Reprinted from September 1985, Vol. ~~107~~, Journal of Applied Mechanics
Vol. 52

First-Order Variation in Elastic Fields Due to Variation in Location of a Planar Crack Front

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The problem explained in the title is formulated generally and given an explicit solution for tensile loadings opening a half-plane crack in an infinite body. For the half-plane crack, changes in the opening displacement between the crack surfaces and in the stress-intensity factor distribution along the crack front are calculated to first order in an arbitrary deviation of the crack-front position from a reference straight line. The deviations considered lie in the original crack plane. The results suggest that in the presence of loadings that would induce uniform conditions along the crack front, if it were straight, small initial deviations from straightness should reduce in size during quasistatic crack growth if of small enough spatial wavelength but possibly enlarge in size if of longer wavelength. The solution methods rely on elastic reciprocity, in terms of a three-dimensional version of weight function theory for tensile cracks, and on direct solution of elastic crack problems. The weight function is derived for the half-plane crack by solving for the first-order variation in the elastic displacement field associated with arbitrary variations of the crack front from a straight reference line. Also, a new three-dimensional weight function theory is developed for planar cracks under general mixed-mode loading involving tension and shears relative to the crack, the connection between weight functions and the Green's function for crack problems is shown, and some results are given for the half-plane crack on the variations of elastic fields for variation of crack-front location in the presence of general loadings including shear.

Theory for Tensile Cracks

A three-dimensional elastic solid contains a planar crack with smooth bounding contour C along the crack front. For the present we assume that the solid is homogeneous, isotropic, symmetric about the crack plane, and subjected to an "original" load system consisting of some distribution of fixed forces and/or imposed boundary displacements that induce "mode 1" tension along the crack front. A cartesian x,y,z coordinate system is attached so that the crack plane lies on $y=0$ (Fig. 1). In addition to the original load system described, a pair of concentrated forces $\pm P$ wedges open the crack at locations $x,0^+,z$ and $x,0^-,z$ on its surfaces. Let $\Delta u(x,z)$ be the opening gap between the crack surfaces at the load location. (So that Δu is bounded when $P\neq 0$, it is convenient at this stage in the development to regard the forces $\pm P$ as being distributed uniformly over a disk of arbitrary small radius ϵ centered on x,z. Then $\Delta u(x,z)$ is understood as the average opening over the same disk, so that $P\delta(\Delta u)$ is the work done by the forces. Later we will be interested in the case $P=0$ and can let $\epsilon\to 0$ with impunity.)

Suppose that while the combined load system (original plus forces $\pm P$ on crack surfaces) is acting, the crack front C is advanced normal to itself on the $y=0$ plane by some variable distance $\delta a(s)$ as in Fig. 1; s is the measure of arc length along C. Thus, treating $\delta a(s)$ as infinitesimal, the change in strain energy U plus potential energy V_0 of the fixed forces of the original load system is

$$\delta(U+V_0)=P\delta[\Delta u(x,z)]-\int_C G(s)\,\delta a(s)\,ds \qquad (1)$$

where G is the energy release per unit crack area of elastic fracture mechanics. G is related to the crack tip stress intensity factor $K[=K(s)]$ by the Irwin relation

$$G=(1-\nu^2)K^2/E \qquad (2)$$

(E = Young tensile modulus, ν = Poisson ratio), whereas K itself appears in expressions for the tensile stress σ across the plane $y=0$ at small distance r ahead of the crack tip and for the opening gap Δu at small distance r behind the tip through the asymptotic forms

$$\sigma\sim K/(2\pi r)^{1/2},\qquad \Delta u\sim 8(1-\nu^2)(K/E)(r/2\pi)^{1/2}. \qquad (3)$$

It is evident that such quantities as U and V_0 can depend only on the magnitude of P or Δu and on the location of the crack front, and thus that the right side of equation (1) is a

Contributed by the Applied Mechanics Division and presented at the 1985 Joint ASME/ASCE Applied Mechanics, Fluids Engineering, and Bioengineering Conference, Albuquerque, N. Mex., June 24-26, 1985 of THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS.

Discussion on this paper should be addressed to the Editorial Department, ASME, United Engineering Center, 345 East 47th Street, New York, N.Y. 10017, and will be accepted until two months after final publication of the paper itself in the JOURNAL OF APPLIED MECHANICS. Manuscript received by ASME Applied Mechanics Division, October, 1984. Paper No. 85-APM-25.

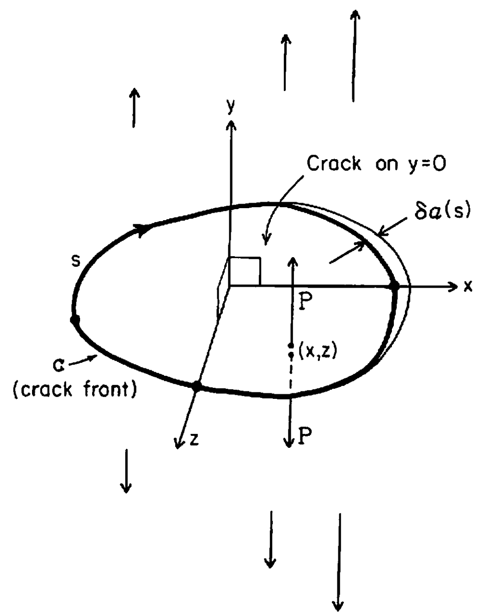

Fig. 1 A planar crack on $y=0$ with front along the arc C. Forces $\pm P$ applied to the upper and lower crack faces at position (x,z). The crack front is advanced normal to itself, in the plane $y=0$, by amount $\delta a(s)$, where s denotes arc length along C.

perfect differential. It then follows (next paragraph) that, to first order in $\delta a(s)$, the variation in opening displacement $\Delta u(x,z)$ when the crack front location is altered while P and the original load system are held fixed is

$$\delta[\Delta u(x,z)] = \int_C \frac{\partial G(s)}{\partial P}\, \delta a(s)\; ds$$

$$= \frac{2(1-\nu^2)}{E} \int_C K(s)\, \frac{\partial K(s)}{\partial P}\, \delta a(s)\; ds, \tag{4}$$

where the derivatives with respect to P are evaluated with the crack front fixed in position along arc C.

To prove this result, introduce an arbitrary function $g(s)$ along C that describes the shape of a crack-front alteration and a quantity A that describes the amplitude of that alteration, such that

$$\delta a(s) = g(s)\,\delta A. \tag{5}$$

Then after a Legendre transformation of equation (1) one has

$$\delta[P(\Delta u) - U - V_0] = (\Delta u)\,\delta P + \left(\int_C Gg\; ds\right)\delta A. \tag{6}$$

Since for any given $g(s)$, the quantities in brackets on the left can all be regarded as functions of P and A, the right side is a perfect differential. Hence the coefficients of δP and δA, regarded as functions of P and A, must satisfy the reciprocal relation

$$\partial(\Delta u)/\partial A = \partial\left(\int_C Gg\; ds\right)/\partial P = \int_C (\partial G/\partial P)\,g\; ds. \tag{7}$$

Upon multiplying this last equation through by δA and recognizing that $g(s)\,\delta A$ represents an arbitrary distribution of crack advance $\delta a(s)$, one verifies equation (4).

