Nearly Circular Connections of Elastic Half Spaces

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In this paper we solve the elasticity problem of two elastic half spaces that are joined together over a region that does not differ much from a circle, i.e., the problem of an external planar crack leaving a nearly circular uncracked connection. The method we use is based on the perturbation technique developed by Rice (1985) for solving the elastic field of a crack whose front deviates slightly from some reference geometry. Quantities such as crack opening displacement and stress intensity factor are derived in detail to the first order of accuracy in the deviation of the shape of the connection from a circle. In addition, some results such as the crack face weight functions and Green's functions for a perfectly circular connection are also discussed under various boundary conditions at infinity. The formulae derived are used to study the configurational stability problem for quasistatic growth of an external circular crack. The results, derived when the crack front is perturbed from circular in a harmonic wave form and is subjected to axisymmetric loading, suggest that a perturbation of wavenumber higher than one is configurationally stable under all boundary conditions at infinity. The perturbation with wavenumber equal to one, which corresponds to a translational shift of the geometric center of the circular connection, turns out to be configurationally stable if any rotation in the remote field is suppressed and configurationally unstable if there is no such restraint.

Introduction

Rice (1985) developed a method of solving the elasticity problem of a planar crack whose front differs slightly in location from that of some reference geometry. It has been applied to cases such as semi-infinite planar cracks with slightly nonstraight fronts (Rice, 1985; Gao and Rice, 1986) and internal somewhat circular cracks (Gao and Rice, 1987). The latter work (Gao and Rice, 1987) has shown that the perturbation method is not only convenient but also remarkably accurate in determining crack opening displacement and stress intensity factors for crack configurations that differ moderately from a circular reference geometry. The internal circular crack problem was addressed much earlier in a perturbation sense by Panasyuk (1962), and Gao and Rice (1987) compare their approach to his. Rice's perturbation method can be carried out immediately for a tensile crack if the solution for the stress intensity factor distribution is known along the reference crack front due to a pair of concentrated wedging forces acting to open the crack at an arbitrary location on its surfaces. Such a point force solution, sometimes called the crack face weight function after Bueckner (1970, 1973) and Rice (1972), was derived by Stallybrass (1981) for an external circular crack, i.e., a circular connection between elastic half-spaces under a traction free boundary condition at infinity. Following Stallybrass's work we are also able to clarify ambiguities in some previously proposed solutions in the literature (e.g., Kassir and Sih, 1975; Tada et al., 1973).

In this paper we therefore solve for the crack opening displacement and tensile mode stress intensity factor for a slightly noncircular connection. The notation $\delta\,(F)$ is used in what follows to denote the variation in some field variable F from its form for the reference circular crack to that for the perturbed crack shape.

Consider two isotropic, homogeneous three-dimensional elastic semi-infinite solids joined over some slightly noncircular connection of bounding contour c. A Cartesian coordinate system x, y, z is attached so that the joining planes lie on $y = 0$ and the origin of the coordinate system is assumed to coincide with the center of some convenient reference circle. This configuration forms an external crack with its front c described by some function $a(s)$ where $a(s)$ is the distance from the origin of the coordinate system to the position s along the crack front; $a(s)$ is nearly constant, and is constant on the reference circle. The crack system is subjected to some distribution of fixed forces that induce "*M*ode 1" tension along the crack front. We may note that in this case when the crack grows into the connecting ligament, $a(s)$ decreases. Therefore, we represent the crack growth from the reference circular shape to the actual shape by $-\delta a(s)$. In this circumstance it can be shown, following Rice (1985), that the variation in opening displacement $\Delta u(x,z)$ between upper and lower crack surfaces at location x, z, when the crack front is

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altered from the reference circular front by $-\delta a(s)$ in presence of the fixed load system, is

$$\delta[\Delta u(x,z)] = -\frac{2(1-\nu^2)}{E} \oint_c K^0(s) k(s;x,z) \delta a(s) ds \qquad (1)$$

to first order in $\delta a(s)$. Here $K^0(s)$ is the Mode 1 intensity factor induced along the reference crack front by the fixed load system and $k(s;x,z)$ is the intensity factor that would be induced at arc length position s along the reference front by a pair of unit wedging forces opening the crack at location at x, z; $k(s; x, z)$ can be called the Mode 1 crack face weight function. This weight function is discussed in detail in Appendix A. Here the intensity factor K is defined so that $K/\sqrt{2\pi\epsilon}$ is the asymptotic form of the tensile stress at small perpendicular distance ϵ from the crack front on the prolongation of the crack plane within the connection.

Crack Opening Displacement

We choose the reference crack as a perfectly circular connection of radius a and adopt polar coordinates for convenience so that $s = a\theta'$ in equation (1). Here the polar coordinate angle θ' is measured from the positive x axis, increasing towards the positive z axis. To emphasize dependence on the reference circular radius a, we introduce the notations $K^0(s) = K^0[\theta;a]$ and $k(s; x, z) = k(\theta'; r, \theta; a)$ for the intensity factors induced at θ' along the reference crack front, respectively, by the given load system and by a pair of unit wedging point forces at polar position r, θ. Then equation (1) becomes

$$\delta[\Delta u(r,\theta)] = -\frac{2(1-\nu^2)}{E} \int_0^{2\pi} \times K^0[\theta'; a] k(\theta'; r, \theta; a)\, a\, \delta a(\theta')\, d\theta' \qquad (2)$$

where $\delta a(\theta') = a(\theta') - a$. Also, we introduce the notation $\Delta u\ (r, \theta) = \Delta u^0\ [r, \theta; a]$ to describe the opening of a perfectly circular connection of radius a under the given loadings.

We can also derive $K^0\ [\theta'; a]$ by the law of superposition when some distributed load $p(r; \theta)$ is acting on the external crack faces

$$K^0[\theta'; a] = \int_0^{2\pi} \int_a^{\infty} p(\rho, \phi)\, k(\theta'; \rho, \phi; a)\, \rho d\rho d\phi \qquad (3)$$

The problem of general tensile loading can also be described in this way when $p(r, \theta)$ is equated to the tensile stress which the general loading would induce at r, θ in the absence of the external crack.

