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Two general integrals of singular crack tip deformation fields

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Abstract. The Eshelby tensor E has vanishing divergence in a homogeneous elastic material, whereas the invariance of the crack tip J integral suggests, in accord with known solutions, that the product rE will have a finite limit at the tip. Here r is distance from the tip. These considerations are shown to lead to two general integrals of the equations governing singular crack tip deformation fields. Some of their consequences are discussed for analysis of crack tip fields in linear and nonlinear materials.

Introduction

Consider a homogeneous elastic solid, linear or nonlinear, containing a planar crack on $x_2 = 0, x_1 < 0$ (Figure 1). The solid is loaded such that the near tip field is two-dimensional in the x_1, x_2 plane, thus consisting of some combination of in-plane and anti-plane deformation with stresses $\sigma_{ij} = \sigma_{ij}(x_1, x_2)$ and displacements $u_k = u_k(x_1, x_2)$. Here Latin indices $i, j, k, \ldots$ range over 1, 2, 3, whereas Greek indices $\alpha, \beta, \gamma, \ldots$ range over 1, 2 only. The analysis which follows applies also to elastic-plastic solids treated within the approximation of the "deformation", or "total strain", formulation.

The governing equations are the three equilibrium conditions (in the absence of body forces)

$$\sigma_{\alpha j,\alpha} = 0 \tag{1}$$

($f_{,\alpha} = \partial f/\partial x_\alpha$) and the stress-displacement gradient relations

$$\sigma_{ij} = \partial W/\partial u_{j,i} \tag{2}$$

where W is the stress work (or strain energy) density, and is a function of displacement gradients that is properly invariant to rigid rotations. Further, a consequence of these equations is that the integrals

$$J_\alpha \equiv \oint_C n_\beta E_{\beta\alpha}\, \mathrm{d}s = 0 \tag{3}$$

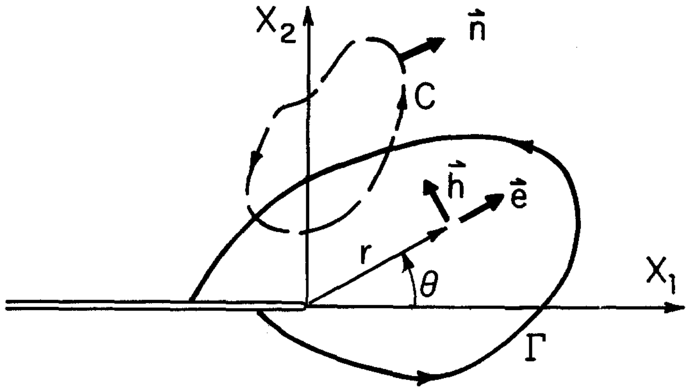

Fig. 1.

for any closed contour C not cutting across the crack surfaces (Fig. 1) where $E_{\beta\alpha} = E_{\beta\alpha}(x_1, x_2)$ is the Eshelby (1956) tensor

$$E_{\beta\alpha} = W\delta_{\beta\alpha} - \sigma_{\beta j}u_{j,\alpha}, \tag{4}$$

n_α is the vector normal to the contour and s is the arc length along it. Hence

$$E_{\beta\alpha,\beta} = 0. \tag{5}$$

For a path Γ surrounding the crack tip (Fig. 1), J_1 reduces to the path-independent crack tip integral J, which may be proven to give the energy release per unit area of crack growth (Rice, 1968a, b; Eshelby, 1970), as shown also by Cherepanov (1967) in the limit when Γ is shrunk onto the tip. Further, following the development of Rice and Drucker (1967), the energy release rate $J > 0$ for any loaded solid for which crack enlargement reduces its overall stiffness. Excepting trivial cases such as crack-parallel uniaxial tension fields, unaffected by the presence of the crack, such response with $J > 0$ must be the case with all solids for which the potential energy is minimum (versus being only stationary) at equilibrium relative to adjacent kinematically admissible states, and that property of the potential energy is, in turn, insured by forms of W satisfying convexity,

$$(W^B - W^A) - \sigma_{ij}^A(u_{j,i}^B - u_{j,i}^A) > 0 \tag{6}$$

for all deformation states A and B differing other than by a rigid rotation. (However, in the context of finite elasticity, the condition just discussed may reasonably be conjectured to be stronger than necessary to ensure $J > 0$).

