Reprinted from March 1988, Vol. ~~110~~, Journal of Applied Mechanics
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Elastic Fracture Mechanics Concepts for Interfacial Cracks

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Elastic fracture mechanics concepts are reexamined for a crack on the interface between dissimilar solids. A derivation by function theory is given of the form of stress and displacement fields in the vicinity of the crack tip, equivalent to complete Williams expansions of both inner and outer (external to a nonlinear or contact zone) type. The complex stress intensity factor K *associated with an elastic interface crack, for which contact is ignored, is discussed and, specifically, its validity as a crack tip characterizing parameter is noted for cases of small scale nonlinear material behavior and/or small scale contact zones at the crack tip. That is, similar values of* K *for two cracked bodies then imply similar states as the crack tip, so that conditions for crack growth can be phrased in terms of* K *reaching a critical failure locus in a complex plane. The maintenance of a similar state at a crack tip under change of crack length is shown to require alteration of both the magnitude and phase angle of a combined tension and shear loading. Some possible definitions of stress intensity factors* K_I *and* K_{II} *of classical type associated with interface cracks are discussed. Also, the scaling of interface crack tip plastic zone size with load under small scale yielding conditions is found to deviate from classical scaling, in proportion to the square of the applied load level, and dependences of the field on distance from the tip and on load phase angle are found to be linked together.*

Introduction

There has been a resurgence of interest in the elastic interface crack problem, for which the characteristic oscillating stress singularity was determined by Williams (1959), and solutions to specific problems given by Cherepanov (1962) (see Cherepanov, 1979, pp. 625–630 and p. 808), England (1965), Erdogan (1965), and Rice and Sih (1965). Works by Park and Earmme (1986), Shih and Asaro (1988), and Hutchinson et al. (1987) provide examples of recent contributions.

Apparently the full form of the near tip field, in the sense of a complete Williams expansion, has not been given for the interface crack. That is, Williams (1959) gives eigenvalues λ (with stresses varying in proportion to r^{λ}) of the form $\lambda = n(\text{integer}) - 1/2 + i\epsilon$ for plane strain or plane stress. Here r is distance from the crack tip,

$$\epsilon = (1/2\pi)\ln[(\kappa_1/\mu_1 + 1/\mu_2)/(\kappa_2/\mu_2 + 1/\mu_1)], \quad (1)$$

subscripts 1 and 2 refer to the materials in $y>0$ and $y<0$, respectively, as in Fig. 1, $\kappa = 3-4\nu$ for plane strain and $(3-\nu)/(1+\nu)$ for plane stress, ν = Poisson ratio, and μ = shear modulus. However, there are evidently other eigenvalues of form $\lambda = n$. The complete form of the near tip field is derived here in an analysis that is not based on the Williams (1959) product solution technique, but rather on an extension to the interface crack of an analysis by Rice (1968, pp. 214–215) based on analytic functions. The procedures follow those of Cherepanov (1962, 1979), England (1965), and Erdogan (1965) in solving specific interface crack problems.

The work also sheds light on how to interpret the elastic solutions discussed, in the spirit of linear elastic fracture mechanics procedures like those developed for cracked homogeneous solids, when there is a small zone of nonlinear material response and/or mechanical contact (Comninou, 1977a,b; Comninou and Schmueser, 1978) at the crack tip. The fact that the elastic interfacial crack solutions discussed here predict interpenetration of the crack walls near the tip is frequently taken as a reason to disregard them. However, it is

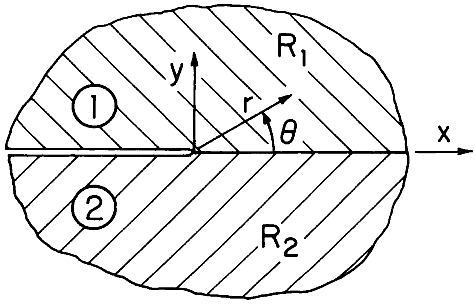

Fig. 1 Region near crack tip along bimaterial interface

Contributed by the Applied Mechanics Division for presentation at the Winter Annual Meeting, Chicago, IL, November 28 to December 2, 1988, of the American Society of Mechanical Engineers.

Discussion on this paper should be addressed to the Editorial Department, ASME, United Engineering Center, 345 East 47th Street, New York, N.Y. 10017, and will be accepted until two months after final publication of the paper itself in the JOURNAL OF APPLIED MECHANICS. Manuscript received by ASME Applied Mechanics Division, June 3, 1987.

Paper No. 88-WA/APM-13.

explained that while the predicted interpenetration means the solutions must be wrong in detail on the scale of the contact zone, they do nevertheless provide a proper characterizing parameter for the near tip state in typical circumstances when that zone size is much smaller than crack length. The characterizing parameter is a complex stress intensity factor K, the same apart from a constant factor as that introduced by Sih and Rice (1964, Appendix) and Rice and Sih (1965), in which tensile and shear effects near the crack tip are intrinsically inseparable into analogues of classical mode I and mode II conditions. The inseparability is sometimes ignored (Malyshev and Salganik, 1965; Cherepanov, 1979). Such is shown to be suitable in limited circumstances but inappropriate, for sufficiently dissimilar materials, when loaded solids with substantially different crack sizes are compared.