To find the opening displacement field for a perfectly circular connection, we impose a uniform crack growth, i.e., $\delta a(\theta') = \delta a$ in equation (2). Then dividing both sides of equation (2) by δa and lettering $\delta a \to 0$, we get

$$\frac{\partial \Delta u^0[r, \theta; a]}{\partial a} = -2\frac{(1-\nu^2)}{E} \int_0^{2\pi} \times K^0[\theta'; a] k(\theta'; r, \theta; a) a\, d\theta' \qquad (4)$$

Noting that $\Delta u^0\ [r, \theta; a] = 0$ when $a \geq r$ (only crack faces open), we integrate over the crack size variable a' and get

$$\Delta u^0[r, \theta; a] = 2\frac{(1-\nu^2)}{E} \int_0^{2\pi} \int_a^r \times K^0[\theta'; a']\, k(\theta'; r, \theta; a')\, a'\, da'\, d\theta' \qquad (5)$$

Substituting equation (3) into (5), we get the following general crack opening displacement for external circular cracks,

$$\Delta u^0[r, \theta; a] = 2\frac{(1-\nu^2)}{E} \int_0^{2\pi} \int_a^r \int_0^{2\pi} \int_{a'}^{\infty} k(\theta'; \rho, \phi; a') \times k(\theta'; r, \theta; a')\, a' p(\rho, \phi)\, \rho d\rho d\phi\, da' d\theta' \qquad (6)$$

If we switch the order of integration with respect to a', θ' and ρ, ϕ, we therefore could rewrite equation (6) as

$$\Delta u^0\ [r, \theta; a] = \int_0^{2\pi} \int_a^{\infty} D(r, \theta; \rho, \phi)\, p(\rho, \phi)\, \rho d\rho d\phi \qquad (7)$$

where

$$D(r, \theta; \rho, \phi) = 2\frac{(1-\nu^2)}{E} \int_a^{\min(r, \rho)} \int_0^{2\pi} \times k(\theta'; \rho, \phi; a')\, k(\theta'; r, \theta; a')\, a'\, d\theta'\, da' \qquad (8)$$

is clearly identified as the crack face Green's function for an external circular crack, and it is further discussed in detail in Appendix B and also in Appendix D.

Equation (6), or equation (7) combined with equation (8), gives us the formula to determine the crack opening displacement for a perfectly circular connection. The integrals in those equations can be carried out once the loading system $p(r, \theta)$ and the crack face weight function $k(\theta'; r, \theta; a)$ is known. The function $k(\theta'; r, \theta; a)$ is discussed in Appendix A and presented under various boundary conditions at infinity. The most general form of $k(\theta'; r, \theta; a)$ is given by equation (A-9) of Appendix A under traction free, completely unrestrained displacement conditions at infinity. For convenience we present it here too:

$$k(\theta'; r, \theta; a) = \frac{1}{(\pi a)^{3/2}} \left\{ \left[\cos^{-1}\left(\frac{a}{r}\right) + \frac{a\sqrt{r^2 - a^2}}{a^2 + r^2 - 2ar\cos(\theta' - \theta)} \right] + 3\left[\frac{r}{a}\cos^{-1}\left(\frac{a}{r}\right) + \left(1 - \frac{a^2}{r^2}\right)^{1/2} \right] \cos(\theta' - \theta) \right\} \qquad (9)$$

When the shape of the connection is slightly noncircular, it is convenient for purposes of calculating the opening $\Delta u\ (r, \theta)$ along the ray at any particular angle θ to take the radius of the reference circular crack front to be a circle of radius equal to $a(\theta)$. We then are able to let r approach simultaneously both the reference front and the actual perturbed front. This procedure, as described in earlier papers (Rice, 1985, Gao and Rice, 1986, 1987), is necessary to retain the correct asymptotic behavior near the crack front as is crucial for the calculation of the stress intensity factor along the perturbed crack front. Then equation (2) becomes,

$$\delta\ [\Delta u(r, \theta)] = 2\frac{(1-\nu^2)}{E} \int_0^{2\pi} K^0[\theta'; a(\theta)]\, k(\theta'; r, \theta; a(\theta)) \times [\, a(\theta) - a(\theta')]\, a(\theta)\ d\theta' \qquad (10)$$

Equation (10) plus equation (6) then gives the total opening displacement as

$$\begin{aligned} \Delta u(r, \theta) &= \Delta u^0[r, \theta; a(\theta)] + \delta\ [\Delta u(r, \theta)] \\ &= 2\frac{(1-\nu^2)}{E} \int_0^{2\pi} \left\{ \int_{a(\theta)}^{r} K^0[\theta'; a']\, k(\theta'; r, \theta; a')\, a'\, da' \right. \\ &\quad \left. + K^0[\theta'; a(\theta)] k(\theta'; r, \theta; a(\theta))\, [a(\theta) - a(\theta')] a(\theta) \right\} d\theta' \\ &\simeq 2\frac{(1-\nu^2)}{E} \int_0^{2\pi} \int_{a(\theta')}^{r} \times K^0[\theta'; a']\, k(\theta'; r, \theta; a')\, a'\, da'\, d\theta' \end{aligned} \qquad (11)$$

where the last $\simeq$ means equal to first order of accuracy in $a(\theta') - a(\theta)$. Equation (11) can be used to evaluate the opening displacement for a slightly noncircular connection if one is given the shape of that connection (i.e., the function $a(\theta')$).

Stress Intensity Factors

Stress intensity factors can be extracted from the near tip behavior of the crack opening displacement, as indicated by equation (B-13) of Appendix B. The same relation holds between $\delta\,[\Delta u(r,\theta)]$ and $\delta K(\theta)$, the first order variation of intensity factor as

$$\delta[\Delta u(r,\theta)] = \frac{8(1-\nu^2)}{E} \times \left\{ \delta K(\theta) \sqrt{\frac{a(\theta)-r}{2\pi}} + O\,[(a(\theta)-r)^{3/2}] \right\} \tag{12}$$

where again the variations are from the reference circular front, of radius equal to $a(\theta)$, to the perturbed front. Substituting crack face weight function (9) into equation (10) and letting $r \rightarrow a(\theta)$, we get the following asymptotic formula

$$\delta[\Delta u(r,\theta)] = \frac{2(1-\nu^2)}{\pi E} \sqrt{\frac{r-a(\theta)}{\pi a(\theta)}} \int_0^{2\pi} \times K^0[\theta';\,a(\theta)] \left\{ \sqrt{\frac{2}{a(\theta)}} + \frac{a(\theta)\sqrt{2a(\theta)}}{a(\theta)^2 4\sin^2[(\theta-\theta')/2]} + \frac{6\sqrt{2}\cos(\theta'-\theta)}{\sqrt{a(\theta)}} \right\} [a(\theta)-a(\theta')]\, d\theta' \tag{13}$$

