi\epsilon \tag{3.1}$$

In the case of a finite crack of length L on the interface between two half-spaces which are subject to remote stresses σ_{22}^{∞} and σ_{12}^{∞}, the complex intensity factor at the right tip is

$$K = (\sigma_{22}^{\infty} + i\sigma_{12}^{\infty})(1 + 2i\epsilon)(\pi L/2)^{1/2} L^{-i\epsilon} \tag{3.2}$$

(See the discussion in Rice and Sih for the behavior of the remote component σ_{11}. Those authors also gave solutions for a periodic row of collinear cracks along the interface of a solid under remote stressing, and for concentrated loads on the faces of a finite length crack.)

In all cases K can be written as

$$K = |K|\, e^{i\gamma} L^{-i\epsilon} \tag{3.3}$$

since $|L^{-i\epsilon}| = |e^{-i\epsilon \ln L}| = 1$. Then, by equation (2.18), the intensity factors of the semi-infinite subinterface crack are

$$K_{\mathrm{I}} = q\,|K| \cos[\gamma + \phi + \epsilon\, \ln(h/L)] \tag{3.4}$$

and

$$K_{\mathrm{II}} = q\,|K| \sin[\gamma + \phi + \epsilon\, \ln(h/L)] \tag{3.5}$$

Before asking what value of h/L is consistent with crack growth parallel to the interface, we first examine an interesting feature of the solution for arbitrary small values of h/L. If the collection of terms $\phi + \epsilon\, \ln(h/L)$ is small, as might easily be the case judging from the systems listed in Table 2, then the stress intensity factors are well approximated by

$$K_{\mathrm{I}} + iK_{\mathrm{II}} \cong qL^{i\epsilon}K = q\,|K|\, e^{i\gamma} \tag{3.6}$$

For example, in the case of the concentrated wedge force (3.1)

$$K_{\mathrm{I}} + iK_{\mathrm{II}} \cong q\,(P + iQ)\,(\pi L/2)^{-1/2} \tag{3.7}$$

assuming ϵ itself is small. Apart from the factor q, this is just

the classical result when no material discontinuity occurs. Similarly, the result for the finite crack (3.2) becomes

$$K_I + iK_{II} \cong q(\sigma_{22}^{\infty} + i\sigma_{12}^{\infty})(\pi L/2)^{1/2} \tag{3.8}$$

which is again the classical result multiplied by q. In other words, in these examples when ϵ and $\phi + \epsilon \ln(h/L)$ are both small, the ratio of the Mode II to Mode I stress intensity factors is the same as in the corresponding classical problem but the energy release-rate is that of the interface crack as reflected by the factor q.

Now consider situations in fatigue, stress corrosion or under monotonic loading when the crack will tend to advance in its own plane approximately parallel to the interface. Assuming the fracture properties of material #2 are homogeneous along with its moduli, the crack will only advance in its plane if $K_{II} = 0$. If $K_{II} > 0$ it will tend to deflect downward away from the interface, while if $K_{II} < 0$ it will tend to grow upward. By equation (3.5), the condition for the crack to advance parallel to the interface in pure Mode I is

$$\sin[\gamma + \phi + \epsilon \ln(h/L)] = 0 \tag{3.9}$$

or

$$\gamma + \phi + \epsilon \ln(h/L) = 2\pi n; \quad n = 0, \pm 1, \ldots \tag{3.10}$$

with the associated Mode I intensity

$$K_I = q|K| \tag{3.11}$$

Values of h/L from equations (3.10) are

$$h/L = \exp[(2\pi n - \gamma - \phi)/\epsilon]; \quad n = 0, \pm 1, \ldots \tag{3.12}$$

but only those values (if any) will be physically meaningful which are small compared to unity but not so small that the parts of the crack faces make contact, as will be discussed below. The crack length L increases as the crack advances and thus h cannot remain strictly constant. However, if h at the tip satisfies equations (3.12) approximately as L increases the slope dh/dL of the crack, the path will be small (and equal to the value given by equations (3.12)), with the crack thus nearly paralleling the interface when h/L is small.