In fact, equation (4) must hold no matter what the magnitude of P and, in particular, it must hold when $P=0$ so that no wedging forces are present at all (we may then let $\epsilon \to 0$). It is evident that in the case $P=0$ we may write for insertion in equation (4) that

$$K(s) = K^\circ(s), \qquad \partial K(s)/\partial P = k(s;x,z), \tag{8}$$

where $K^\circ(s)$ is the intensity factor distribution induced along the crack front by the original load system and $k(s;x,z)$ is defined such that $Pk(s;x,z)$ is the intensity factor at position s along the crack front due to wedge forces $\pm P$ opening the crack at location x,z. Hence, the variation in crack-opening displacement at location x,z when the crack-front position is altered by $\delta a(s)$ in presence of (only) the original load system is

$$\delta[\Delta u(x,z)] = \frac{2(1-\nu^2)}{E} \int_C K^\circ(s)\,k(s;x,z)\,\delta a(s)\; ds \tag{9}$$

to first order in $\delta a(s)$.

It may be noticed that the original load system enters into this result only through the distribution $K^\circ(s)$ of intensity factor that it causes but not otherwise. Thus two "original" load systems that induce identical K° distributions will induce identical first-order alterations in crack opening for a given advance $\delta a(s)$. The derivation of equation (9) follows essentially the author's (Rice, 1972) formulation of the theory of Bueckner's (1970) "weight functions" for two-dimensional elastic crack mechanics, and particularly parallels the Appendix of Rice (1972) in which three-dimensional weight function theory is introduced; equation (9) may be recognized as a special case of equation (*A*7) in that work. Bueckner (1972) gave an equivalent three-dimensional extension of his weight function theory based on "fundamental fields" that satisfy the equations of elasticity but have normally inadmissable singularities of arbitrarily prescribed strength around the crack rim. In fact, Bueckner's (1972) fundamental fields have displacements that may be regarded as specific realizations of $\delta(\Delta u)$ in the foregoing and his strength measure may be regarded as being proportional to the product $K^\circ(s)\,\delta a(s)$. The significance of that product or of the related product $K^\circ(s)g(s)$ will be clear from several subsequent expressions.

Equation (9) may be put to use in two ways. First, in a manner paralleling the typical two-dimensional applications of weight function theory, suppose $\delta(\Delta u)$ can be calculated to first order for arbitrary $\delta a(s)$ in presence of the original load system. Then equation (9) evidently enables one to infer the expression for the unknown function $k(s;x,z)$, namely, the intensity factor at s per unit wedging forces applied to the crack faces at x,z. This is easy to use in two dimensions because then δa has no dependence on distance s along the crack front and we have only to calculate $\partial(\Delta u)/\partial a$ to extract the desired function $k(x)$. Analogous three-dimensional applications have not been much considered, apparently because it always seemed impossible to solve the three-dimensional elasticity equations in enough generality to calculate $\delta(\Delta u)$ for arbitrary $\delta a(s)$. It will be shown here a few sections later, however, that such a solution can be constructed for the simple geometry of a half-plane crack in an infinite body. Also, Parks and Kamenetzky (1979) have outlined a numerical implementation of three-dimensional weight function theory in a finite element procedure and Bueckner (1972) has pointed out that, effectively, the solution to $\delta(\Delta u)$ for simpler $\delta a(s)$ amounting, say, to expansion in radius or translation in center location for a three-dimensional crack, allows certain weighted averages of k around the crack rim to be obtained. The latter idea was also developed by Besuner (1974).

For the second use of equation (9), suppose that we know from some other source the function $k(s;x,z)$ for a given crack geometry. Then equation (9) lets us construct the first-order variation $\delta[\Delta u(x,z)]$ for arbitrary variation $\delta a(s)$ from the given crack geometry. The equation is applied in that way in the next section. As will be seen, such an application enables one to calculate the corresponding variation $\delta K(s)$ in intensity factor along the crack front to first order in $\delta a(s)$, as considered recently by Meade and Keer (1984*b*), and the calculations allow study of the configurational stability of a three-dimensional crack front.

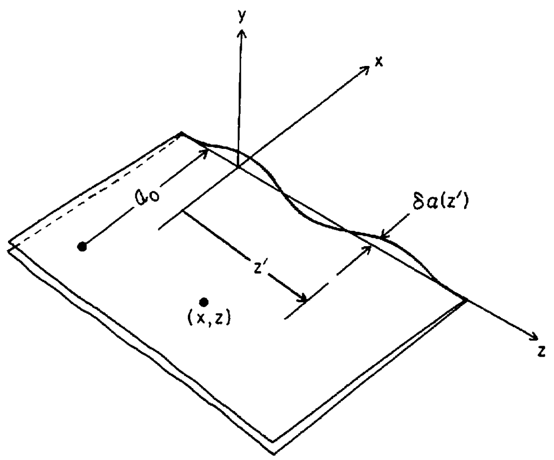

Fig. 2 Half-plane crack on $y=0$ in an infinite body. Reference straight crack front along z axis; z' denotes location along front and $\delta a(z')$ denotes advance of crack front; a is distance from some fixed point on a crack surface to reference straight crack front and $a = a_0$ when front lies along z axis as shown.

Before turning to the applications, we note the following extensions of equation (9). Suppose, for example, that we wish to know the variation of a certain weighted average of the crack opening when the crack front is advanced. Thus for weighting with some function $p(x,z)$

$$\delta\left[\int\int p(x,z)\,\Delta u(x,z)\;dx\;dz\right] = \frac{2(1-\nu^2)}{E}\int_C K^\circ(s)\,\tilde{k}(s)\,\delta a(s)\;ds \tag{10}$$

to first order in δa where evidently

$$\tilde{k}(s) = \int\int p(x,z)\,k(s;x,z)\;dx\;dz, \tag{11}$$

and hence $\tilde{k}(s)$ is the stress-intensity factor distribution due to a "loading" $p(x,y)$ considered as a pressure distribution on the crack faces. A similar expression is obtained for other deformation measures than crack-opening displacement. For example, if Q measures the intensity of some arbitrary additional force distribution acting symmetrically about the crack plane, and if q is the conjugate deformation quantity (i.e., defined so that $Q\delta q$ is a work increment of that additional force distribution) then

$$\delta q = 2\frac{1-\nu^2}{E}\int_C K^\circ(s)\,\tilde{k}(s)\,\delta a(s)\;ds \tag{12}$$

is the change in q to first order when the crack is advanced in presence of the original load system but with $Q=0$. Here $\tilde{k}(s)$ is the intensity factor induced by unit Q.