Comparing equation (13) with (12), we see immediately that the variation in stress intensity factor is

$$\delta K(\theta) = K(\theta) - K^0[\theta;\,a(\theta)] = \frac{1}{2\pi} PV \int_0^{2\pi} K^0[\theta';\,a(\theta)]\,[1-a(\theta')/a(\theta)] \times \left\{ 1 + \frac{1}{4\sin^2[(\theta'-\theta)/2]} + 6\cos(\theta'-\theta) \right\} d\theta' \tag{14}$$

Here PV denotes principal value. Equation (14) gives the formula to evaluate the stress intensity factor when the shape of the connection, i.e., $a(\theta')$, and the loading configuration, i.e., $K^0\,[\theta';\,a(\theta)]$, are known.

In fact, equation (14) is correct only when we do not have a displacement-restraint type of boundary condition at infinity, i.e., when the crack system is subjected only to fixed forces. Similar to the discussion in Appendix A, we treat some typical displacement boundary conditions at infinity in the following.

(*i*) "Clamped" at Infinity, i.e., Fixed Against Any Displacement. In this case, the crack face weight function should be $k_d(\theta';\,r,\,\theta;\,a)$ of equation (A-6) of Appendix A. Following the similar steps leading to equation (14), we have

$$\delta K(\theta) = \frac{1}{8\pi} PV \int_0^{2\pi} \frac{K^0[\theta';\,a(\theta)]1-a(\theta')/a(\theta)]}{\sin^2[(\theta'-\theta)/2]}\, d\theta' \tag{15}$$

(*ii*) Free Vertical Motion But Fixed Against Rotation. In this case, the crack face weight function should be $k_v(\theta';\,r,\theta;a)$ of equation (A-7) of Appendix A. Similarly we have

$$\delta K(\theta) = \frac{1}{2\pi} PV \int_0^{2\pi} K^0\,[\theta';a(\theta)]\,[1-a(\theta')/a(\theta)] \times \left\{ 1 + \frac{1}{4\sin^2[(\theta'-\theta)/2]} \right\} d\theta' \tag{16}$$

(*iii*) Free Rotation But Fixed Against Vertical Displacement Along y Axis. In this case, the crack face weight function should be $k_r(\theta';\,r,\,\theta;\,a)$ of equation (A-8). Therefore,

$$\delta K(\theta) = \frac{1}{2\pi} PV \int_0^{2\pi} K^0\,[\theta';a(\theta)]\,[1-a(\theta')/a(\theta)] \times \left\{ \frac{1}{4\sin^2[(\theta'-\theta)/2]} + 6\cos(\theta'-\theta) \right\} d\theta' \tag{17}$$

From now on, for conciseness we will refer to the above different cases of boundary conditions at infinity by their case number, e.g., case (*i*) represents fixed displacement at infinity, and the case of equations (9), (13) and (14), for which there is no restraint against displacement at infinity, will be called case (*iv*).

Growth Mode of an External Circular Crack

The previous elastic analysis of somewhat circular connections may be used to study the configurational stability of the fracturing process of a bonded circular area between two large elastic solids, at least when this occurs quasistatically (e.g., by fatigue load cycling or sustained load corrosion) under elastic fracture mechanics conditions. We study the configurational stability of the mode of growth as a concentric circle of diminishing radius for an external, initially circular, crack under some spatially fixed axisymmetric loading system. Since any somewhat noncircular crack growth profile could be represented in terms of a Fourier series, it will be sufficient to consider the following perturbation of the front in a harmonic wave form:

$$a(\theta) = a_0 - Re\,[Ae^{in\theta}] \tag{18}$$

where a_0 is a real constant, n is an integer, A is a constant (possibly complex) and $|A|/a_0 << 1$. We assume that the quasistatic growth rate of the crack increases with the intensity factor at the same location along the front. Then a small harmonic perturbation of wave number n can be said to be configurationally unstable (increase in amplitude $|A|$) during subcritical crack growth if the intensity factor $K(\theta)$ is decreased from $K^0[\theta;\,a_0]$ when $a(\theta)$ exceeds a_0 and increased when $a(\theta)$ is less than a_0, and configurationally stable if the opposite is true. That is, crack growth is likely to amplify the forms of those unstable wave configurations, if any exist. Of course, the growth or decay of the harmonic perturbations is understood to be superposed on the uniform axially symmetric diminuation of a_0 in describing the total crack growth.

Since the applied loading is now considered axially symmetric relative to the reference crack center, $K^0[\theta';\,a] = K^0[a]$, i.e., it is independent of angle. Substituting equation (18) into equations (14), (15), (16), and (17), carrying out the integrations, and expanding $K^0[a]$ to the linear term in a Taylor series about a_0, we have to the first order in $|A|$,

$$K(\theta) = K^0[a_0] - \left\{ \frac{dK^0[a_0]}{da_0} + \frac{n_1}{2a_0} K^0[a_0] \right\} Re\,[Ae^{in\theta}] \tag{19}$$

where for case (*i*), $n_1 = n$; for case (*ii*), $n_1 = n + 2$; for case (*iii*),

$$n_1 = \begin{cases} -5 & n=1 \\ n & \text{otherwise} \end{cases}$$

and for case (*iv*),

$$n_1 = \begin{cases} -3 & n=1 \\ n+2 & \text{otherwise} \end{cases} \tag{20}$$

Clearly if the sum within the curly brackets in equation (19) is positive, any perturbation from circular of the corresponding wavenumber would be diminishing, i.e., configurationally stable since K attains the smallest value at the places where the crack has grown most, i.e., where $Ae^{in\theta} = |A|$. For convenience we name this sum by $H(n_1)$ to emphasize its dependence on the number n_1 (which further relates to wave number n) so that the critical, neutrally stable situation can be said to be reached at a number n_1^c (not necessarily integer) satisfying $H(n_1^c) = 0$. It is easy to see that when $n_1 > n_1^c$ the quantity within the curly bracket in equation (19) becomes positive and it becomes negative if the opposite is true. Therefore, $n_1 < n_1^c$ must be satisfied for a configurationally unstable wavy mode perturbation. It can also be noticed that the translational shift mode, i.e., $n = 1$ is most likely to be unstable for cases (*iii*) and (*iv*) since $n_1 < 0$ in those cases, and higher modes ($n > 1$) are more likely to be unstable for cases (*i*) and (*iii*) since $n_1 = n$ in those cases. Hence it might be suitable to conclude here that case (*ii*) when points at infinity can only move freely in the vertical direction and are fixed against rotation is the most stable crack system while case (*iii*) when points at infinity can only rotate freely about a fixed point on the central axis y is the most unstable system, especially for the translational mode $n = 1$.