For linear elasticity, positive definiteness of W, as always assumed, suffices for $J > 0$.

At this point it is useful to distinguish two contexts in which the system of equations above will apply: (i) the "small strain" formulation (or approximation), in which W is assumed to depend only on the strain-type combinations $u_{i,j} + u_{j,i}$ of displacement gradients. Such a model incorporates linear elasticity but also includes non-linear stress-strain behavior as frequently assumed in crack analysis, e.g., of HRR type singular fields in power-law hardening materials (Hutchinson, 1968; Rice and Rosengren, 1968). (ii) The arbitrary strain formulation, in which the x_α are regarded as material coordinates in an undeformed reference configuration, contours C and Γ and their normals n_α and arc length s are measured in that configuration, σ_{ij} is the unsymmetrical nominal stress (or "Lagrangian" or "first Piola-Kirchhoff" stress, the force per unit undeformed area; e.g., Malvern (1969), Sect. 5.3), and W is reckoned per unit undeformed volume. In this interpretation rotation invariance requires that W depend only on combinations of displacement gradients of the form $u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}$.

Crack tip singular fields and general integrals

Following Rice and Rosengren (1968), we take Γ to be a circular path of radius r, write $E_{\alpha\beta} = E_{\alpha\beta}(r, \theta)$, and introduce unit vectors e_α and h_α (Fig. 1) in the respective r and θ directions (note that for context (ii) above, r, θ, e_α and h_α are defined in the undeformed configuration):

$$(e_1, e_2) = (\cos\theta, \sin\theta), \quad (h_1, h_2) = (-\sin\theta, \cos\theta). \tag{7}$$

Thus

$$J = \int_{-\pi}^{+\pi} r e_\alpha(\theta) E_{\alpha 1}(r, \theta)\, \mathrm{d}\theta \tag{8}$$

where r is understood to be fixed during the integration on θ. Since the result must be independent of r, we see that in angular average sense $re_\alpha E_{\alpha 1}$ must be independent of r and, in particular, must remain non-zero and finite as $r \to 0$.

Thusly motivated, some properties of crack tip singular fields of a type such that the limit

$$rE_{\alpha\beta}(r, \theta) \to F_{\alpha\beta}(\theta) \quad \text{as} \quad r \to 0 \tag{9}$$

exists are examined here, i.e., for fields such that the asymptotic form $E_{\alpha\beta} \sim F_{\alpha\beta}(\theta)/r$ applies near the tip, where the θ dependence, $F_{\alpha\beta}(\theta)$, is to be determined (or constrained).

General statements about the existence of singular solutions of this type are difficult to make. We may note, however, that known crack tip singularities for linear elasticity, power-law hardening within the small strain formulation (HRR field), and general nonlinear hardening solids in anti-plane shear (Rice, 1967, 1968a; Knowles, 1977), and in plane stress (Wong and Shield, 1969) and plane strain (Knowles and Sternberg, 1973; Stephenson, 1982) within finite strain elasticity are all of this type.

The cases just mentioned involve distributions of stress σ_{ij} and displacement gradient $u_{j,i}$ which vary smoothly with θ at any fixed $r > 0$. In other cases the near tip field is less smooth. For example, anti-plane crack solutions for certain types of anisotropic ideally plastic solids (Rice, 1967b, 1984), notably for ideally plastic single crystals (Rice and Nikolic, 1985), involve solutions for which σ_{ij} is finite at $r = 0$ whereas u_j varies discontinuously with θ across one or more rays emanating from $r = 0$. These solutions, for stationary cracks, involve proportional plastic straining and hence can also be interpreted as valid nonlinear elastic solutions. Similar features of crack tip fields have been found for nonlinear materials which lose ellipticity, and strain soften, when deformed beyond a certain threshold (Knowles and Sternberg, 1981; Fowler, 1984). It seems that the results which follow apply to such cases, with the understanding that certain components of $F_{\alpha\beta}(\theta)$ may have a Dirac singular pulse across the ray of discontinuity. These cases are discussed in the second to last subsection of the paper.