Possible definitions of stress intensity fractors K_I and K_{II} of classical type (and physical units) for interfacial crack problems are discussed. Some of these characterize the near tip field just as does the complex K. Also, the nonclassical scaling of plastic zone size and stress field at the interface crack tip under small scale yielding is noted.

Near Tip Stress Field

The well-known Muskhelishvili representation of 2D elastic displacement (u) and stress (σ) fields in isotropic solids can be put in the form

$$2\mu(u_x+iu_y)=\kappa\phi(z)+(\bar{z}-z)\bar{\phi}'(\bar{z})-\bar{\Omega}(\bar{z}) \tag{2}$$

$$\sigma_{xx}+\sigma_{yy}=2[\phi'(z)+\bar{\phi}'(\bar{z})] \tag{3}$$

$$\sigma_{yy}-\sigma_{xx}+2i\sigma_{xy}=2[(\bar{z}-z)\phi''(z)-\phi'(z)+\Omega'(z)] \tag{4}$$

where $z=x+iy$, $\phi(z)$ and $\Omega(z)$ are analytic, $\phi'(z)=d\phi/dz$, and the overbar denotes complex conjugate. We seek the form of solution in some region $R(=R_1+R_2$, Fig. 1) surrounding a traction-free interface crack tip. Let ϕ_1, Ω_1, and ϕ_2, Ω_2 denote solutions in the two regions.

Since $(\sigma_{yy}-i\sigma_{xy})_1=(\sigma_{yy}-i\sigma_{xy})_2$ along all the interface,

$$\phi_1'(x)^+ +\bar{\Omega}_1'(x)^- =\phi_2'(x)^- +\bar{\Omega}_2'(x)^+ \tag{5}$$

everywhere along the x axis in R. Observing that since $\phi_1(z)$ and $\Omega_1(z)$ are analytic in R_1, $\bar{\phi}_1(z)$, and $\bar{\Omega}_1(z)$ are analytic in R_2, etc., this equation shows that

$$\phi_1'(z)-\bar{\Omega}_2'(z)=\phi_2'(z)-\bar{\Omega}_1'(z)=2g(z) \tag{6}$$

where $g(z)$ is analytic throughout R (including points along all the interface). The result also analytically continues the definition of $\phi_1'(z)-\bar{\Omega}_2'(z)$ to R_2 and of $\phi_2'(z)-\bar{\Omega}_1'(z)$ to R_1.

On the bonded portion of interface, $y=0$, $x>0$ in R, $(u_x+iu_y)_1=(u_x+iu_y)_2$. Thus, after differentiating both with respect to x,

$$[\kappa_1\phi_1'(x)^+ -\bar{\Omega}_1'(x)^-]/\mu_1=[\kappa_2\phi_2'(x)^- -\bar{\Omega}_2'(x)^+]/\mu_2 \tag{7}$$

on $x>0$. This equation allows analytic continuation of different linear combinations of $\phi'(z)$ and $\bar{\Omega}'(z)$ across the interface such that

$$(\kappa_1/\mu_1)\phi_1'(z)+(1/\mu_2)\bar{\Omega}_2'(z)$$
$$=(\kappa_2/\mu_2)\phi_2'(z)+(1/\mu_1)\bar{\Omega}_1'(z) \tag{8}$$

holds everywhere in R. Finally, on the cracked portion of the interface $y=0^+$, $x<0$ in R we set $(\sigma_{yy}-i\sigma_{xy})_1=0$ (by equation (5), this will also imply $(\sigma_{yy}-i\sigma_{xy})_2=0$ on $y=0^-$, $x<0$). Thus

$$\phi_1'(x)^+ +\bar{\Omega}_1'(x)^- =0 \tag{9}$$

on $x<0$ in R.

Now by using equations (6) and (8), we may express the various functions $\bar{\Omega}_1'(z)$, $\phi_2'(z)$, and $\bar{\Omega}_2'(z)$ in terms of $\phi_1'(z)$ and $g(z)$. The expressions are not written out explicitly here but when that for $\bar{\Omega}_1'(z)$ is substituted into equation (9) there results

$$(\kappa_2/\mu_2+1/\mu_1)\phi_1'(x)^+ +(\kappa_1/\mu_1+1/\mu_2)\phi_1'(x)^-$$
$$=2(\kappa_2+1)g(x)/\mu_2 \tag{10}$$

on $x<0$ in R. A homogeneous solution of this equation, exhibiting the strongest singularity compatible with bounded total strain energy, is provided by $\phi_1'(z)=z^{-1/2-i\epsilon}$, with ϵ defined by equation (1). Also, a particular solution is $\phi_1'(z)=2c_2g(z)/(c_1+c_2)$ where