Variation of Front of a Half-Plane Crack in an Infinite Body

Suppose that the crack front C, or at least the segment of interest whose location is to be perturbed, is initially straight. Provided that we are interested only in the opening displacements in the vicinity of this straight front, at points x,z that are much closer to it than to external boundaries of the cracked body or to other segments of the crack front, it will suffice to use in equation (9) the result for $k(s;x,z)$ for an infinite body containing a half-plane crack. In particular, the crack front C is chosen to coincide with the z axis and the crack plane with the plane $y=0$, $x<0$ as in Fig. 2. Now we let z', which runs along the z axis, replace s and rewrite equation (9) as

$$\delta[\Delta u(x,z)] = \frac{2(1-\nu^2)}{E}\int_{-\infty}^{+\infty} K^\circ(z')\,k(z';x,z)\,\delta a(z')\;dz'. \tag{13}$$

The formulation is complete once we have a formula for $k(z';x,z)$, which can depend on only $z'-z$ and x, and this can be taken from published work. The stress intensity factors for arbitrary point forces on a semi-infinite crack face are given in the handbook by Tada et al. (1973) and referenced to prior elasticity analyses by Uflyand (1965), Sih and Liebowitz (1968), and Kassir and Sih (1973). The problem has recently been given a convenient reformulation by Meade and Keer (1984*a*). The result for the mode I stress intensity at z' along the crack tip due to unit opening forces applied on the crack faces at $x(<0), z$ is

$$k(z';x,z) = (-2x/\pi^3)^{1/2}/[x^2+(z'-z)^2], \tag{14}$$

where positive square root is implied, and thus the alteration of the opening displacement of the crack faces is

$$\delta[\Delta u(x,z)] = \frac{8(1-\nu^2)}{E}\left(\frac{-x}{2\pi}\right)^{1/2}\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{K^\circ(z')\,\delta a(z')dz'}{x^2+(z-z')^2}\right]. \tag{15}$$

At this point it is useful to recall, as suggested by equation (3), that the opening displacement very near a point along the (altered) crack tip will have the form

$$\Delta u(x,z) = \frac{8(1-\nu^2)}{E}\left(\frac{\delta a - x}{2\pi}\right)^{1/2} K + O(\delta a - x)^{3/2} \tag{16}$$

where K is the stress intensity factor at that point. If we now let $x\rightarrow 0^-$ in equation (15) at a location z where $\delta a(z) \neq 0$, it is easy to show that the bracketed term becomes infinite as

$$[\ldots] \sim K^\circ(z)\,\delta a(z)/(-2x), \tag{17}$$

and thus equation (15) predicts that $\delta[\Delta u]$ behaves as

$$\delta[\Delta u(x,z)] \sim \frac{4(1-\nu^2)}{E}\,\frac{\delta a(z)}{(-2\pi x)^{1/2}}\,K^\circ(z). \tag{18}$$

This is an obviously correct asymptotic form as $x\rightarrow 0^-$, as we can check from equation (16); it corresponds to evaluating $d(\Delta u)/d(\delta a)$ at $\delta a = 0$ from that equation and then multiplying the result by δa to give the first-order result.

Suppose now that at a particular value of z, $\delta a(z) = 0$. Assuming that $d[\delta a(z)]/dz$ exists at that point, one may show that the bracketed term in equation (15) has a well-defined limit as $x\rightarrow 0$ and that this limit is

$$\lim_{x\rightarrow 0}\,[\ldots] = \frac{1}{2\pi}PV\int_{-\infty}^{+\infty}\frac{K^\circ(z')\delta a(z')}{(z'-z)^2}dz' \tag{19}$$

where PV in front of the improper integral denotes the principal value. (To prove this, break the integral on z' up into one over $z-\eta$ to $z+\eta$ and another over the rest of the axis. One shows that the limit as $x\rightarrow 0$ of the integral from $z-\eta$ to $z+\eta$ vanishes when one takes the subsequent limit $\eta\rightarrow 0$, and the limit $\eta\rightarrow 0$ of the integral over the rest of the axis with $x=0$ defines the PV.) However, if we compare equation (15) to equation (16) at a point z where $\delta a(z) = 0$, it is evident that the then well-defined value of the bracketed term of equation (19) can be interpreted as the variation of intensity factor K at z due to some or all of the rest of the crack front being altered in position. Hence to first order in δa,

$$\delta K(z) = \frac{1}{2\pi}PV\int_{-\infty}^{+\infty}\frac{K^\circ(z')\,\delta a(z')}{(z'-z)^2}\,dx' \tag{20}$$

is the variation in stress-intensity factor at a location z where $\delta a(z) = 0$ but $d[\delta a(z)]/dz$ exists, and the total stress intensity factor $K(z)$ at that location is

$$K(z) = K^\circ(z) + \delta K(z). \tag{21}$$

To find K at some location z where $\delta a(z) \neq 0$ we simply relocate the reference straight crack front by moving it along the x direction an amount $\delta a(z)$ and then apply the formulas just derived. To express this in an equation, a variable a is introduced to measure the distance of the reference straight crack front from some given point of the body on one of the crack surfaces (a has the value a_0 shown in Fig. 2 when the reference straight crack front coincides with the z axis). Further, to emphasize dependence on a, the intensity distribution $K^\circ(z)$ induced along a straight crack front by the original loadings is written henceforth as $K^\circ(z;a)$. Evidently, to calculate the stress-intensity factor at location z, we want to move the reference straight crack front to $a = a_0 + \delta a(z)$, and then use equation (20) to account for the nonstraightness of the crack with $\delta a(z')$ replaced with $\delta a(z') - \delta a(z)$. It is convenient to write the result using the notation

$$a(z) = a_0 + \delta a(z), \tag{22}$$

and the stress-intensity factor at location z is therefore

$$K(z) = K^\circ[z;a(z)] + \frac{1}{2\pi} PV \int_{-\infty}^{+\infty} \frac{K^\circ[z';a(z)][a(z')-a(z)]}{(z'-z)^2} dz'. \tag{23}$$

This formula is accurate to first order in $a(z) - a(z')$.

The result just obtained appears not to be consistent with one presented by Meade and Kerr (1984*b*) for a half-plane crack in an infinite body with a wavy crack front in the form

$$a(z) = a_0 - A[1 - \cos(2\pi z/\lambda)]. \tag{24}$$

Here A is a small parameter. They consider opening loadings applied to the crack faces as uniform line loads, applied along lines parallel to z on the crack faces at distances a_0 from the tip. For their method of loading, $K^\circ = K^\circ(a)$, independent of z (and $K^\circ = 2^{1/2} P_0/\pi^{1/2}a^{1/2}$, where P_0 is the intensity of their line loading). They propose in their equation (110) that

$$K(z) = K^\circ[a(z)] + O(A^2/\lambda^2), \tag{25}$$

and this is at conflict with equation (23) in the foregoing, where the PV integral contributes an effect of first order in A. This particular case is discussed further in the next section. Also, in the section after that the convenient formulation presented by Meade and Keer (1984*a*,*b*) for three-dimensional problems of plane cracks in infinite elastic bodies is used to give an alternative derivation of equation (15) in the foregoing which is the basis of equation (23) for $K(z)$, and also to discuss an apparent inconsistency of the solution to the elastic field equations for a wavy crack front proposed by Meade and Keer (1984*b*). As a bonus, that alternate derivation leads to a new derivation of the intensity factor distribution of equation (14), and also to the general three-dimensional weight function for a half-plane tensile crack in an infinite body.