Remotely Applied Centered Force; Imposed Remote Displacement

Consider, for example, that a remotely applied tensile force F is transmitted across a circular connection with no net moment about the center of the connection. The case (*iv*) formulae of the last section apply here and

$$H(n_1) = \frac{dK^0}{da_0} + \frac{n_1}{2a_0} K^0 \; [a_0] = \frac{n_1 - 3}{2a_0} K^0 \; [a_0] \qquad (21)$$

By equations (20), we know that $n_1 = n + 2$ for $n > 1$. Therefore, $H > 0$ for $n > 1$ so that all perturbations of wavenumber greater than one are configurationally stable. For the translational mode, i.e., when $n = 1$ and $n_1 = -3$, it is obvious that $H < 0$ so that this mode is configurationally unstable. In fact, equation (19) becomes when $n = 1$,

$$K = K^0 \; [1 + 3 \; \mathrm{Re}(Ae^{i\theta})/a_0] \qquad (22)$$

We get the same relation by applying equation (B-8) of Appendix B, for a connection under remotely applied force and moment, as in this case the center of the connection has simply been shifted by an amount $|A|$ so as to generate a net moment equal to $F|A|$ about the $\theta = 90° - \arg(A)$ axis (here arg (A) is the phase angle of A). Therefore, equation (22) is valid even for a shift of any finite amount. This suggests that translational shift is very likely to occur when the crack system is subjected only to a centered force. It should be noted that the shape the crack will take after finite amount of growth is hard to predict because once the translational shift occurs the net moment thus generated has to be considered. The stress intensity factor will become nonuniform along the shifted circle, and thus it will not remain circular.

A case is studied in Appendix C for which the crack system is subjected to a fixed vertical displacement of amount equal to c at infinity and the stress intensity factor and crack opening displacement thus induced are also derived there. Under this displacement boundary condition, the crack face weight function should be $k_d\,(\theta';r,\theta;a)$ of equation (A-6). Hence by equation (C-4) of Appendix C, we have

$$K^0 \; [\theta; a] = \frac{Ec}{(1 - \nu^2)\sqrt{\pi a}} \qquad (23)$$

Therefore,

$$H_d = \frac{dK^0}{da_0} + \frac{n_1}{2a_0} K^0 = = \frac{(n-1)\,Ec}{2(1 - \nu^2)\sqrt{\pi a_0^3}} \geq 0 \qquad (24)$$

for $n \geq 1$. Equation (34) indicates that the translational mode is neutrally stable while perturbations of higher modes are stable. Growth in a circular shape should occur in this case.

The above results suggest that in a displacement controlled tensile test where we fix the amount of remote vertical displacement of a specimen which is constrained against rotation, growth in a circular shape should occur. In the load controlled tensile tests where a fixed, originally centered load is applied to the specimen (weight load, for example), a nonuniformity of growth which begins as an amplification of any initial nonuniformity in the translational shift mode is likely to take place, so that the crack could hardly grow in a uniform manner.

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APENDIX A

Crack Face Weight Function for an External Circular Crack

From equation (6) to (8) of the text it is clear that the formulae for the crack face opening displacement all require the knowledge of crack face weight function $k(\theta';r,\theta;a)$, i.e., the stress intensity factor induced at θ' along the reference circular front by a pair of unit wedging forces at r, θ on the crack faces outside the circular connection. As pointed out by Stallybrass (1981), this weight function solution depends on the boundary condition at infinity. Examples of such boundary conditions at infinity could be vanishing displacements or traction free conditions. In the following we follow Stallybrass (1981) and categorize the forms of crack face weight function under different boundary conditions at infinity.

The normal stress distribution within the circular connection on the prolongation of the crack plane due to a unit point wedging force pair acting to open the crack faces at location r, θ is the same as the stress field induced within a circular punch area due to an externally applied unit concentrated force on the surface of the half space at the corresponding location. That stress distribution was derived by Galin (1953) using potential theory and later by Stallybrass (1981) using an integral equation approach and, for the case when the solid is restrained against displacement at infinity, the stress at location ρ, ϕ in the connection is

$$\sigma_{yy}^{(u)}(\rho,\phi;r,\theta)=\frac{1}{\pi^2\sqrt{a^2-\rho^2}}\,\frac{\sqrt{r^2-\rho^2}}{\rho^2-2r\rho\cos(\phi-\theta)+r^2} \qquad (A\text{-}1)$$

Equation (A-1) enables us to calculate the net force and moments hence generated on each horizontal plane (planes parallel to the crack plane), i.e.,

$$P(r)=-\int_0^a\int_0^{2\pi}\sigma_{yy}^{(u)}(\rho,\phi;r,\theta)\,\rho d\rho d\phi+1$$

$$=\frac{2}{\pi}\cos^{-1}\left(\frac{a}{r}\right) \qquad (A\text{-}2)$$

The net moment generated about the $\theta + 90°$ axis is

$$M(r)=-\int_0^a\int_0^{2\pi}\rho\cos(\phi-\theta)\sigma_{yy}^{(u)}(\rho,\phi;r,\theta)\rho d\rho d\phi+r$$

$$=\frac{2}{\pi}r\left[\cos^{-1}\left(\frac{a}{r}\right)+\frac{a}{r}\left(1-\frac{a^2}{r^2}\right)^{1/2}\right] \qquad (r>a) \qquad (A\text{-}3)$$

where we also explicitly emphasized the dependence of P and M on position variable r, where the unit point force acts.