$$c_1=(\kappa_1+1)/\mu_1,\ c_2=(\kappa_2+1)/\mu_2. \tag{11}$$

Thus the general solution for $\phi_1'(z)$ is

$$\phi_1'(z)=e^{-\pi\epsilon}z^{-1/2-i\epsilon}f(z)+2c_2g(z)/(c_1+c_2) \tag{12}$$

where $f(z)$, like $g(z)$, is also analytic everywhere in R. Further, by using the expressions discussed at the outset of this paragraph the other functions are given by

$$\bar{\Omega}_1'(z)=e^{\pi\epsilon}z^{-1/2+i\epsilon}\bar{f}(z)-2c_2\bar{g}(z)/(c_1+c_2) \tag{13}$$

$$\phi_2'(z)=e^{\pi\epsilon}z^{-1/2-i\epsilon}f(z)+2c_1g(z)/(c_1+c_2) \tag{14}$$

$$\bar{\Omega}_2'(z)=e^{-\pi\epsilon}z^{-1/2+i\epsilon}\bar{f}(z)-2c_1\bar{g}(z)/(c_1+c_2). \tag{15}$$

For a pure number ζ, $\zeta^{i\epsilon}$ may be evaluated as $e^{i\epsilon\ln\zeta}$.

Williams Expansion; Complex Stress Intensity Factor

A Williams type expansion of the near tip field is generated from equations (2)-(4) and (12)-(15) by writing f and g as local Taylor series expansions

$$f(z)=\sum_{n=0}^{\infty}a_nz^n,\ g(z)=\sum_{n=0}^{\infty}b_nz^n. \tag{16}$$

Then a_0 represents the strength of the crack tip singularity and may be written as

$$a_0=\bar{K}/2\sqrt{2\pi}\cosh(\pi\epsilon) \tag{17}$$

where K is a complex stress intensity factor which uniquely characterizes the singular field. It has been introduced, following Hutchinson et al. (1987), such that along the interface ahead of the crack tip

$$(\sigma_{yy}+i\sigma_{xy})_{\theta=0}=Kr^{i\epsilon}/\sqrt{2\pi r} \tag{18}$$

and along the crack faces

$$(u_y+iu_x)_{\theta=\pi}-(u_y+iu_x)_{\theta=-\pi}$$
$$=(c_1+c_2)Kr^{i\epsilon}\sqrt{r}/2\sqrt{2\pi}(1+2i\epsilon)\cosh(\pi\epsilon), \tag{19}$$

whereas the energy release per unit of new crack area is (e.g., Malyshev and Salganik, 1965)

$$G=(c_1+c_2)K\bar{K}/16\cosh^2(\pi\epsilon). \tag{20}$$

This K is related to the complex intensity factor k_1+ik_2 introduced by Sih and Rice (1964) and Rice and Sih (1965) by $K=(k_1+ik_2)\sqrt{\pi}\cosh(\pi\epsilon)$, and reduces to K_I+iK_{II} for a homogeneous solid ($c_1=c_2$, $\epsilon=0$). For the interface crack of length L subject to remotely uniform stresses σ_{yy}^{∞} and σ_{xy}^{∞} (Fig. 2)

$$K=(\sigma_{yy}^{\infty}+i\sigma_{xy}^{\infty})(1+2i\epsilon)L^{-i\epsilon}\sqrt{\pi L/2} \tag{21}$$

at the right-hand crack tip.

The coefficient b_0 in equations (16) represents a stress field of type σ_{xx} that is uniform but different in each of the two phases, in the manner discussed by Rice and Sih (1965). Further, equations (16) confirm that the full set of Williams eigenvalues have form $\lambda=n-1/2+i\epsilon$ and $\lambda=n$.

Nonlinear Material Response and Crack Surface Contact

Inevitably there will be a zone near the crack tip for which the present solution representation fails, whether due to material nonlinearity (e.g., a plastic zone) or to the well-known surface interpenetration predicted by equation (19), leading to Comninou contact zones. Let r_1 be the maximum radius of any such zone and let r_2 be the minimum radius from the crack tip of Fig. 1 to the specimen surface, or other crack tip, or place of load application. Assume that the excluded zone near the crack tip is small enough that $r_1 < r_2$; note also that r_2 will be the radius of convergence for the Taylor series in equations (16). Then the analysis presented above may be reviewed with R reinterpreted as the annular ring between r_1 and r_2. Every step is valid, so that the representations in equations (2)-(4) and (12)-(15) continue to hold with the understanding that $f(z)$ and $g(z)$ are analytic in the reinterpreted R, and thus have the Laurent representation

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n z^n, \quad g(z) = \sum_{n=-\infty}^{+\infty} b_n z^n \tag{22}$$

on $r_1 < |z| < r_2$. The terms with negative n coincide to what, in the homogeneous case, has been called an outer Williams expansion (Rice, 1974) and, just as shown for that case, the term $b_{-1} = 0$ since there is no net force acting on the inner zone at the crack tip.