A similar approach to that leading to equation (23) from (20) is useful for rewriting the result for the crack-surface opening displacement. As it stands, equation (15) for $\delta(\Delta u)$ is rigorously correct to first order in $\delta a(z)$ but, because of the effect of transport of the crack tip singularity in changing the $(-x)^{1/2}$ of equation (16) to $(-x)^{-1/2}$ where indicated in equation (15) is not very useful near the tip of a crack at a location where $\delta a(z) \neq 0$. Indeed, the first order expression for $\delta(\Delta u)$ becomes highly inaccurate when $(-x)$ is of the same order or smaller than $\delta a(z)$. By moving the reference crack tip to $a = a_0 + \delta a(z)$ we eliminate this problem.

Thus, we introduce the notation $\Delta u^\circ(x,z;a)$ for the distribution of opening displacement when the crack front is straight and at distance a from the fixed point on the crack surface. One notes incidently from equation (15) with δa uniform in z that the a dependence of this opening distribution must satisfy

$$\frac{\partial \Delta u^\circ(x,z;a)}{\partial a} = \frac{8(1-\nu^2)}{E}\left(\frac{a-a_0-x}{2\pi}\right)^{1/2} \times \left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{K^\circ(z';a)dz'}{(a-a_0-x)^2+(z-z')^2}\right]; \tag{26}$$

here $-x$ has been replaced by $a-a_0-x$ to accommodate general location of the crack front, and the derivative on the left is calculated as the limit of $\delta(\Delta u)/\delta a$. When K° is uniform in z, this reduces to

$$\frac{\partial \Delta u^\circ(x;a)}{\partial a} = \frac{4(1-\nu^2)}{E} \frac{K^\circ(a)}{[2\pi(a-a_0-x)]^{1/2}}, \tag{27}$$

which is essentially K° times the two-dimensional Bueckner weight function for the semi-infinite crack in an infinite body.

Now, the total crack-opening displacement $\Delta u(x,z)$ along the crack faces at a given location z can be obtained by adding Δu° when $a = a_0 + \delta a(z)$ to $\delta(\Delta u)$ as calculated from equation (15) with $\delta a(z')$ replaced with $a(z') - a(z)$. Thus there results

$$\Delta u(x,z) = \Delta u^\circ[x,z;a(z)] + \frac{8(1-\nu^2)}{E}\left[\frac{a(z)-a_0-x}{2\pi}\right]^{1/2} \times \left\{\frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{K^\circ[z';a(z)][a(z')-a(z)]}{[a(z)-a_0-x]^2+(z-z')^2} dz'\right\} \tag{28}$$

which is accurate to first order in $a(z') - a(z)$ and remains well defined as x approaches the crack tip at $a(z) - a_0$.

Wavy Crack Front and Stability

Consider the crack profile

$$a(z) = a_0 + A\cos(2\pi z/\lambda), \tag{29}$$

$\lambda > 0$, and suppose that the cracked body is loaded such that the intensity factor is uniform along the crack front when the front is straight, i.e., that $K^\circ(z;a) = K^\circ(a)$. Then insertion of the preceding $a(z)$ in equation (23) and calculation of the integral shows that

$$K(z) = K^\circ[a_0 + A\cos(2\pi z/\lambda)]\{1 - \pi(A/\lambda)\cos(2\pi z/\lambda)\} \tag{30}$$

to first-order accuracy in A. If we rewrite the equation to exhibit only those terms that are of zeroth and first order in A, then

$$K(z) = K^\circ(a_0) + [dK^\circ(a_0)/da_0 - \pi K^\circ(a_0)/\lambda]A\cos(2\pi z/\lambda). \tag{31}$$

This solution enables one to address the following stability problem. Suppose a small deviation from straightness in the form of equation (29) exists along a uniformly stressed crack front, that is, along one that would be uniformly stressed if it were straight. Will the deviation grow or decay as cracking advances quasistatically by, say, fatigue or corrosion? It seems reasonable to conclude that the deviation will grow if K is higher at the most advanced parts of the crack front ($z=0$, $\pm\lambda$, $\pm 2\lambda$, etc.) than at the most retarded ($z = \pm\lambda/2$, $\pm 3\lambda/2$, etc.), and that the deviation will decay if the opposite is true. Thus the sign of the bracket governs and decay will occur, i.e., deviations from straightness will be smoothed out, if the bracket is negative. We conclude that the straight crack-front configuration is stable to small-amplitude perturbations of wavelength satisfying

$$\lambda dK^\circ(a_0)/da_0 < \pi K^\circ(a_0). \tag{32}$$

Further, if $dK^\circ/da_0 < 0$, as is the case, e.g., with localized wedging forces applied on or near the crack surfaces, then the straight configuration is stable to perturbations of all wavelengths. On the other hand, if $dK^\circ/da_0 > 0$ as is normally the case when loads are applied remotely from the crack tip, then there exists a critical wavelength $\lambda_c = \pi K^\circ/(dK^\circ/da_0)$ above which perturbations will tend to grow.

As a practical matter, λ_c as just defined will typically be of the order of the crack length itself for disk-like or tunnel cracks, and hence conditions may not generally be met to consider perturbations on the scale of λ_c by use of the model of a half-plane crack in an infinite body, i.e., by neglect of proximity of the perturbed region to boundaries or to other differently oriented segments of crack front. For $\lambda << \lambda_c$ such is not a concern, and in that case the term involving λ in equation (31) dominates the first order effect. Thus K is decreased by $\pi K^\circ A/\lambda$ at the most advanced penetrations, and increased by the same amount at the most retarded locations. This increase may be taken to describe the stress intensification at microscale heterogeneities when a crack tip begins to advance by and surround a fracture-resistant zone of material.

We may also consider the perturbation of equation (24) considered by Meade and Keer (1984*b*). In this case equation (23) implies that

$$K(z) = K^\circ(a_0) + [dK^\circ(a_0)/da_0]A[\cos(2\pi z/\lambda) - 1] - \pi K^\circ(a_0)(A/\lambda)\cos(2\pi z/\lambda). \tag{33}$$

when only zeroth and first orders in A are retained. By contrast, the solution proposed by Meade and Keer (1984*b*) when expanded to the same order is

$$K(z) = K^\circ(a_0) + [dK^\circ(a_0)/da_0]A[\cos(2\pi z/\lambda) - 1]. \tag{34}$$

This is missing the last term of equation (33), which is the result of the principal value integral in equation (23). Given the form of perturbation from straightness in equation (24), it would seem evident that K at $z = 0$ should indeed decrease with increasing A. This is because more of the crack surfaces are attached back together when $A > 0$, compared to the situation when $A = 0$, and the most advanced parts of the crack front ($z = 0$, $\pm\lambda$, $\pm 2\lambda$, etc.) are then shielded by the rejoined elements of material to their sides. This effect of decrease of K at $z = 0$ with increase of A is predicted by equations (23) and (33) but not (at least to first order) by equations (25) and (34).