Equation (5) with use of polar coordinates requires that

$$E_{\beta\alpha,\beta} = (h_\beta/r)\partial E_{\beta\alpha}/\partial\theta + e_\beta \partial E_{\beta\alpha}/\partial r = 0, \tag{10}$$

and this may be rewritten, after multiplication by r^2 and recognition that $\partial h_\alpha/\partial\theta = e_\alpha$, as

$$\partial(rh_\beta E_{\beta\alpha})/\partial\theta + r\partial(re_\beta E_{\beta\alpha})/\partial r = 0, \tag{11}$$

Now let $r \to 0$ and use eq. (9), recognizing also that the limit of the second term here must be zero (if it were non-zero, the integration would show $re_\beta E_{\beta\alpha}$ to be logarithmically singular in r at $r = 0$, which contradicts the assumption made above), to obtain

$$d[h_\beta(\theta)F_{\beta\alpha}(\theta)]/\mathrm{d}\theta = 0. \tag{12}$$

Thus one has the pair of integrals for ($\alpha = 1, 2$)

$$h_\beta(\beta)F_{\beta\alpha}(\theta) = -C_\alpha \text{ (= constants).} \tag{13}$$

These may be rewritten, using eqs. (4) and (9), as the two general integrals

$$r(Wh_\alpha - h_\beta\sigma_{\beta j}u_{j,\alpha}) = -C_\alpha \quad (\alpha = 1, 2) \tag{14}$$

of the crack tip singular field where here, and in the equations which follow, the understanding of the left hand side is as a limit with $r \to 0$.

One of the constants must be zero for a stress free crack surface. That is because on $\theta = \pi$ or $-\pi$, h_1 vanishes and so also does the traction vector $h_\beta\sigma_{\beta j}$. Thus

$$C_1 = 0. \tag{15}$$

An equivalent form for the pair of integrals follows after taking the scalar product of eq. (14) first with h_α and then with e_α. Thus

$$r(W - h_\beta\sigma_{\beta j}u_{j,\alpha}h_\alpha) = rW - T_j\partial u_j/\partial\theta = -C_2 \cos\theta, \tag{16a}$$

and

$$rh_\beta\sigma_{\beta j,\alpha}e_\alpha = T_j r\partial u_j/\partial r = C_2 \sin\theta, \tag{16b}$$

where

$$T_j = h_\beta\sigma_{\beta j} \tag{17}$$

is the traction vector acting on the ray θ = constant, according to the direction of increasing θ. Note that in calculating $\partial u_j/\partial\theta$, the components u_j are understood to be cartesian (versus polar); the distinction does not matter for $\partial u_j/\partial r$.

Some consequences

1. Energy density on the crack faces

Let $\theta = \pm\pi$, at which $T_j = 0$. Then (16a) implies

$$rW(r, \pi) = rW(r, -\pi) = C_2, \tag{18}$$

which relates C_2 to the singularity in energy density along the crack faces (i.e., to the limit as $r \to 0$ of rW), and shows coincidently that the strength of the singularity in W must be the same on both crack faces. The latter was proved in a related manner by Budiansky and Rice (1973), and has remarkable consequences when one realizes that it must hold for arbitrary mixed-mode loading.

For example, consider an isotropic material, treated within the small strain formulation, for which W is a homogeneous function of degree $(1 + n)$ in the stresses, at least for the large stress levels prevailing near the crack tip. This describes a material for which, in uniaxial tension σ_{11}, $u_{1,1} \propto \sigma_{11}^n$, and is the type of material for which HRR fields have been derived (with the additional assumption of incompressibility): $n = 1$ corresponds to perfect elaticity and $n \to \infty$ to perfect plasticity. Following Budiansky and Rice (1973), for general mixed mode I and II loading of a crack in a material with symmetrical response in tension and compression, the energy density has the form

$$W = b|\sigma_{11}|^{1+n} \quad (b = \text{constant}) \tag{19}$$

on the crack faces (where all other stresses vanish except for the reaction stress σ_{33}, which must be a fixed multiple of σ_{11}). Thus