Small Scale Nonlinear or Contact Zones

For homogeneous materials, linear elastic fracture mechanics procedures (i.e., characterizing crack growth in terms of $K_I + iK_{II}$) are valid when the inevitable nonlinear zone at the crack tip is sufficiently small that, when r is rescaled on that zone size, the radius r_2 is effectively infinite. In that case the "small scale yielding" approach is valid: The actual crack problem is replaced by that of a semi-infinite crack in an infinite solid with asymptotic boundary condition that, at large r, the field approaches that of the standard Irwin-Williams elastic singularity of a strength characterized by $K_I + iK_{II}$ for the problem. Thus, even though nonlinearities cause the actual field to differ locally from the Irwin-Williams stress and deformation distributions, that actual field is uniquely characterized by $K_I + iK_{II}$, which provides the boundary conditions and hence determines the onset of crack growth.

The same remarks apply for the interface crack problem in terms of the complex intensity factor K. It uniquely characterizes the actual field in the small scale nonlinear or contact zone case, even though expressions such as (18) and (19) are not accurate within that zone. Conditions for onset of crack growth for the interface crack are thus properly phrased in terms of K for the small scale regime. Within that small scale nonlinear or contact zone description, the field for $|z| > r_1$ has the representation as in equations (2)-(4) and (12)-(15) with

$$f(z) = \sum_{n=-\infty}^{0} a_n z^n, \quad g(z) = \sum_{n=-\infty}^{-2} b_n z^n \tag{23}$$

where a_0 is specified in terms of K for the crack problem, as in (17). The remaining coefficients of the small scale solution are determined only as part of a full nonlinear analysis, e.g., as by Sham (1984) for the homogeneous elastic ideally plastic solid. They are all unique, if unknown, functions of K.

An improvement which, in the homogeneous material case, is known to significantly enlarge the range of load levels over which the small scale yielding analysis procedure gives accurate results is to include in the latter of equations (23), as a specified term, the term b_0 representing the crack-parallel near tip σ_{xx} stress of the elastic crack solution (Larsson and Carlsson, 1973; Rice, 1974).

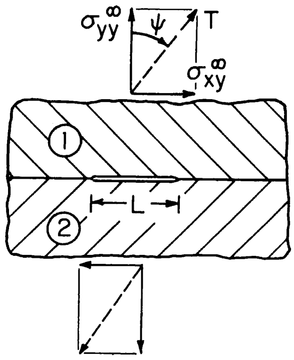

Fig. 2 Remote tension and shear loading of interface crack

Near Tip Contact Zone

An elementary estimate of the contact zone size, assuming that it is small compared to crack size, is given by finding the largest r for which the opening gap, $u_y(r,\pi) - u_y(r,-\pi)$, predicted by equations (19) vanishes, that is, for which

$$\mathrm{Re}[Kr^{i\epsilon}/(1+2i\epsilon)] = 0. \tag{24}$$

For the remotely loaded crack of Fig. 2, let

$$\sigma_{yy}^{\infty} + i\sigma_{xy}^{\infty} = Te^{i\psi} \tag{25}$$

where T is the magnitude of the traction vector on remote surface $y =$ constant $>> L$, and where the phase angle ψ gives its direction. Thus, $\psi = 0$ corresponds to tension; $\psi = \pm\pi/2$ to shear in the $\pm x$ directions. In this case equation (24), with equations (21) and (25), becomes

$$\mathrm{Re}[e^{i\psi}(r/L)^{i\epsilon}] = \cos[\psi - \epsilon\, \ln(L/r)] = 0 \tag{26}$$

Assume now that $\epsilon > 0$ (if not, we can just exchange the labels "1" and "2," which changes ϵ to $-\epsilon$, and then change ψ to $-\psi$ to describe the same physical problem but with $\epsilon > 0$). Let ψ lie in the range $-\pi/2 < \psi < +\pi/2$, so that some tensile component always acts. Then the contact zone size r_c is estimated by $\psi - \epsilon \ln(L/r) = -\pi/2$, or

$$r_c = L\exp[-(\psi + \pi/2)/\epsilon]. \tag{27}$$

Since ϵ is typically small, r_c/L is a rapidly varying function of ψ and is very much smaller than unity over most of the range cited above, including, say, $-\pi/4 < \psi < \pi/2$. It will not remain small for any $\epsilon > 0$ when ψ approaches $-\pi/2$.

In general ϵ increases with increase of the stiffness ratio μ_2/μ_1. For example, if we take material "1" as cork (with $\nu_1 \approx 0$) and bond it to a stiff substrate like alumina (Al_2O_3) for "2," so that $\mu_1/\mu_2 \approx 0$, then ϵ has its largest feasible value (at least for solids with $\nu \geq 0$), namely, $\epsilon = 0.175$. Among the harder, nonpolymeric solids, a relatively extreme stiffness contrast is provided by fused silica (SiO_2) or soda lime glass for "1" and Al_2O_3 for "2," in which case $\epsilon = 0.075$. Representative values of ϵ are considerably lower for various "1"/"2" combinations of interest for practical metal and nonmetal interfaces. For example, Hutchinson et al. (1987) give $\epsilon = 0.039$ for Ti/Al_2O_3, 0.028 for Cu/Al_2O_3, 0.019 for Nb/Al_2O_3, 0.011 for Si/Cu, 0.005 for MgO/Ni, and 0.004 for Au/MgO based on elastic parameters that they tabulate.