Direct Solution for Half-Plane Crack in Infinite Body

Since there is disagreement with other work, it seems appropriate to reexamine the problem of a wavy half-plane crack in a full space here by directly solving the Navier equations of elasticity. Meade and Kerr (1984*a,b*) show, following Sections 5.8 of Green and Zerna (1954), that these equations are satisfied in a manner compatible with tensile loading that is symmetric relative to the crack plane if displacement components are written as

$$\begin{aligned} u_y &= -2[(1-\nu^2)/E]Y + [(1+\nu)/E]y\,\partial Y/\partial y \\ u_x &= [(1+\nu)/E]\partial(F+yY)/\partial x \\ u_z &= [(1+\nu)/E]\partial(F+yY)/\partial z \end{aligned} \tag{35}$$

where F and Y are harmonic functions related by $\partial F/\partial y = (1-2\nu)Y$. In addition, the stress components that enter crack-surface boundary conditions are calculated from the stress-strain relations as

$$\begin{aligned} \sigma_{yy} &= -\partial Y/\partial y + y\partial^2 Y/\partial y^2 \\ \sigma_{yx} &= y\partial^2 Y/\partial y\partial x, \qquad \sigma_{yz} = y\partial^2 Y/\partial y\partial z \end{aligned} \tag{36}$$

and cause no shear traction on $y = 0$. Thus the problem of loading on the crack face is one of finding a function Y satisfying $\nabla^2 Y = 0$, having vanishing derivatives at infinity, and generating stress σ and opening gap Δu on $y = 0$ given by

$$\sigma = -\partial Y/\partial y\Big|_{y=0} \quad \text{and} \quad \Delta u = -[4(1-\nu^2)/E]Y\Big|_{y=0^+} \tag{37}$$

respectively, such that boundary conditions are satisfied. If the crack tip lies along

$$x = Ag(z), \tag{38}$$

where $g(z)$ is some arbitrary function of z, and the crack faces are loaded by normal pressure $p(x,z)$ $[= -\sigma(x,z)]$, these conditions are

$$\partial Y/\partial y\Big|_{y=0} = p(x,z) \quad \text{for} \quad x < Ag(z),$$

and

$$Y\Big|_{y=0} = 0 \quad \text{for} \quad x > Ag(z) \tag{39}$$

as discussed by Meade and Keer (1984*b*) for their particular case. The problem cannot be solved explicitly in this generality but it is known that solutions to it, when restricted to have bounded energy in any finite region about the crack tip, exhibit characteristic inverse square root stress singularities whose nature is indicated in such equations as (3) and (16). The harmonic function Y generating such a singularity necessarily has the form

$$Y \sim -2K(r/2\pi)^{1/2}\sin(\theta/2) \tag{40}$$

(plus another part generating finite ∇Y) near the tip, where r and θ are polar coordinates in planes locally perpendicular to the tip; $\theta = \pm\pi$ on the crack surfaces.

Let the solution of the boundary value problem as formulated be $Y(x,y,z;A)$. We now turn to calculation of the function

$$W(x,y,z) = [\partial Y(x,y,z;A)/\partial A]\Big|_{A=0}. \tag{41}$$

It is evident that $\nabla^2 W = 0$ and, since the crack-face loading has no dependence on A,

$$\partial W/\partial y\Big|_{y=0} = 0 \quad \text{for} \quad x < 0,$$

$$W\Big|_{y=0} = 0 \quad \text{for} \quad x > 0. \tag{42}$$

This apparently homogeneous boundary value problem for W would have only the solution $W = 0$ if restricted to the bounded energy class in the same way as for conventional elastic solutions. However, W has a stronger singularity along the crack front than normally allowed. We can calculate this by calculating $\partial r/\partial A$ and $\partial\theta/\partial A$, corresponding to fixed x,y,z, in equation (40) at $A = 0$ so as to calculate $\partial Y/\partial A$. Of course, K at any particular z also varies with A but this does not affect the most strongly singular term. The result of a straightforward calculation is that

$$W \sim -K^\circ(z)g(z)(2\pi r)^{-1/2}\sin(\theta/2) \tag{43}$$

where, in this limit, the crack front is straight and coincident with the z axis, and $K^\circ(z)$ is the K distribution induced by the given loadings $p(x,z)$ for that straight configuration. Note that the nature of the original loading and shape g of the incipient growth appear only in the product $K^\circ(z)g(z)$.

To obtain a solution of $\nabla^2 W = 0$ with the singular form indicated, introduce the Fourier representation

$$K^\circ(z)g(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \alpha(\omega)e^{i\omega z}d\omega. \tag{44}$$

Evidently, we wish to seek solutions of $\nabla^2 W = 0$ with a z dependence of form $e^{i\omega z}$, and to meet the homogeneous boundary conditions of equations (42) on the crack faces one begins by looking for harmonic functions in the form

$$f(r)\sin(m\theta/2)e^{i\omega z} \tag{45}$$

with $m = 1, 3, 5, \ldots$. As it happens, all conditions on the problem can be met with $m = 1$. In that case it is easy to show that the possible forms for $f(r)$ are

$$f(r) = r^{-1/2}e^{\pm|\omega|r}, \tag{46}$$

and obviously the $+$ sign must be rejected in the present case. Thus we see that

$$r^{-1/2}e^{-|\omega|r}\sin(\theta/2)e^{i\omega z} \quad (47)$$

is harmonic, and this has precisely the singular structure as $r \to 0$ required in equation (43). Therefore the solution is

$$W = -(1/2\pi)(2\pi r)^{-1/2}\sin(\theta/2)\int_{-\infty}^{+\infty} \alpha(\omega)e^{-|\omega|r}e^{i\omega z}d\omega \quad (48)$$

since, by equation (44), this reduces to equation (43) when $r \to 0$.

To complete the solution, we write by Fourier inversion of equation (44) that

$$\alpha(\omega) = \int_{-\infty}^{+\infty} K^\circ(z')g(z')e^{-i\omega z'}dz', \quad (49)$$

insert this into equation (48), change the order of integration, and note that

$$\int_{-\infty}^{+\infty} e^{-i\omega z'}e^{-|\omega|r}e^{i\omega z}d\omega = 2r/[r^2 + (z-z')^2]. \quad (50)$$

The result for W is therefore that

$$W(r,\theta,z) = -(1/\pi)(r/2\pi)^{1/2}\sin(\theta/2)\int_{-\infty}^{+\infty} \frac{K^\circ(z')g(z')dz'}{r^2 + (z-z')^2}. \quad (51)$$

Remembering that W is $\partial Y/\partial A$ at $A = 0$ and that the opening gap Δu is related to Y as in equation (37) on $\theta = 0, \pi$, we have therefore derived that

$$\left.\frac{\partial \Delta u(x,z,A)}{\partial A}\right|_{A=0} = \frac{8(1-\nu^2)}{E}\left(\frac{-x}{2\pi}\right)^{1/2}\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{K^\circ(z')g(z')}{x^2 + (z-z')^2}dz'\right]. \quad (52)$$

Here θ has been set equal to π and thus r to $-x$. This last equation is precisely the same as equation (15); $\delta a(z)$ is just $\delta A g(z)$. Thus an independent derivation of equation (15) is now at hand, and the result given earlier in equation (23) for K along a nonstraight crack tip follows from there.