$$|\sigma_{11}| = (C_2/br)^{1/(1+n)} \tag{20}$$

on the crack faces. For pure mode I (tensile) loading, one knows from the HRR tensile fields that $C_2 \neq 0$ if $n \neq 1$. (The case $n = 1$, linear elasticity, is exceptional in that C_2 vanishes for pure mode I loading; i.e., σ_{11} is then not singular as $r \to 0$ along the crack faces). Also, the symmetry of mode I requires that $\sigma_{11}^+ = \sigma_{11}^-$ ($+$, $-$ denote top and bottom crack face). However the symmetry of mode II (in-plane shear) loading requires that $\sigma_{11}^+ = -\sigma_{11}^-$. Evidently, for general mixed mode loading one must have $|\sigma_{11}|^+ = |\sigma_{11}|^-$ and hence, as the ratio of mode II to mode I loading is increased from 0 to ∞, there must be at least one transition from a range of mixed mode loading in which $\sigma_{11}^+ = \sigma_{11}^-$ to another range in which $\sigma_{11}^+ = -\sigma_{11}^-$. In an attempt to locate the transition point numerically, Shih (1974) reports that it must correspond to a very small mode II/mode I ratio, and speculates that the transition may be at 0^+. For the same far-field load intensity, as measured by J, C_2 for mode I is much smaller than C_2 for mode II and thus transition is not readily determined.

2. Constant C_2 for isotropic linear elastic crack tip fields

In this case one may evaluate C_2 from the formula for energy density along the crack faces, eq. (18). In terms of stress, and noting the combination of plane and anti-plane fields,

$$W = (1 - \nu)\sigma_{11}^2/4\mu + \sigma_{13}^2/2\mu \quad (\theta = \pm\pi) \tag{21}$$

on the crack faces. Here μ = shear modulus, ν = Poisson ratio. From the standard expresions for crack tip elastic fields expressed in terms of intensity factors, where those factors satisfy

$$(\sigma_{22}, \sigma_{21}, \sigma_{23})_{\theta=0} = (K_{\mathrm{I}}, K_{\mathrm{II}}, K_{\mathrm{III}})/\sqrt{2\pi r}, \tag{22}$$

one finds that

$$(\sigma_{11}, \sigma_{13}) \equiv \mp (2K_{\mathrm{II}}, K_{\mathrm{III}})/\sqrt{2\pi r} \quad \text{at} \quad \theta = \pm\pi. \tag{23}$$

Thus, for the isotropic linear solid the constant C_2 of the two integrals identified here is

$$C_2 = (1 - \nu)K_{\mathrm{II}}^2/2\pi\mu + K_{\mathrm{III}}^2/4\pi\mu, \tag{24}$$

whereas it is known that

$$J = (1 - \nu)(K_{\mathrm{I}}^2 + K_{\mathrm{II}}^2)/2\mu + K_{\mathrm{III}}^2/2\mu. \tag{25}$$

Comparison of these results, and comments in the previous section and eq. (37) to follow, confirm that C_2 is not a universal function of J, but depends on the loading mode and the nature of material response.

3. Orthogonality of traction and displacement gradient ahead of crack

When $\theta = 0$, eq. (16b) reduces to $rT_j\partial u_j/\partial r = 0$, or to

$$r\sigma_{2j}\partial u_j/\partial r = 0 \quad \text{on} \quad \theta = 0. \tag{26}$$

This too must hold for arbitrary mixed mode loading. Consider the near tip field in an isotropic linear elastic material. The displacements (apart from unessential rigid motions) along the prolongation of the crack must, from symmetry considerations, have the form

$$(u_1, u_2, u_3) \sim (gK_{\mathrm{I}}, hK_{\mathrm{II}}, 0)\sqrt{r} \quad \text{on} \quad \theta = 0 \tag{27}$$

where g and h are constants. Their values can be looked up in all the standard sources where crack tip stress and displacement fields are tabulated (e.g., Rice, 1968a). However, eq. (26) with eqs. (27) and (22) requires that g and h satisfy

$$K_{\rm II}K_{\rm I}g + K_{\rm I}K_{\rm II}h = 0 \tag{28}$$

for arbitrary $K_{\rm I}$ and $K_{\rm II}$, so it can be anticipated that the factors g and h to be found in those sources will satisfy $h = -g$. (They do!).

The underlying orthogonality of $r\partial u_j/\partial r$ and T_j (or σ_{2j}) on $\theta = 0$ is, however, more general and must hold whether the material is nonlinear, anisotropic or both, no matter what mixed mode loading combination acts.