If one adopts $r_c/L < 0.01$ as a suitable restriction on r_c so that the small scale contact zone concept may be applied, that is, so that the field may be regarded as being characterized by the complex K, then one requires that $\psi > -\pi/2 + 4.605\ \epsilon$. Thus it is required for validity of the linear elastic fracture mechanics approach outlined above that $\psi > -50$ deg when

$\epsilon=0.15$, $\psi>-77$ deg when $\epsilon=0.05$, and $\psi>-87$ deg when $\epsilon=0.01$. These restrictions will generally be met in practical cases for which there is some nonnegligible tensile component of the loading relative to the crack.

Although a finite, if extremely small, contact zone is predicted by the procedure here for tensile loading, $\psi=0$, nonlinear elastic analysis of a particular class of joined materials in plane stress conditions for that loading (Knowles and Sternberg, 1983) does not predict contact at the tip.

Coupled Size and Load Phase Effects

Suppose linear elastic fracture mechanics conditions are met so that crack growth conditions are controlled by the complex K. A remarkable result is that if we wish to duplicate the conditions near the tip of a crack of length L loaded, for example, in tension ($\psi=0$), by testing another body with crack length $L'\neq L$, then that other body must be loaded by a combination of tension and shear. This differs from the ordinary fracture mechanics of homogeneous solids and it means that separate tensile and shear modes cannot be unambiguously defined.

Consider the geometry of Fig. 2 and rewrite K from equation (21) as

$$K=Te^{i\psi}(1+2i\epsilon)L^{-i\epsilon}\sqrt{\pi L/2}. \tag{28}$$

Thus if one is to duplicate, in a solid with crack length L', the same conditions near the tip as in a solid with crack length L loaded by traction T at angle ψ, then

$$T'e^{i\psi'}(L')^{-i\epsilon}\sqrt{L'}=Te^{i\psi}L^{-i\epsilon}\sqrt{L}. \tag{29}$$

Thus the traction T' must be altered according to the usual inverse square root dependence

$$T'=T\sqrt{L/L'}, \tag{30}$$

but the phase angle of the loading must be altered also, to

$$\psi'=\psi+\epsilon\,\ln(L'/L). \tag{31}$$

Usually the phase angle change, $\psi'-\psi$, is small. For example, if one compares specimens for which the crack lengths differ by a factor of 10, the change in ψ is 2.3 ϵ, that is, 10 deg for $\epsilon=0.075$ (the value for fused SiO_2/Al_2O_3) but only 1.4 deg for $\epsilon=0.011$ (Si/Cu). More extreme changes in L such as comparison of a 50 μm defect in service with a 5 mm crack in a test specimen ($L'/L=100$) changes ψ by 4.6 ϵ, and this may be nonnegligible for materials of significant difference in stiffness (e.g., 20 deg for $\epsilon=0.075$, and 46 deg for the largest possible ϵ, 0.175).

Another perspective is in terms of a failure locus. Imagine that from mixed tension and shear loading tests on a solid as in Fig. 2 with crack length L a locus of stress states σ_{yy}^{∞}, σ_{xy}^{∞} at onset of crack growth have been determined. Such results define a curve in the σ_{yy}^{∞}, σ_{xy}^{∞} plane or, equivalently, define a polar plot of traction magnitude T versus load angle ψ. Equations (30) and (31) then show that when the crack length is increased to L', the failure locus both contracts self-similarly and rotates. Evidently the rotation will usually be small unless there are extreme changes in L and a large ϵ value.

Awkward as it may seem, the proper physical units of complex K, when one measures stress in MPa and length in m, are such that a particular value should be reported as

$$K=C\ MPa\ \sqrt{m}\ m^{-i\epsilon} \tag{32}$$

where C is a pure complex number. Also, if one changes the units for measuring length from m to mm, then not only is the magnitude of C increased by $\sqrt{1000}\approx32$, but so also is its phase angle changed by $-\epsilon\,\ln(1000)\approx-6.9\,\epsilon$ (i.e., by -20 deg when $\epsilon=0.050$). Thus a value of K for which C is real in some system of units cannot properly be regarded as a "mode I" K, because C would not remain real with some other choice of units. One must conclude that tension and shear effects are inherently coupled near interface crack tips. Particular external loadings cannot be said without ambiguity to produce separate "mode I tension" or "mode II shear" conditions near the crack tip.

Once a particular system of units is chosen, the onset of crack growth may be characterized by C reaching a failure locus in a complex plane whose axes are the real and imaginary parts of C. Change of units will both self-similarly scale (as in conventional fracture mechanics) and also rotate the failure locus, so that the real and imaginary parts of C cannot be interpreted as "mode I" and "mode II" components.