Since the stress distribution (A-1) represents the case when all displacements vanish at infinity, the above calculated net force P in (A-2) and net moment M in (A-3) are balanced by "reaction" force and moment from the restraint at infinity. In the situation when we have traction free boundary condition at infinity, i.e., when there is no restraint against displacement there, the "reaction" force and moment should be taken off by superposing equal, oppositely sensed force and moment at infinity to achieve such boundary conditions. Therefore, two auxiliary problems should be discussed prior to the full presentation of crack face weight functions, namely, the circular connection subjected to remote net centered force P and net moment M about the $\theta + 90°$ axis at infinity. Fortunately the stress distribution induced within the circular connection due to these loadings have been derived by Sneddon (1951) as

$$\sigma_{yy}^{(P)}(\rho,\phi;r,\theta)=\frac{P(r)}{2\pi a\sqrt{a^2-\rho^2}} \qquad (r>a,\rho<a) \qquad (A\text{-}4)$$

and

$$\sigma_{yy}^{(M)}(\rho,\phi;r,\theta)=\frac{3M(r)\rho\cos(\phi-\theta)}{2\pi a^3\sqrt{a^2-\rho^2}} \qquad (r>a,\rho<a) \qquad (A\text{-}5)$$

Equation (A-4) and (A-5) represents the stress distribution induced within the circular connection by net tensile force $P(r)$ and net moment $M(r)$ about $\theta + 90°$ axis at infinity. By the rule of superposition discussed before, the total stress distribution within the circular connection area would be (i) equation (A-1) if infinity is "clamped", i.e., with no displacements; (ii) equations (A-1) plus (A-4) if the solid is allowed only to move freely in the y direction (or vertically) but is fixed against rotation at infinity; (iii) equations (A-1) plus (A-5) if the solid could only rotate freely without displacement of points lying along the y axis at infinity; (iv) equations (A-1) plus (A-4) plus (A-5) if the solid is free to move without any restraint against displacement at infinity.

Assembling all the discussion made so far, we list the crack face weight function at the following typical boundary conditions at infinity:

(*i*) "Clamped" at Infinity, i.e., Fixed Against Any Displacements. In this case, the crack face weight function is simply

$$k_d(\theta';r,\theta;a)=\lim_{\rho\to a^-}\sqrt{2\pi(a-\rho)}\,\sigma_{yy}^{(u)}(\rho,\theta';r,\theta)$$

$$=\frac{\sqrt{(r^2-a^2)/a\pi^3}}{a^2+r^2-2ar\cos(\theta'-\theta)} \qquad (A\text{-}6)$$

Former discussion shows that the difference between solution under this condition and the solution under the condition of traction free, unrestrained displacement conditions at infinity lies only in terms representing the effect of a net force P as in (A–2) and a net moment M as in (A–3). Therefore, in solving the elasticity problems of a circular connection, in the first step we use above k_d as crack face weight function and in the second step we study separately the effect of the remote tensile forces and/or or moments and combine the results with those of the first step. An example of this way of thinking will be shown in Appendix B in deriving the crack face Green's function.

Solution (A-6) matches the point force solution proposed by Kassir and Sih (1975), although they failed to specify the limitation of the boundary condition at infinity on their solution.

(*ii*) Free Vertical Motion But Fixed Against Rotation at Infinity. In this case, the remote tensile centered force P of equation (A–2) should be superposed. Therefore, the crack face weight function is

$$k_v(\theta';r,\theta;a)=\lim_{\rho\to a^-}\sqrt{2\pi(a-\rho)}\,[\sigma_{yy}^{(u)}(\rho,\theta';r,\theta)$$

$$+\sigma_{yy}^{(P)}(\rho,\theta';r,\theta)]=\frac{1}{(\pi a)^{3/2}}$$

$$\times\left[\cos^{-1}\left(\frac{a}{r}\right)+\frac{a\sqrt{r^2-a^2}}{a^2+r^2-2ar\cos(\theta'-\theta)}\right] \qquad (A\text{-}7)$$

Note that this equation (A-7) coincides with the solution proposed by Tada et al. (1973), although they also did not specify the condition under which their solution would be valid.

(*iii*) Free Rotation But Fixed Against Vertical Displacement Along y Axis at Infinity. In this case, there is an additional contribution from the superposed net moment only. Therefore,

$$k_r(\theta';r,\theta;a)=\lim_{\rho\to a^-}\sqrt{2\pi(a-\rho)}\,[\sigma_{yy}^{(u)}(\rho,\theta';r,\theta)$$

$$+\sigma_{yy}^{(M)}(\rho,\theta';r,\theta)]=\frac{1}{(\pi a)^{3/2}}\left\{\frac{a\sqrt{r^2-a^2}}{a^2+r^2-2ar\cos(\theta'-\theta)}\right.$$

$$\left.+3\left[\frac{r}{a}\cos^{-1}\left(\frac{a}{r}\right)+\left(1-\frac{a^2}{r^2}\right)^{1/2}\right]\cos(\theta'-\theta)\right\} \qquad (A\text{-}8)$$

(*iv*) Traction Free at Infinity. In this case, contributions from both net force and net moment should count, and we have the following solution by Stallybrass (1981)

$$k_v(\theta';r,\theta;a)=\lim_{\rho\to a^-}\sqrt{2\pi(a-\rho)}\,[\sigma_{yy}^{(u)}(\rho,\theta';r,\theta)$$

$$+\sigma_{yy}^{(P)}(\rho,\theta';r,\theta)+\sigma_{yy}^{(M)}(\rho,\theta';r,\theta)]=\frac{1}{(\pi a)^{3/2}}$$

$$\times\left\{\left[\cos^{-1}\left(\frac{a}{r}\right)+\frac{a\sqrt{r^2-a^2}}{a^2+r^2-2ar\cos(\theta'-\theta)}\right]+3\left[\frac{r}{a}\cos^{-1}\right.\right.$$

$$\left.\left.\left(\frac{a}{r}\right)+\left(1-\frac{a^2}{r^2}\right)^{1/2}\right]\cos(\theta'-\theta)\right\} \qquad (A\text{-}9)$$

APPENDIX B

Crack Face Green's Function; Opening Displacement for Circular Connection under Remotely Applied Force and Moment

Equation (9) of the text gives the expression for the crack face Green's function as

$$D(r,\theta;\rho,\phi)=2\frac{1-\nu^2}{E}\int_a^{\min(r,\rho)}\int_0^{2\pi}$$

$$\times k(\theta';\rho,\phi;a')k(\theta';r,\theta;a')a'da'd\theta' \qquad (B\text{-}1)$$

Temporarily let us impose the condition that all the displacements vanish at infinity. In this case, we could replace crack face weight function k in equation (B-1) by k_d in equation (A-6). Therefore, replacing k by k_d and D by D_d in equation (B-1), we have

$$D_d(r,\theta;\rho,\phi)=2\frac{1-\nu^2}{E\pi^3}\int_a^{\min(r,\rho)}\int_0^{2\pi}$$

$$\times\frac{\lambda(r/a',\theta-\theta')\lambda(\rho/a',\phi-\theta')}{\sqrt{r^2-a'^2}\sqrt{\rho^2-a'^2}}d\theta\, da' \qquad (B\text{-}2)$$

where

$$\times\lambda(k,\phi)=(1-k^2)/(1-2k\cos\phi+k^2) \qquad (B\text{-}3)$$

has been introduced for conciseness of the formulae, following Sankar and Fabrikant (1982).