Now, in what amounts to an application of weight function ideas of the first type described in the opening section, let us note that equation (52), just independently derived, and equation (13) must be consistent with one another for arbitrary $\delta a(z)$. Their consistency requires that $k(z';x,z)$ have a certain definite form, which is exactly that given in equation (14). Thus the solution derived in this section together with the fundamental equation (13) of weight function theory gives the solution for the stress intensity distribution k along the crack front due to unit concentrated forces opening the crack faces.

It has been shown (Rice, 1972, equations (A8) to (A11) put in the notation for the present problem) that if the displacement vector variation is expressed as

$$\delta \mathbf{u}(x,y,z) = \int_{-\infty}^{+\infty} \mathbf{U}^\circ(x,y,z;z')\delta a(z')dz' \quad (53)$$

to first order in $\delta a(z)$ for a crack under some "original" tensile loading, then

$$\mathbf{h}(x,y,z;z') = E\mathbf{U}^\circ(x,y,z;z')/2(1-\nu^2)K^\circ(z') \quad (54)$$

is the three-dimensional weight function. This is unique within rigid motions, is independent of the nature of the original loading system, and has the property that for arbitrary distributions of body force $\mathbf{f}(x,y,z)$ compatible with tensile loadings, the stress intensity factor induced along the crack front is

$$K(z') = \int \mathbf{f}(x,y,z)\cdot\mathbf{h}(x,y,z;z')\, dxdydz. \quad (55)$$

The integral extends over all locations where the body force acts and loads on the crack faces can, of course, be included as concentrations of body force.

By substituting from equation (54) in (53), we may write the latter in a manner to show that the weight function can be read off when we have obtained a representation for $\delta\mathbf{u}$ in the form

$$\delta\mathbf{u}(x,y,z) = \frac{2(1-\nu^2)}{E}\int_{-\infty}^{+\infty} \mathbf{h}(x,y,z;z')K^\circ(z')\delta a(z')dz'. \quad (56)$$

But $\delta\mathbf{u}$ can be written from equations (35) as a linear operation on δY, the same operation that forms $\mathbf{u}$ from Y. Also, recognizing that δY is just $W\delta A$ with the product $g(z')\delta A$ written as $\delta a(z')$, we may write from equation (51) that

$$\delta Y(x,y,z) = -\int_{-\infty}^{+\infty} H(x,y,z;z')K^\circ(z')\delta a(z')dz' \quad (57)$$

with

$$H(x,y,z;z') = (r/2\pi^3)^{1/2}\sin(\theta/2)/[r^2 + (z-z')^2]. \quad (58)$$

Thus recognizing that $\delta\mathbf{u}$ of equation (56) is formed from δY of (57) by the linear operation (35), we conclude that $2(1-\nu^2)\mathbf{h}/E$ is formed from $-H$ by the same linear operation. Thus the components of the weight function are

$$h_y = H - [1/2(1-\nu)]y\,\partial H/\partial y$$
$$h_x = -[1/2(1-\nu)]\partial(L + yH)/\partial x$$
$$h_z = -[1/2(1-\nu)]\partial(L + yH)/\partial z \quad (59)$$

where

$$L(x,y,z;z') = -(1-2\nu)\int_y^\infty H(x,\bar{y},z;z')d\bar{y}. \quad (60)$$

This seems to be the first case for which a three-dimensional weight function has been determined. Regrettably, since the integral defining L cannot be reduced to a simple form, the x and z components of the weight function are not in a very convenient form for applications. However, h_y has no similar difficulties and it is all that is needed when loads act in directions perpendicular to the crack plane. As a check, letting $y \to 0^+$ or 0^-, one observes that

$$h_y(x,\pm 0,z;z') = H(x,\pm 0,z;z') = \pm(1/2)k(z';x,z) \quad (61)$$

as expected, where k is the intensity factor of equation (14).

Mead and Keer (1984b) attempt to directly calculate the function Y defined by equations (39), for their particular shape function

$$g(z) = -[1 - \cos(2\pi z/\lambda)], \quad (62)$$

and for loading of the crack faces by line loads of intensity P_0 at distance a_0 from the tip as described earlier. Shortly after their equation (47), they propose that the solution for Y (in present notation) is

$$Y = -\frac{P_0}{2\pi}\log\left\{\frac{r + a_0 + Ag(z) + 2[ra_0 + rAg(z)]^{1/2}\sin(\theta/2)}{r + a_0 + Ag(z) - 2[ra_0 + rAg(z)]^{1/2}\sin(\theta/2)}\right\} + O(A^2/\lambda^2) \quad (63)$$

where

$$r^2 = [x - Ag(z)]^2 + y^2, \qquad \tan\theta = y/[x - Ag(z)]. \quad (64)$$

The difficulty is that when this solution is used to calculate $\partial Y(x,y,z,A)/\partial A$ at $A = 0$, it does not give a result that coincides with the exact solution derived here. In particular, when the $g(z)$ of equation (62) is used and we use $K^\circ(a) = 2^{1/2}P_0/\pi^{1/2}a^{1/2}$ as appropriate for the line-load problem, there results from the solution in equation (51) for W that

$$W \equiv \partial Y/\partial A\Big|_{A=0} = (P_0/\pi)(a_0 r)^{-1/2}\sin(\theta/2)[1$$

$$-\cos(2\pi z/\lambda)e^{-2\pi r/|\lambda|}]. \qquad (65)$$

By contrast, the corresponding quantity calculated from equation (63) for Y as proposed by Meade and Keer does not show the exponential decay of the effect of the $\cos(2\pi z/\lambda)$ wiggle in crack-front position at large distances r, nor does it show a dependence on a_0 only in the form of a factor. Meade and Kerr develop their analysis by asymptotic methods based in part on multiple (double) scaling in the z direction, one length scale being based on A and the other on λ. Equation (65) together with the fix-up of type in going from equation (15)-(28), based on placing the reference straight crack front at $a(z)$, i.e., at $x = Ag(z)$, suggest that there may not in fact be a double scale of the type they assumed for the z dependence of the solution, but that instead such occurs for the x and y dependence.

Cracks Under General Mixed Mode Conditions

As a generalization, we can consider a possibly anisotropic body of arbitrary shape containing a planar crack on $y=0$ and suppose that the original loading system induces opening as well as shear displacements in the x and z directions on the crack faces. In this case there are three stress-intensity factors, given subscripts 1,2,3 where 1 refers to tension, 2 to in-plane shear, and 3 to antiplane shear relative to a plane locally perpendicular to the crack front. Thus in the plane $y=0$ at distance r ahead of the tip there is the asymptotic stress distribution

$$(\sigma_{yy},\sigma_{yn},\sigma_{ys}) \sim (K_1,K_2,K_3)/(2\pi r)^{1/2}, \qquad (66)$$

where n and s denote directions in the plane $y=0$ that are, respectively, normal and tangential to the crack front C. The Irwin relation for the energy release per unit area of crack extension is

$$G = \Lambda_{\alpha\beta}K_\alpha K_\beta \qquad (67)$$

where α,β range over 1,2,3 with summation on repeated indices, where $\Lambda_{\alpha\beta} = \Lambda_{\beta\alpha}$, and for an isotropic material

$$\Lambda_{11} = \Lambda_{22} = (1-\nu^2)/E, \qquad \Lambda_{33} = (1+\nu)/E, \qquad \text{other} \quad \Lambda_{\alpha\beta} = 0. \qquad (68)$$

For anisotropic solids the calculation of $\Lambda_{\alpha\beta}$ is discussed by Stroh (1958) and Barnett and Asaro (1972); $\Lambda_{\alpha\beta}$ can be expressed as a numerical factor times the inverse of a matrix appearing in the prelogarithmic energy factor of dislocation theory. In general, the $\Lambda_{\alpha\beta}$ will depend on the local tangent direction to the crack front.