It may be noted that a circular failure locus, whether in the σ_{yy}^{∞}, σ_{xy}^{∞} plane or the complex C plane, will be unaffected by rotation. Such a circular failure locus would correspond to a fixed energy release rate G at onset of crack growth (see equation (20)), irrespective of the phase angle of the applied loading.

Small ϵ and Possible Definitions of Stress Intensity Factors of Classical Type

Despite the intrinsically mixed tension and shear fields at the crack tip, and the related rotational effects discussed, for very many material combinations of interest ϵ will be very small, say, of order 0.01 to 0.03. In such cases the rotation angle, $\psi'-\psi$, associated with a factor of 10 change in L lies only between 1.3 deg and 4.0 deg. One is therefore inclined in such cases to neglect the rotation and seek a description without the complexities associated with complex K. Such has evidently been the motivation for proposals advanced by Malyshev and Salganik (1965) and Cherepanov (1979).

Observe that for any interfacial crack problem the complex K will have the form

$$K=\Lambda T\sqrt{L}L^{-i\epsilon} \tag{33}$$

where T is an applied traction loading, L is a relevant length describing the geometry (say, the shorter of crack length, uncracked ligament width, and distance from crack tip to a point of load application) and Λ is a complex number which depends on the phase angle of the applied loading, on ratios of the various other lengths to L and, in general, on ν_1, ν_2 and μ_1/μ_2. In describing any component of the singular near tip stress and displacement field, K always appears as the factor $Kr^{i\epsilon}$, i.e., as

$$Kr^{i\epsilon}=\Lambda T\sqrt{L}(r/L)^{i\epsilon}=\Lambda T\sqrt{L}e^{-i\epsilon\ln(L/r)}. \tag{34}$$

Equation (18) for the stresses σ_{yy} and σ_{xy} ahead of the crack has the same form as for a homogeneous solid if we replace $Kr^{i\epsilon}$ by K_I+iK_{II}. So also does equation (19) for the crack surface displacements, under the same replacement, at least to the disregard of terms of order ϵ.

Thus to the extent that $Kr^{i\epsilon}$ is sensibly independent of r over some range of interest for the application of fracture mechanics methodology, we might choose a value (any value) of r in that range, say $\hat{r}$, and characterize the crack tip fields by intensity factors of classical type defined by

$$K_I+iK_{II}=K\hat{r}^{i\epsilon}=\Lambda T\sqrt{L}(\hat{r}/L)^{i\epsilon}. \tag{35}$$

How should $\hat{r}$ be chosen? It hardly matters within very broad limits, for small ϵ. For example, considering two choices of $\hat{r}$ that differ by a factor of 10, the two resulting values for K_I+iK_{II} as defined above will have a ratio to one another of

$$(10)^{i\epsilon}=e^{2.3i\epsilon}=(0.9997+0.0230i)\ \text{to}\ (0.9976+0.0689i) \tag{36}$$

for $\epsilon=0.01$ to 0.03. These are sufficiently close to unity that factor of 10 precision in choosing $\hat{r}$ is more than adequate. Also, if one intends applications over a range of geometrical sizes so that L varies by a factor of order 10 or less, then the term $(\hat{r}/L)^{i\epsilon}$ in equation (35) will likewise vary by no more than a term of order $(10)^{i\epsilon}$, which has just been seen to differ negligibly from unity for the small ϵ values considered.

It is tempting to choose $\hat{r}$ as a fixed fraction of L, say, as $\hat{r}=L/50$ which gives

$$K_I+iK_{II}=\Lambda e^{-3.91i\epsilon}T\sqrt{L} \tag{37}$$

and, in fact, differs little numerically for small ϵ from what one gets by formally setting $\hat{r}=L$ (despite the asymptotic character of the expressions containing K) so that

$$K_I+iK_{II}=\Lambda T\sqrt{L}. \tag{38}$$

The latter choice is consistent with the discussion by Cherepanov (1979).

Of course, both of these choices for $\hat{r}$ are objectional in principle. They go against the spirit of elastic fracture mechanics where the intention (as in the earlier interpretation of complex K) is to define parameters which fully characterize the effects of load and geometry on the crack tip field. Extremely large changes in L, with T altered accordingly, so that $T\sqrt{L}$ is invariant, definitely affect the character of the near tip field (the rotation effect discussed earlier). This effect is lost in equations (37) and (38) since $\hat{r}$ has been scaled to L in those expressions and hence does itself change. If, however, one limits attention to more modest changes in L, on the order of a factor of 10 or less, and to material combinations giving small ϵ values (as seems typical), then either of equations (37) or (38) is suitable for operational use in fracture mechanics analysis within that context. For clarity, it is proposed that one should refer to these as classical stress intensity factors based on distance $\hat{r}=L/50$ or L, respectively.