4. HRR fields in anti-plane shear

Let $W = na\gamma^{(n+1)/n}/(n + 1)$ at sufficiently large strain, with $\gamma^2 = u_{3,\alpha}u_{3,\alpha}$ for states of anti-plane strain. This describes an isotropic material for which $\tau = a\gamma^{1/n}$ at large enough strain where $\tau^2 = \sigma_{\alpha 3}\sigma_{\alpha 3}$, and for which the ratio σ_{13}/σ_{23} is the same as $u_{3,1}/u_{3,2}$. The usual application is to elastic-plastic materials with $\gamma \propto \tau^n$ in the plastic range.

To characterize the deformation, let ϕ be the angle, measured positive anti-clockwise, between the vector with components $u_{3,\alpha}$ and the x_2 axis, as in previous analyses of this case by Rice (1967, 1968a). Then

$$\partial u_3/\partial r = -\gamma \sin(\phi - \theta), \quad (1/r)\partial u_3/\partial\theta = \gamma\cos(\phi - \theta), \tag{29}$$

and the stress-strain relations give

$$T_3 = h_\beta\sigma_{\beta 3} = \sigma_{03} = a\gamma^n\cos(\phi - \theta). \tag{30}$$

Thus the two integrals of eqs. (16a) and (16b) are (in formulae again valid only as $r \to 0$)

$$ar\gamma^{(n+1)/n}[n/(n + 1) - \cos^2(\phi - \theta)] = -C_2\cos\theta, \tag{31a}$$

$$-ar\gamma^{(n+1)/n}\cos(\phi - \theta)\sin(\phi - \theta) = C_2\sin\theta. \tag{31b}$$

These completely solve for γ and ϕ in terms of r and θ (and C_2, which scales the field and which can be determined in terms of J). Thus the two integrals, used together with stress-strain relations, have replaced the two equations expressing equilibrium and strain compatibility, i.e., the two equations

$$\sigma_{13,1} + \sigma_{23,2} = 0 \quad \text{and} \quad (u_{3,1})_{,2} = (u_{3,2})_{,1}, \tag{32}$$

which would normally be required to determine the field. (Note that the approach taken in deriving eqs. (31) above, from eqs. (16), never explicitly recognized a compatibility condition between the $u_{3,\alpha}$, hence the need for the latter equation of the set just written).

For comparison, the near tip singular solution derived for this case by Rice (1967a, 1968a) is (when converted to the present notation and normalized by J)

$$r \cos \theta = [(n - 1)/(n + 1) + \cos 2\phi] J/\pi a \gamma^{1+1/n} \tag{33a}$$

$$r \sin \theta = (\sin 2\phi) J/\pi a \gamma^{1+1/n}. \tag{33b}$$

It is hardly obvious that this is the same solution as derived above, but by taking the ratio of eq. (31b) to (31a) one may solve for ϕ in terms of θ, and the same may be done by taking the ratio of eq. (33b) to (33a). After considerable manipulation, these are both found to give the same result that

$$\phi = (1/2)\left[\theta + \arcsin\left(\frac{n-1}{n+1}\sin\theta\right)\right]. \tag{34}$$

The first set of equations then yields, e.g., on $\theta = 0$,

$$\gamma = [(n + 1)C_2/ar]^{n/(n+1)} \quad \text{on} \quad \theta = 0, \tag{35}$$

whereas the second yields

$$\gamma = [2nJ/(n + 1)\pi a r]^{n/(n+1)} \quad \text{on } \theta = 0. \tag{36}$$

These will be identical if

$$C_2 = 2nJ/(n + 1)^2\pi, \tag{27}$$

which is a relation that can be extracted directly from the solution of eqs. (31) when it is used to calculate the J integral.