Suppose that in addition to the original loading system, a distribution of forces with intensity proportional to Q acts, as in the discussion preceding equation (12), and that q is the conjugate deformation quantity. Then with the original load system fixed, one has

$$Q\delta q - \int_C \Lambda_{\alpha\beta}(s)K_\alpha(s)K_\beta(s)\delta a(s)\, ds = \delta(U+V_0), \qquad (69)$$

as in equation (1), and from this we infer analogously to equations (9) and (12) that the variation in q when the crack is advanced in presence of only the original loading system ($Q=0$) is

$$\delta q = \int_C 2\Lambda_{\alpha\beta}(s)\bar{k}_\alpha(s)K^\circ_\beta(s)\delta a(s)\, ds \qquad (70)$$

to first order in $\delta a(s)$. Here K°_α are the intensity factors induced by the original loading and $\bar{k}_\alpha$ the intensity factors induced by unit value of Q. As before, if the $\bar{k}_\alpha$ are known, one can calculate δq, whereas if δq is known for arbitrary "source" terms $K^\circ_\alpha\delta a$ along C, one can calculate the $\bar{k}_\alpha$.

To proceed to a three-dimensional weight function formulation appropriate for arbitrary mixed mode conditions, consider the problem of calculating the first-order variation $\delta\mathbf{u}$ in the elastic displacement field at a general position $\mathbf{r} = (x,y,z)$ when the crack advances in presence of the original loading. It is clear that $\delta\mathbf{u}$ satisfies the equations of elasticity in presence of zero body force, zero boundary traction on the crack faces and wherever else tractions were prescribed in the original loading, and zero boundary displacement where displacements were prescribed. Despite the apparent homogeneity of the problem, $\delta\mathbf{u}$ is nonzero because its three components have prescribed singularities of strengths that are linear in the three terms of form $K^\circ_\beta(s)\delta a(s)$ along the crack front. Each such source term can be regarded as generating a vector displacement field at $\mathbf{r}$ which is conveniently written as $2\Lambda_{\alpha\beta}\mathbf{h}_\alpha(\mathbf{r}, s)$ weighted with the source strength $K^\circ_\beta(s)\delta a(s)$ along C. Thus

$$\delta\mathbf{u}(\mathbf{r}) = \int_C 2\Lambda_{\alpha\beta}(s)\mathbf{h}_\alpha(\mathbf{r},s)K^\circ_\beta(s)\delta a(s)\, ds. \qquad (71)$$

Note that the vector functions $\mathbf{h}_\alpha$ thus defined are universal in that they have no dependence on the nature of the original loading system, other than depending on which portions of the external boundary had tractions versus displacements prescribed. Of course, the functions also depend on the shape and location of the crack. The vector functions $\mathbf{h}_\alpha(\mathbf{r},s)$ defined by equation (71) are sensibly called "weight functions." To see why, denote the components of $\mathbf{h}_\alpha$ as $h_{\alpha j}$ where $j = x, y$, or z, and observe that $\delta u_j(\mathbf{r})$ is generated by $h_{\alpha j}(\mathbf{r},s)$. But δu_j is just δq when Q measures the intensity of a concentrated force in the j direction at $\mathbf{r}$, and hence $h_{\alpha j}(\mathbf{r},s)$ must equal $\bar{k}_\alpha(s)$ of equation (70) where the $\bar{k}_\alpha$ are then the intensity factors induced by a unit point force in the j direction at $\mathbf{r}$. Thus for an arbitrary distribution of body force $\mathbf{f}(\mathbf{r})$, the intensity factors induced at location s along the crack front are

$$K_\alpha(s) = \int \mathbf{f}(\mathbf{r})\cdot\mathbf{h}_\alpha(\mathbf{r},s)d^3\mathbf{r} \qquad (72)$$

integrated over the region of loading, confirming the weight function interpretation of the $\mathbf{h}_\alpha$. Paris et al. (1976) and Bortman and Banks-Sills (1983) introduced two-dimensional weight functions for shear modes in isotropic solids.

To summarize, if we know from some other work the stress intensity factors induced by concentrated forces at $\mathbf{r}$, then from equation (72) it is clear that we know the $\mathbf{h}_\alpha(\mathbf{r},s)$. We can then use these $\mathbf{h}_\alpha$ in equation (71) to calculate $\delta\mathbf{u}(\mathbf{r})$ and, from it, variations in stress intensity factors K_α along the crack front to first order in $\delta a(s)$. This is what was done for the half-plane tensile crack in the second section of this paper. On the other hand, if we can calculate directly $\delta\mathbf{u}(\mathbf{r})$ for arbitrary source terms $K^\circ_\beta(s)\delta a(s)$ along the crack front, as done for the half-plane tensile crack in the preceding section, then we have a more basic result. That calculation, when put in the form of equation (71), amounts to fundamental calculation of the weight functions $\mathbf{h}_\alpha(\mathbf{r},s)$. I leave for future work the full exploration of these ideas for the half-plane crack in an infinite isotropic body, under general tensile and shear loadings, and discuss here only some aspects of that case.

Specifically, let us develop the expression for general mixed-mode loading that is analogous to equations (13) and (15) for the tensile crack problem. Suppose that the faces of the half-plane crack are loaded by concentrated forces P_j at $x,0^+,z$ and $-P_j$ at $x,0^-,z$, with $j = x,y$, or z. The stress-intensity factors $k_{\alpha j} = k_{\alpha j}(z';x,z)$ induced at z' by unit P_j are (Tada et al., 1973; Meade and Keer, 1984*a*)

$$k_{1x} = k_{1z} = k_{2y} = k_{3y} = 0$$

$$k_{1y} = k$$

$$k_{2x} = \left[1 + \frac{2\nu}{2-\nu}\frac{x^2-(z'-z)^2}{x^2+(z'-z)^2}\right]k$$

$$k_{3z} = \left[1 - \frac{2\nu}{2-\nu}\frac{x^2-(z'-z)^2}{x^2+(z'-z)^2}\right]k$$

$$k_{2z} = k_{3x} = -\frac{4\nu}{2-\nu}\frac{x(z'-z)}{x^2+(z'-z)^2}k \qquad (73)$$

where $k = k(z';x,z)$ is given by equation (14). Of course the $k_{\alpha j}(z';x,z)$ are functions only of x and $z-z'$. Evidently, the $k_{\alpha j}$ must be related to the weight functions defined in the foregoing by

$$k_{\alpha j}(z';x,z) = h_{\alpha j}(x,0^+,z;z') - h_{\alpha j}(x,0^-,z;z'). \qquad (74)$$

Thus the previous formulas, written out fully for the isotropic case, imply that

$$\delta[\Delta u_j(x,z)] = \int_{-\infty}^{+\infty}\left\{\frac{2(1-\nu^2)}{E}[k_{1j}(z';x,z)K_1^\circ(z') + k_{2j}(z';x,z)K_2^\circ(z')] + \frac{2(1+\nu)}{E}k_{3j}(z';x,z)K_3^\circ(z')\right\}\delta a(z')\,dz' \qquad (75)$$

is the first-order variation in j component of the relative displacement $\Delta\mathbf{u}$ of the crack faces at x,z when the crack front advances by $\delta a(z')$ in presence of (only) the original loadings, which have induced intensity factors $K_\alpha^\circ(z')$ along the reference straight crack front.