Preferred Definitions of Stress Intensity Factors of Classical Type

It is preferable in principle to choose $\hat{r}$ in equation (35) as some material length, invariant to differences of crack size or other overall geometric dimensions in different applications. Such an $\hat{r}$ might be chosen as some large multiple of atomic dimensions or as a multiple of some less well defined "fracture process zone size" in certain specified test conditions. Upon reflection it may be realized that so long as we wish to use K_I and K_{II} as characterizing parameters, it really doesn't matter how we choose $\hat{r}$ so long as it is fixed for a given material combination, i.e., unaffected by changes in crack size or other geometric dimensions. Thus a suitable procedure is that one adopt a fixed $\hat{r}$ for all material combinations and that this be taken as $\hat{r}=1\mu$m so that

$$K_I+iK_{II}=K(\mu m)^{i\epsilon}=\Lambda T\sqrt{L}(\mu m/L)^{i\epsilon}. \tag{39}$$

The K_I and K_{II} so defined should be referred to as classical stress intensity factors based on distance $\hat{r}=1\mu$m.

Plainly to the extent that the complex K uniquely characterizes the crack tip field, so also do K_I and K_{II} based on $\hat{r}=1\mu$m (or on any other fixed $\hat{r}$), and this is so no matter how much L varies or however large or small is the value of ϵ. Essentially, the proposed definition of K_I+iK_{II} lets one avoid dealing with the awkwardly complex physical dimensions (MPa $\sqrt{m}\ m^{-i\epsilon}$) of complex K, in favor of those (MPa $\sqrt{m}$) of classical stress intensity factors, but retains all the features of complex K as a crack tip characterizing parameter. Its drawback is that it seems to carry the implication that "mode I" and "mode II" have unambiguous meaning.

Finally, one may also observe that if the crack did not quite lie on the interface but, rather, if its tip was in one of the two joined solids, then classical K_I and K_{II} could be defined at its tip. In this connection Hutchinson et al. (1987) have recently shown that if a crack lies parallel to the interface, in material "2" at $y=-h$, and if $h<<L$, then the classical K_I and K_{II} at its tip are given by

$$K_I+iK_{II}=qe^{i\phi}Kh^{i\epsilon}=q\Lambda e^{i\phi}T\sqrt{L}(h/L)^{i\epsilon}. \tag{40}$$

Here K is the complex intensity for the associated interface crack problem,

$$q=[(1+c_1/c_2)/2\cosh^2\pi\epsilon]^{1/2} \tag{41}$$

and ϕ is a function of ν_1, ν_2 and μ_1/μ_2 which, for typical material combinations like those enumerated in a previous section, satisfies (like ϵ) $|\phi|<<1$. For those cases of small ϵ and small ϕ one may write

$$K_I+iK_{II}\approx q\Lambda T\sqrt{L}(h/L)^{i\epsilon} \tag{42}$$

and, unless L/h is large in a logarithmic sense (i.e., so long as $\epsilon\ln(L/h)$ is small compared to unity), this can be rewritten, as Hutchinson et al. note, as

$$K_I+iK_{II}\approx q\Lambda T\sqrt{L}. \tag{43}$$

The coincidence, apart from the factor q (or $qe^{i\phi}$) of expression (43) with (38), and of (40) and (42) with (35) and (39) is evident. In particular, the K_I+iK_{II} proposed in equation (39) to be associated with an interface crack coincides, apart from the factor $qe^{i\phi}(h/\mu m)^{i\epsilon}$ (which will often differ little from q), with the actual K_I+iK_{II} of equation (40) for an interface-parallel crack.

Small Scale Yielding at Bimaterial Crack Tips

In the small scale yielding problem, when one or both of the joined solids deforms plastically, the characteristic dimension of the plastic zone must depend only on the complex K and material properties. The properties include the yield strength σ_0 of the weaker of the two solids, the ratios of yield strengths, and dimensionless properties describing strain hardening and ratios of elastic constants. Let r_p be that characteristic dimension. It could represent the maximum radius of the plastic region or (if different) the radius along $\theta=0$.

Dimensional analysis shows that r_p must enter through the complex dimensionless combination

$$K/\sigma_0 r_p^{1/2-i\epsilon}=|\Lambda|(T/\sigma_0)\sqrt{L/r_p}\ e^{i\omega}(r_p/L)^{i\epsilon} \tag{44}$$

where, in the latter version, Λ in the generic form of complex K in equation (33) has been written as $|\Lambda|e^{i\omega}$. This is actually a pair of combinations (real and imaginary part or, more conveniently, amplitude and phase). Thus r_p must satisfy an equation of the form

$$r_p=\frac{K\bar{K}}{\sigma_0^2}f[\text{phase}(Kr_p^{i\epsilon})]=\Lambda\bar{\Lambda}\frac{T^2}{\sigma_0^2}Lf[\omega-\epsilon\ln(L/r_p)] \tag{45}$$

where $f[\ldots]$ is a dimensionless function. This is, of course, an implicit equation for r_p. Furthermore, since the argument, $\omega-\epsilon\ln(L/r_p)$, enters the dimensional analysis as the phase angle of a complex quantity, $f[\ldots]$ must have a periodic (with period 2π) dependence on that argument.