5. Finite stress fields and possibility discontinuous displacement fields near the crack tip

Consider ideally plastic solids or solids which exhibit a peak strength and then soften with increasing deformation in a post-elliptic regime. When solutions exist they would seem, as for the known solutions, to necessarily have finite T_j near the tip and, at least in context (i) of the Introduction, fully finite σ_{ij}. Let $U_j(\theta)$ be the crack tip displacement field, i.e.,

$$u_j(r, \theta) \to U_j(\theta) \quad \text{as} \quad r \to 0. \tag{38}$$

It is characteristic of ideally plastic solids that $U_j(\theta)$ does indeed vary with θ. In fact, as mentioned just before eq. (10), some of the known anti-plane ideally plastic solutions (but not, e.g., that for an isotropic solid), and also the solutions presented for nonlinear materials with a finite peak strength and softening, exhibit discontinuous $U_j(\theta)$ at one or more θ (other than along the crack itself at $\theta = \pm\pi$).

Equations (16) should apply when there are no such discontinuities, and we shall assume tentatively that they apply also when there are discontinuities, subject to later re-examination. Observe first that the finiteness of u_j at $r = 0$ requires that

$$r\partial u_j/\partial r \to 0 \quad \text{as} \quad r \to 0; \tag{39}$$

if the limit were instead non-zero, u_j would become logarithmically infinite at the tip. Thus eq. (16b) together with the fact that T_j is finite for the class of materials considered here requires that

$$C_2 = 0. \tag{40}$$

The argument based on eq. (39) applies, so that eq. (16b) with $C_2 = 0$ will be valid, whether $U_j(\theta)$ is discontinuous or not.

Equation (16a) now requires that

$$rW \to T_j \, \mathrm{d}U_j/\mathrm{d}\theta \quad \text{as} \quad r \to 0, \tag{41}$$

where T_j is understood as the limit of $T_j(r, \theta)$ as $r \to 0$.

When $U_j(\theta)$ is discontinuous by amount D_j at $\theta°$, $\mathrm{d}U_j/\mathrm{d}\theta \sim D_j\delta_D(\theta - \theta°)$, where δ_D is the Dirac pulse function, and thus eq. (41) requires that

$$rW \sim T_j^{\circ} D_j \delta_D(\theta - \theta°) \quad \text{near} \quad \theta = \theta° \tag{42}$$

where T_j° is T_j at $\theta°$. This may be verified separately as follows. Observe that consistent definition of W in the presence of discontinuities must accord with the weak form of eq. (2). Such a form can be obtained by first rewriting that equation as

$$\delta W = \sigma_{ij}\delta u_{j,i} = \sigma_{\alpha j}\delta u_{j,\alpha}, \tag{43}$$

and then integrating over the region within a contour like C in Figure 1, using $\sigma_{\alpha j,\alpha} = 0$, to write the weak form as

$$\delta \int_A Wr \, d\theta \, dr = \oint_C n_\alpha \sigma_{\alpha j} \delta u_j \, ds \tag{44}$$

where A is bounded by C. Let $\varepsilon > 0$ and let C be the pie-sliver contour bounded by $\theta = \theta^\circ - \varepsilon$, $r = r_1$, and $\theta = \theta^\circ + \varepsilon$. Here for simplicity it is assumed that the discontinuity remains on the ray $\theta = \theta^\circ$ for some finite r, as in the anisotropic single crystal ideally plastic solutions (e.g., Rice and Nikolic, 1985), although such does not fully describe other cases of interest (e.g., Fowler, 1984) with post-elliptic softening. Thus letting $\varepsilon \to 0$ and observing that T_j must be continuous, eq. (44) gives

$$\delta \int_0^{r_1} \int_{\theta^\circ -}^{\theta^\circ +} (rW) \, d\theta \, dr = \int_0^{r_1} T_j^\circ \, \delta D_j \, dr, \tag{45}$$

where now T_j° and D_j are understood to be defined at finite r along the ray of discontinuity. With T_j° presumed to take on a fixed value once the displacement discontinuity develops (i.e., corresponding to an appropriate yield stress in the anti-plane ideally plastic cases), the δ prefixes may be removed from both sides. Then the above equation is equivlent to an integrated (from $\theta^\circ -$ to $\theta^\circ +$) version of eq. (42), providing the verification sought.

6. *Other consequences*

The integrals have greatly simplified the calculation of HRR type crack tip singular fields for ductile single crystals which harden according to a power-law form. The results of such work will be reported separately by the author and M. Saeedvafa. They might also find application in simplifying the determination of other types of HRR fields for tensile or mixed mode cracks in power-law hardening materials, and of finite strain fields at crack tips.

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