Now consider the following transformation

$$r^*=1/r;\ \rho^*=1/\rho;\ a^*=1/a;\ x=1/a' \qquad (B\text{-}4)$$

It may be shown that equation (B-2) becomes,

$$D_d(r,\theta;\rho,\phi)=2\frac{1-\nu^2}{E\pi^3r\rho}\int_{\max(r^*,\rho^*)}^{a^*}\int_0^{2\pi}$$

$$\times\frac{\lambda(x/r^*,\theta-\theta')\lambda(x/\rho^*,\phi-\theta')}{\sqrt{x^2-r^{*2}}\sqrt{x^2-\rho^{*2}}}\,d\theta' dx \qquad (B\text{-}5)$$

The integral in equation (B-5) has been studied by Gao and Rice (1987). They pointed out that above integral can be reduced to a pseudo-elliptic integral, and by a standard transformation they proved

$$\int_{\max(r^*,\rho^*)}^{a^*}\int_0^{2\pi}\frac{\lambda(x/r^*,\theta-\theta')\lambda(x/\rho^*,\phi-\theta')}{\sqrt{x^2-r^{*2}}\sqrt{x^2-\rho^{*2}}}\,d\theta' dx$$

$$=\frac{2\pi}{d^*}\arctan\left(\frac{\sqrt{(a^{*2}-r^{*2})(a^{*2}-\rho^{*2})}}{a^*d^*}\right) \qquad (B\text{-}6)$$

where $d^*=\sqrt{r^{*2}-2r^*\rho^*\cos(\theta-\phi)+\rho^{*2}}$. Using equations ($B$-4) again to transform back to the original variables, we finally have,

$$D_d(r,\theta;\rho,\phi)=4\frac{1-\nu^2}{E\pi^2d}\arctan\left(\frac{\sqrt{(r^2-a^2)(\rho^2-a^2)}}{ad}\right) \qquad (B\text{-}7)$$

where $d=\sqrt{r^2-2r\rho\cos(\theta-\phi)+\rho^2}$ is the distance between r, θ and ρ, ϕ. Equation (B-7) matches the corresponding formulae given by Galin (1953) and Stallybrass (1981). It is discussed also in Appendix D. The above result looks very similar to the crack face Green's function for internal circular cracks given by Gao and Rice (1987), which is not unexpected because of the similarity of crack face weight functions in this case.

Recall that equation (B-7) represents the crack face Green's function when the condition of vanishing displacement field at infinity is imposed. This has been called case (*i*). We know from Appendix A that by superposing a net force P of equation (A-2) and net moment M of equation (A-3) about $\theta+90°$ axis, we can get rid of the restriction on the displacement field at infinity and achieve the traction free boundary condition there with no restraint against displacement, case (*iv*), and can similarly deal with cases (*ii*) and (*iii*). Therefore, to calculate the crack face Green's function for cases (*ii*), (*iii*), and (*iv*) we need to study two auxiliary problems, namely, crack opening displacement under remote applied centered for F and moment M, where the moment M is now assumed to be about the 90° axis for convenience.

Consider that the described external circular crack system is subjected to a remotely applied tensile force F with a net moment M about the $\theta=90°$ axis. In this case, $K^0\,[\theta';a]$ is given by Tada et a. (1973) (also, see Neuber, 1937, and Sneddon, 1951) as

$$K^0\,[\theta';a]=\frac{aF+3M\cos\theta'}{2a^2\sqrt{\pi a}} \qquad (B\text{-}8)$$

Substituting equation (B-8) into equations (5), (6), and using (A-8), one may easily find

$$\frac{\partial\Delta^0u[r,\theta;a]}{\partial a}=-\frac{(1-\nu^2)}{E(\pi a)^2}\int_0^{2\pi}(F+3M\cos\theta'/a)$$

$$\times\left\{\left[\cos^{-1}\left(\frac{a}{r}\right)+\frac{a\sqrt{r^2-a^2}}{a^2+r^2-2ar\cos(\theta-\theta')}\right]\right.$$

$$\left.+3\left[\frac{r}{a}\cos^{-1}\left(\frac{a}{r}\right)+\left(1-\frac{a^2}{r^2}\right)^{1/2}\right]\cos(\theta'-\theta)\right\}d\theta'$$

$$=-2\,\frac{(1-\nu^2)}{\pi E}\left\{\frac{F}{a^2}\left[\cos^{-1}\left(\frac{a}{r}\right)+\frac{a}{\sqrt{r^2-a^2}}\right]\right.$$

$$\left.+\frac{3M\cos\theta}{a^3}\left[\frac{3r}{2a}\cos^{-1}\left(\frac{a}{r}\right)+\frac{3r}{2\sqrt{r^2-a^2}}-\frac{a^2}{2r\sqrt{r^2-a^2}}\right]\right\} \qquad (B\text{-}9)$$

Now we integrate over the radius of the connection between a and r, as in going from equation (5) to (6), and find that

$$\Delta u^0[r,\theta;a]=2\,\frac{(1-\nu^2)}{\pi E}\left\{\frac{F}{a}\cos^{-1}\left(\frac{a}{r}\right)\right.$$

$$\left.+\frac{3M\cos\theta}{2a^2}\left[\frac{r}{a}\cos^{-1}\left(\frac{a}{r}\right)+\left(1-\frac{a^2}{r^2}\right)^{1/2}\right]\right\} \qquad (B\text{-}10)$$