For the geometry of Fig. 2, $\Lambda=(1+2i\epsilon)e^{i\psi}\sqrt{\pi/2}$, and thus $\omega=\psi+\arctan(2\epsilon)$ where ψ is the phase angle of the remotely applied traction, $\sigma_{yy}^\infty+i\sigma_{xy}^\infty=Te^{i\psi}$. Thus in that case

$$r_p=\frac{\pi}{2}(1+4\epsilon^2)\frac{T^2}{\sigma_0^2}Lf[\psi+\arctan(2\epsilon)-\epsilon\ln(L/r_p)]. \tag{46}$$

It is plausible that r_p should vary with the phase angle ψ of the applied traction, and hence that $f[\ldots]$ is not independent of its argument (i.e., not a constant). In such cases it is then required that for fixed ψ, r_p cannot have the classical small scale yielding proportionality, $r_p\propto T^2L/\sigma_0^2$, because f itself changes as r_p increases.

As an example, Shih and Asaro (1988) have recently done plane strain elastic-plastic numerical calculations for a bimaterial crack as in Fig. 2 with "2" being rigid, $\mu_2/\mu_1=\infty$, $(\sigma_0)_2/\sigma_0=\infty$, and $\nu_1=0.3$, in which case $\epsilon=0.0935$. Table 1 summarizes their results for the cases they report which are definitively in the small scale yielding range (either because $r_p<<1$ or because they used the small scale yielding formula-

Table 1

T/σ_0	ψ	r_p/L	$f[\ldots]$	$\omega-\epsilon\ln(L/r_p)$
2×10^{-4}	0	2.8×10^{-8}	0.431	-1.44
2.236×10^{-1}	$-1.107(-63.4°)$	2.5×10^{-3}	0.308	-1.27
6×10^{-3}	0	1.6×10^{-5}	0.273	-0.85

tion for a semi-infinite crack, with the complex K for the geometry of Fig. 2 used in asymptotic boundary conditions). These cases are for a "deformation theory" treatment of plasticity in Ramberg-Osgood form with hardening exponent $n=3$, and r_p corresponds to the zone size along $\theta=0$. The loading is described in the first two columns, their finite element results for r_p/L in the third, and the implied value of $f[\ldots]$ in the fourth. The argument of $f[\ldots]$ is reported in the fifth column. Note that $f[\ldots]$ inferred from results with a strong negative shear component ($\psi=-63.4$ deg) fits between those for $\psi=0$ but with much smaller plastic zones, the results being ordered according to values of $\omega-\epsilon\ln(L/r_p)$.

A similar dimensional analysis reveals the form of the stress field in the small scale yielding formulation. This must have the form

$$\sigma_{ij}=\sigma_0 g_{ij}[\sigma_0^2 r/K\bar{K},\theta,\text{phase}(Kr^{i\epsilon})]$$

$$=\sigma_0 g_{ij}[\sigma_0^2 r/\Lambda\bar{\Lambda}T^2L,\theta,\omega-\epsilon\ln(L/r)] \quad (47)$$

where there is periodic dependence on the third group and, as earlier, the g_{ij} depend also on dimensionless material properties or property ratios.

Notes

Since preparation of this manuscript Symington (1987) has confirmed the expected result that the original Williams (1959) analysis leads to crack tip solutions with integer eigenvalues λ, represented here by the series for $g(z)$ in equation (16). Those λ did not appear as a conclusion of the Williams paper since he had evidently cancelled out a certain trigonometric factor, vanishing when $\lambda=$ integer, in evaluating the zeros of the 8 by 8 determinant in his analysis.

Also, in a recent pre-publication revision of their manuscript, Shih and Asaro (1988) have incorporated a dimensional analysis of the small scale yielding elastic-plastic problem similar to that in the last section of this paper. They simplify the presentation by noting that within the arbitrariness of arrangement of the results of a dimensional analysis, the r_p which appears within the argument of function f in equations (45) and (46) could equally be replaced by $K\bar{K}/\sigma_0^2=\Lambda\bar{\Lambda}T^2L/\sigma_0^2$, it then being understood that the function f is different from that here. This nicely removes the implicit nature of equations (45) and (46) for r_p, but likewise shows that classical scaling with T^2L/σ_0^2 cannot apply. Similarly, the r within the third argument of g_{ij} in equation (47) could be replaced also by $K\bar{K}/\sigma_0^2=\Lambda\bar{\Lambda}T^2L/\sigma_0^2$ with the understanding that different functions g_{ij} apply.

Acknowledgment

This study was supported by Office of Naval Research contract N00014-86-K-0753 through sub-agreement VB38639-0 from the University of California. The impetus to clarify the interpretation of the complex K came from discussions with A. G. Evans. That to determine the form of terms corresponding to the integer Williams eigenvalues came from queries by M. Symington and D. M. Parks. Discussions with J. W. Hutchinson and B. Budiansky have been helpful.

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