As an illustration, consider the symmetric wedge loading ($Q = 0$) of the geometry in Fig. 2(a). By equations (3.1) and (3.3), $\gamma = 0$. For the material systems listed in Table 2, the largest magnitude of ϵ is 0.04, and it is readily seen that the only physically meaningful solution from equations (3.12), if any, is that associated with $n = 0$, i.e.,

$$h/L = \exp[-\phi/\epsilon] \tag{3.13}$$

Of the systems in Table 2, only Cu/Si, Si/Cu, Ni/MgO, and MgO/Ni have positive values of ϕ/ϵ and might therefore propagate a subinterface crack of the kind envisioned here for this particular geometry and loading. For Cu/Si, $h/L = 0.26$; while for Si/Cu, $h/L = 0.32$. The accuracy of these estimates may be somewhat questionable since they probably lie outside the range of h/L where the asymptotic analysis is accurate. For Ni/MgO, $h/L = 0.009$ and for MgO/Ni, $h/L = 0.017$, and these estimates should be accurate. Evidently the crack could satisfy a $K_{II} = 0$ criterion by propagating near the interface in either phase. We do not investigate here the configurational stability of those paths but expect, following Cotterell and Rice (1980), that only a path with a negative crack-parallel nonsingular stress term at the tip is stable. The conclusions for a finite crack paralleling the interface in Fig. 2(b) under remote tensile loading ($\sigma_{12}^{\infty} = 0$) are similar. Now, $\gamma \cong 2\epsilon$ and $h/L = \exp[-2 - \phi/\epsilon]$. Only the systems noted above will permit propagation of the subinterface crack parallel to the interface under the pure tensile loading. In general, however, it is important to note that the possibility of propagating a subinterface crack depends on both the material properties *and* the loading combination, so that subinterface cracks in the other systems may occur for other geometries and loadings.

The discussion and the analysis given above assume that contact between the crack faces of the subinterface crack does not occur. In applications where the near tip conditions of the subinterface crack is in pure Mode I and where h/L is not ludicrously small, it is unlikely that contact of the crack faces will be an issue. If the corresponding interface crack problem does indicate contact well away from the tip, at distances as large as h or greater, then the possibility of contact in the subinterface crack problem should certainly be checked. Solutions to (3.12) for h/L are only physically meaningful when h/L is not so small that contact will certainly occur or, what is more likely, that h is not so small that the crack lies so close to the interface that the material at the tip has properties which are affected by the existence of the interface.

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APPENDIX

Integral Equation Formulation and Solution

A dislocation formulation of the integral equation for the semi-infinite crack problem of Fig. 1 is used. The general procedure for formulating such problems is outlined by Bilby and Eshelby (1968) and Rice (1968) and the formulation and solution of a similar problem was given by Thouless et al. (1987).

The depth h of the crack below the interface will be taken to be unity here since the dependence of the solution on h is explicitly given in the body of the paper. With reference to Fig. 1, let $x = x_1$ and $y = x_2 + 1$ be coordinates centered at the crack tip with $y = 0$ parallel to the interface. Let $b_x(\xi)$ and $b_y(\xi)$ be the x and y components of an edge dislocation located on the x axis at $x = \xi$. This problem was first solved by Head (1953). At a point $(x, 0)$ the traction on a plane parallel to the x axis induced by the dislocation at $(\xi, 0)$ can be compactly derived us-

ing Muskhelishvili methods for plane strain elasticity and is given by

$$\sigma_{yy}(x)+i\sigma_{xy}(x)=2\bar{B}(\xi)(x-\xi)^{-1}+B(\xi)H_1(x-\xi)+\bar{B}(\xi)H_2(x-\xi) \quad (A.1)$$

where $i=\sqrt{(-1)}$, $\bar{}$ denotes the complex conjugate, and where

$$H_1(\zeta)=-16\delta\zeta/(4+\zeta^2)^2$$

$$H_2(\zeta)=-[(\lambda+\delta)\zeta-2i(\delta-\lambda)]/(4+\zeta^2)+8\delta/(\zeta-2i)^3$$

$$B(\xi)=G_2[b_x(\xi)+ib_y(\xi)]/[4\pi i(1-\nu_2)]$$

Here,

$$\delta=(\beta-\alpha)/(\beta+1)$$

and

$$\lambda=(\alpha+\beta)/(\beta-1)$$

where α and β are defined by equations (2.13) and (2.14).

The distribution of dislocations is chosen to give zero tractions on $y=0$ for $x<0$, i.e., $B(\xi)$ with b_x+ib_y now reinterpreted to correspond to a dislocation density must satisfy

$$\int_{-\infty}^{0}\{B(\xi)H_1(x-\xi)+\bar{B}(\xi)[2(x-\xi)^{-1}+H_2(x-\xi)]\}d\xi=0 \quad (A.2)$$

where the contribution with the $(x-\xi)^{-1}$ integrand is the Cauchy principal value integral.