If we replace F by $P(\rho)$ of equation (A-2) and M by $M(\rho)$ (also the axis of the moment is changed to $\phi+90°$) of equation (A-3) of Appendix A, we therefore could rewrite equation (B-10) as

$$\Delta u^0[r,\theta;a]=4\,\frac{(1-\nu^2)}{\pi^2Ea}\left\{P(r)P(\rho)\right.$$

$$\left.+\frac{3M(r)M(\rho)}{2a^2}\cos(\phi-\theta)\right\} \qquad (B\text{-}11)$$

If we combine equation (B-11) and equation (B-7), we have, for case (*iv*)

$$D(r,\theta;\rho,\phi)=4\,\frac{1-\nu^2}{E\pi^2a}\left\{\frac{a}{d}\arctan\left(\frac{\sqrt{(r^2-a^2)(\rho^2-a^2)}}{ad}\right)\right.$$

$$\left.+P(r)P(\rho)+\frac{3M(r)M(\rho)}{2a^2}\cos(\phi-\theta)\right\} \qquad (B\text{-}12)$$

where $P(r)$, $M(r)$ are given by equations (A-2) and (A-3). We also observe that the symmetry is indeed preserved in equation (B-12). The M terms are deleted in (B-12) to give D for case (*ii*), and the P terms are deleted to give D for case (*iii*).

In going from equation (B-8) to equation (B-10), we have just shown an example of how to calculate the crack opening displacement from the knowledge of the distribution of stress intensity factor along the crack front. It is, of course, also possible to go in the reverse direction, i.e., to extract the intensity factor distribution from the near front behavior of the opening displacement. When $r \to a^+$, $\cos^{-1}(a/r) \to \sqrt{2(r-a)/a}$ and $(1-a^2/r^2)^{1/2} \to \sqrt{2(r-a)/a}$, so that equation ($B$-10) shows

$$\Delta u^0[r,\theta;a] = 8\,\frac{(1-\nu^2)}{E}\left(\frac{F}{2a\sqrt{\pi a}} + \frac{3M\cos\theta}{2a^2\sqrt{\pi a}}\right)\sqrt{\frac{r-a}{2\pi}} + O[(r-a)^{3/2}]$$

$$= 8\,\frac{(1-\nu^2)}{E}\sqrt{\frac{r-a}{2\pi}}\,K^0[\theta;a] + O[(r-a)^{3/2}] \qquad (B\text{-}13)$$

The latter version of equation (B-13) represents the known asymptotic behavior of crack opening displacement near the crack front for any tensile crack.

When $r \to \infty$, $\cos^{-1}(a/r) \to \pi/2$,

$$\Delta u^0[r,\theta;a] \sim \frac{(1-\nu^2)}{E}\left\{\frac{F}{a} + \frac{3M}{2a^2}\,\frac{r\cos\theta}{a}\right\} \qquad (B\text{-}14)$$

Equation (B-14) shows that the crack faces far from the front would tend to be linear flat planes (free of stress) with slopes about $\theta = 90^\circ$ of $\pm 3M(1-\nu^2)/(2a^3E)$. These planes would ultimately contact when $M \neq 0$, invalidating the present solution, if the jointed solids are truly unbounded half spaces, although this need not be a problem in practice for finite joined bodies, especially if the mathematically planar crack represents a shallow notch cut-out.

APPENDIX C

Stress Intensity Factor of an External Circular Crack with Fixed Displacement at Infinity

Assume that the elastic solid is subjected to a fixed amount of vertical displacement at infinity as following

$$u_{\pm\infty} = (c+\alpha x)\,\mathrm{sgn}(y) \qquad (C\text{-}1)$$

where $\mathrm{sgn}(y) = y/|y|$ for $y \neq 0$, and the same coordinate system as used in the text is adopted and $x = r\cos\theta$. We solve here for the stress intensity factor induced by this displacement in equation (C-1) and the crack opening displacement function.

Referring to equation (B-14) of Appendix B, we know that under remotely applied centered force and moment the crack faces far from the crack front would tend to become linear flat planes (free of stress). We also observe that under the imposed remote dispalcement field (C-1), crack faces far from the front should approach the same displacement field at infinity because the stresses approach zero there. Therefore imposing a fixed displacement field at infinity ($y \to \pm\infty$) is equivalent to imposing a net tensile force and a net moment at infinity for an external circular crack. Now consider a crack system subject only to a tensile force F and a net moment M at infinity but otherwise traction free. From Appendix B, we know crack opening displacement far from the crack front is

$$\Delta u^0[r,\theta;a] \sim \frac{(1-\nu^2)}{E}\left\{\frac{F}{a} + \frac{3M}{2a^2}\,\frac{r\cos\theta}{a}\right\}$$

Now let

$$\Delta u^0[r,\theta;a] = u_{+\infty} - u_{-\infty} \qquad (C\text{-}2)$$

Comparing both sides of equation (C-2), we find the following relations

$$F = \frac{2Eac}{1-\nu^2}\,. \qquad M = \frac{4Ea^3\alpha}{3(1-\nu^2)} \qquad (C\text{-}3)$$

Therefore by equation (B-8) and (B-10) the stress intensity factor induced is

$$K^0[\theta;a] = \frac{E}{1-\nu^2}\,\frac{c+2a\alpha\cos\theta}{\sqrt{\pi a}} \qquad (C\text{-}4)$$

and the crack opening displacement is

$$\Delta u^0[r,\theta;a] = \frac{4}{\pi}\left\{c\cos^{-1}\left(\frac{a}{r}\right) + \alpha a\cos\theta\left[\frac{r}{a}\cos^{-1}\left(\frac{a}{r}\right) + \left(1-\frac{a^2}{r^2}\right)^{1/2}\right]\right\} \qquad (C\text{-}5)$$

APPENDIX D

General Displacement Green's Function and Stress Field for Internal and External Circular Cracks

When a three-dimensional crack system is subjected to tensile loading that is symmetric relative to the crack plane, it is known that the elasticity equations and boundary conditions can be satisfied if the displacement and stress field are written as (Galin, 1953; Green and Zerna, 1954; Meade and Keer, 1984)

$$\begin{aligned} u_y &= -2[(1-\nu^2)/E]Y + [(1+\nu)/E]y\partial Y/\partial y \\ u_x &= [(1+\nu)/E]\partial(F+yY)/\partial x \\ u_z &= [(1+\nu)/E]\partial(F+yY)/\partial z \end{aligned} \qquad (D\text{-}1)$$

where F and Y are harmonic functions related by $\partial F/\partial y = (1-2\nu)Y$. The coordinates are set up in the same manner as in the text with the crack on the $y = 0$ plane. The stress components that enter crack surface boundary conditions are calculated from stress-strain relations as

$$\begin{aligned} \sigma_{yy} &= -\partial Y/\partial y + y\partial^2 Y/\partial y^2 \\ \sigma_{yz} &= y\partial^2 Y/\partial y\partial x, \quad \sigma_{yz} = y\partial^2 Y/\partial y\partial z \end{aligned} \qquad (D\text{-}2)$$

It is seen from equations (D-2) that there is no shear traction on $y = 0$. Thus the problem of loading on the crack face is one of finding a function Y satisfying $\nabla^2 Y = 0$, vanishing at infinity (at least for case (i)), and generating stress $\sigma(x,z)$ and opening gap $\Delta u(x,z)$ on $y = 0$ given by

$$\sigma(x,z) = -\partial Y/\partial y|_{y=0} \text{ and } \Delta u = -[4(1-\nu^2)/E]Y|_{y=0^+} \qquad (D\text{-}3)$$

(a) Internal Circular Cracks. Now we consider the elasticity problem of a three-dimensional elastic solid with an internal circular crack of radius equal to a subjected to a point force pair in the $\pm y$ directions acting at ξ, η on the crack faces. According to equations (D-1), (D-2), and (D-3), we formulate following problem,

$$\begin{aligned} &\nabla^2 Y = 0 \\ &Y = 0 \text{ when } x^2+z^2 \geq a; \quad y = 0 \\ &\partial Y/\partial y = -\delta(x-\xi)\delta(z-\eta) \text{ when } x^2+z^2 \leq a; \; y = 0 \\ &Y = 0 \quad \text{at} \quad \infty \end{aligned} \qquad (D\text{-}4)$$

Let us denote the solution to equation (D-4) as $Y = H(x,y,z;\xi,\eta)$. It is known (Galin, 1953) that

$$H(x,y,z;\xi,\eta) = -\frac{1}{\pi^2 d}\arctan\left\{\frac{\sqrt{(a^2-\xi^2-\eta^2)(a^2-x^2-y^2-z^2+R)}}{\sqrt{2}ad}\right\} \qquad (D\text{-}5)$$

where

$$d^2 = (x-\xi)^2 + (z-\eta)^2 + y^2$$

$$R^2 = (a^2 - x^2 - y^2 - z^2)^2 + 4a^2y^2 \qquad (D\text{-}6)$$

Replacing Y by H in equations (D-1) and (D-2), we then get the displacement Green's function and stress field at an arbitrary location in space. Specifically, the crack face Green's function is seen to be

$$D(x,z;\xi,\eta) = \frac{4(1-\nu^2)}{E} H(x,0,z;\xi,\eta)$$

$$= \frac{4(1-\nu^2)}{\pi^2 Ed} \arctan\left\{\frac{\sqrt{(a^2-\xi^2-\eta^2)(a^2-x^2-y^2)}}{ad}\right\} \qquad (D\text{-}7)$$

If expressed in polar coordinates, equation (D-7) becomes,

$$D(r,\theta;\rho,\phi) = \frac{4(1-\nu^2)}{\pi^2 Ed} \arctan\left\{\frac{\sqrt{(a^2-\rho^2)(a^2-r^2)}}{ad}\right\} \qquad (D\text{-}8)$$

where d reduced to $\sqrt{r^2 - 2r\rho\cos(\theta-\phi) + \rho^2}$. Equation ($D$-8) coincides with the solution derived through the perturbation analysis by Gao and Rice (1987, Appendix A).

(b) External Circular Cracks. Now we consider a similar crack system but with an external circular crack, or a circular connection of radius equal to a subjected to a point force pair in the $\pm y$ directions acting at ξ, η on the crack faces with zero displacement at infinity. In an analogous way we formulate the problem as solving

$$\nabla^2 Y = 0$$

$$Y = 0 \text{ when } x^2 + z^2 \le a;\ y = 0$$

$$\partial Y/\partial y = -\delta(x-\xi)\delta(z-\eta) \text{ when } x^2 + z^2 \ge a;\ y = 0 \qquad (D\text{-}9)$$

$$Y = 0 \quad \text{at } \infty$$

Note that in this formulation we imply that there is no displacements at infinity and hence the solution thus generated can only be applied to case (i) of the text. The solution to equation (D-9), dentoed here as $Y = L(x, y, z; \xi, \eta)$, was given also by Galin (1953) as

$$L(x,y,z;\xi,\eta) = -\frac{1}{\pi^2 d} \arctan$$

$$\left\{\frac{\sqrt{(\xi^2+\eta^2-a^2)(x^2+y^2+z^2-a^2+R)}}{\sqrt{2}ad}\right\} \qquad (D\text{-}10)$$

where d and R are given by equations (D-6). Similarly if we replace Y by L in equations (D-1) and (D-2) the displacement Green's function and stress field at an arbitrary location in an elastic solid with an external crack are generated. Specifically the crack face Green's function can be extracted and expressed in polar coordinates as

$$D_d(r,\theta;\rho,\phi) = \frac{4(1-\nu^2)}{\pi^2 Ed} \arctan\left\{\frac{\sqrt{(\rho^2-a^2)(r^2-a^2)}}{ad}\right\} \qquad (D\text{-}11)$$

Equation (D-11) coincides with (B-7) of Appendix B, where the crack face Green's function is derived by the perturbation formalism.

Let us note that by using the solutions of equations (D-5) and (D-10) for Y in equations (D-1), we can compute the displacements u_z, u_y, and u_z at (x, y, z) due to unit opening point forces acting on the crack faces at (ξ, η, 0), for the respective internal and external circular crack cases. By the elastic reciprocal theorem, those very same results for u_z, u_y, and u_z also represent the opening gaps Δu on the crack faces induced at (ξ, η, 0) by unit point forces at (x, y, z) in the respective x, y, and z directions. But from the knowledge of that opening gap Δu in the vicinity of the crack front, one may also calculate (e.g., equation (B-13)) the tensile mode stress intensity factors induced by the unit point forces at (x, y, z) in the respective x, y, and z directions. These stress intensity factor defined the x, y, and z components of the tensile mode vector weight function **h** as introduced by Rice (1972, 1985). Hence, although we do not further pursue the details here, the results of this Appendix allow calculation of the vector tensile mode weight function at general field points for internal and external circular cracks.