The integral equation (A.2) is supplemented by the condition that the crack opening displacements far from the tip are the same as those for the interface crack. That is for large negative ξ, from equation (2.3),

$$b_x(\xi)+ib_y(\xi)=-\frac{d}{d\xi}[\delta_x(\xi)+i\delta_y(\xi)]$$

$$=\frac{[(1-\nu_1)/G_1+(1-\nu_2)/G_2]}{\cosh\,\pi\epsilon}\frac{i\bar{K}(-\xi)^{-i\epsilon}}{\sqrt{2\pi(-\xi)}} \quad (A.3)$$

and, therefore, as $\xi\to-\infty$

$$B(\xi)=(2\pi)^{-3/2}(1-\beta^2)^{1/2}(1+\alpha)^{-1}\bar{K}(-\xi)^{-1/2-i\epsilon} \quad (A.4)$$

To put the integral equation into a form suitable for numerical solution, make the change of variables

$$x=(u-1)/(u+1),\quad -1<u<1$$

$$\xi=(t-1)/(t+1),\quad -1<t<1$$

and let

$$\zeta\equiv x-\xi=2(u-t)/[(u+1)(t+1)]$$

Then, with $A(t)=B(\xi)$, equation (A.2) can be reduced to

$$\int_{-1}^{1}\bar{A}(t)(u-t)^{-1}dt+\int_{-1}^{1}\{A(t)H_1(\zeta)+\bar{A}(t)[1+t+H_2(\zeta)]\}(1+t)^{-2}dt=0 \quad (A.5)$$

The approximation for $A(t)$ was taken as

$$A(t)=(2\pi)^{-3/2}\frac{\bar{K}(1-\beta^2)^{1/2}}{2^{i\epsilon}(1+\alpha)}\frac{(1+t)^{1/2+i\epsilon}}{\sqrt{1-t}}+\frac{(1+t)}{\sqrt{1-t}}\sum_{k=1}^{N}c_kT_{k-1}(t) \quad (A.6)$$

where the c's are complex coefficients which must be obtained by the solution process and $T_j(t)$ is the Chebyshev polynomial of the first kind of degree j. The lead term in equation (A.6) gives the correct asymptotic behavior (A.4) as $\xi\to-\infty$ or, equivalently, as $t\to-1$. The stress intensity factors are given by

$$K_I+iK_{II}=(2\pi)^{3/2}\lim_{\xi\to 0-}\{(-\xi)^{1/2}\bar{B}(\xi)\}$$

or

$$K_I+iK_{II}=K(1-\beta^2)^{1/2}(1-\alpha)^{-1}+(2\pi)^{3/2}\sqrt{2}\sum_{k=1}^{N}\bar{c}_k \quad (A.7)$$

The solution procedure is essentially the same as that employed by Thouless et al. (1987). When the representation (A.6) is substituted into (A.5) the integral equation becomes an equation of the form

$$\sum_{k=1}^{N}[c_kI_1(u,k)+\bar{c}_kI_2(u,k)]=KI_3(u)+\bar{K}I_4(u) \quad (A.8)$$

where the integral expressions for the I's are readily identified. For example,

$$I_1(u,k)=\int_{-1}^{1}H_1(\zeta)T_{k-1}(t)(1+t)^{-1}(1-t)^{-1/2}dt \quad (A.9)$$

These integrals are evaluated numerically for specific values of u and k. Some further reduction of the integrals is necessary to render them in a form suitable for efficient numerical evaluation. Moreover, great care must be taken to ensure that the integration scheme provides a sufficiently accurate estimate of each integral. Accurate evaluation of these integrals is the major obstacle to accurate evaluation of the stress intensity factors.

With the real and imaginary parts of c_k for $k=1, N$ denoting the set of $2N$ unknowns, the real and imaginary parts of equation (A.8) are satisfied at N points u_i on the interval $-1<u<1$. The numerical results reported in Table 1 were computed using Gauss-Legendre points for the u_i. The solution procedure produces both K_I and K_{II}, yet from energy-release considerations the sum of the squares of the intensity factors is known (2.10). This provides an independent check on the accuracy of the numerical solution. The results reported in Table 1 were computed with $N=20$. The independent check (2.10) was satisfied to better than 0.1 percent for essentially all the (α, β) pairs reported in the Table. It is believed that the accuracy of ϕ is comparable.