Let us note the relation between the weight functions and Green's function (i.e., the dyadic $\mathbf{G}(\mathbf{r},\bar{\mathbf{r}})$ giving, by $\mathbf{u} = \mathbf{G}\cdot\mathbf{P}$, the displacement $\mathbf{u}$ at $\mathbf{r}$ induced by concentrated force $\mathbf{P}$ at $\bar{\mathbf{r}}$) for a cracked body. Suppose that the "original loading" is just the concentrated force described, with null body forces and boundary tractions or displacements elsewhere. Then equation (72) requires

$$K_\beta^\circ(s) = \mathbf{h}_\beta(\bar{\mathbf{r}},s)\cdot\mathbf{P}, \qquad (76)$$

and hence equation (71) reads that

$$\delta\mathbf{u}(\mathbf{r}) = \left[\int_C 2\Lambda_{\alpha\beta}(s)\mathbf{h}_\alpha(\mathbf{r},s)\mathbf{h}_\beta(\bar{\mathbf{r}},s)\delta a(s)\,ds\right]\cdot\mathbf{P}. \qquad (77)$$

Since $\mathbf{u} = \mathbf{G}\cdot\mathbf{P}$, this shows that the bracketed term is the variation $\delta\mathbf{G}(\mathbf{r},\bar{\mathbf{r}})$ to first order in $\delta a(s)$.

For the half-plane crack in an infinite body, this expression leads to a formula for the Green's function in terms of the weight functions. Note that if a denotes the x coordinate of the straight crack front parallel to the z axis (this means $a_0 = 0$ in Fig. 2) then, representing s by z' as before, one must have

$$\mathbf{h}_\alpha(\mathbf{r},z') = \mathbf{h}_\alpha(x-a,y,z-z'). \qquad (78)$$

The Green's function must depend parametrically on a, and if we let δa denote a uniform $\delta a(z)$ along the crack front and divide by δa in equation (77), one has that

$$\frac{\partial\mathbf{G}(\mathbf{r},\bar{\mathbf{r}};a)}{\partial a} = 2\Lambda_{\alpha\beta}\int_{-\infty}^{+\infty}\mathbf{h}_\alpha(x-a,y,z-z')\mathbf{h}_\beta(\bar{x}-a,\bar{y},\bar{z}-z')\,dz'. \qquad (79)$$

But when $a = -\infty$, $\mathbf{G}$ reduces to the function $\mathbf{G}(\mathbf{r},\bar{\mathbf{r}};-\infty)$ obtained from the Kelvin-type solution for a point force in an infinite uncracked body. This depends only on $\mathbf{r}-\bar{\mathbf{r}}$. Thus the Green's function when the crack tip lies along the z axis ($a=0$) is

$$\mathbf{G}(\mathbf{r},\bar{\mathbf{r}}) = \mathbf{G}^{\text{Kelv.}}(\mathbf{r}-\bar{\mathbf{r}}) + 2\Lambda_{\alpha\beta}\int_{-\infty}^{0}\int_{-\infty}^{+\infty}\mathbf{h}_\alpha(x-a,y,z-\bar{z}+t)\mathbf{h}_\beta(\bar{x}-a,\bar{y},t)\,dt\,da. \qquad (80)$$

For the half-plane crack in an infinite isotropic body, perhaps the simplest way of solving for the weight functions will be to solve the Navier elasticity equations directly for $\delta\mathbf{u}(\mathbf{r})$, with arbitrary $\delta a(z')$, putting the result in the form of equation (71) so that the $\mathbf{h}_\alpha$ can be read off. However, we can also make use of what is now available in the literature to derive an integral representation for the $\mathbf{h}_\alpha$ as follows. First note that from details of the point force solution given by Love (1927, articles 130, 131, and 141), the components of the Kelvin Green's function for the isotropic solid are

$$G_{ij}^{\text{Kelv.}}(\mathbf{r}-\bar{\mathbf{r}}) = \frac{(\lambda+3\mu)\delta_{ij} + (\lambda+\mu)X_iX_j/R^2}{8\pi\mu(\lambda+2\mu)R} \qquad (81)$$

and if $\Sigma_{pqj}^{\text{Kelv.}}$ denotes the associated stress components, i.e., stresses σ_{pq} at $\mathbf{r}$ due to unit concentrated force in the j direction at $\bar{\mathbf{r}}$, then

$$\Sigma_{pqj}^{\text{Kelv.}}(\mathbf{r}-\bar{\mathbf{r}}) = \frac{1}{4\pi(\lambda+2\mu)R^3}[\mu(\delta_{pq}X_j - \delta_{pj}X_q - \delta_{qj}X_p) - 3(\lambda+\mu)(X_pX_q/R^2)X_j]. \qquad (82)$$

Here p,q,i,j range over x,y, and z, δ_{pq} is the Kronecker symbol,

$$X_x = x-\bar{x}, \qquad X_y = y-\bar{y}, \qquad X_z = z-\bar{z}, \qquad (83)$$

and $R^2 = X_pX_p$. We wish to solve for the weight functions $h_{\alpha j}(\bar{\mathbf{r}}, z')$, which can be interpreted as the stress intensities K_α at z' generated by a unit point force in the j direction at $\bar{\mathbf{r}}$. If there were no crack, this loading would cause traction stresses σ_{yq} (i.e., σ_{yx}, σ_{yy}, σ_{yz}) on the plane $y = 0$ given as

$$\Sigma_{yqj}^{\text{Kelv.}}(x-\bar{x},-\bar{y},z-\bar{z}). \qquad (84)$$

To represent the effect of the crack, we must remove these tractions. This can be done by regarding $\mp\Sigma_{yqj}^{\text{Kelv.}}\,dxdy$ as concentrated forces applied to the upper and lower crack surfaces, and superposing all such forces as weighted by $k_{\alpha q}(z';x,z)$ of equations (73) to calculate K_α. Thus, using the notations $h_{\alpha j}(\bar{\mathbf{r}},z') = h_{\alpha j}(\bar{x},\bar{y},\bar{z}-z')$ and $k_{\alpha q}(z';x,z) = k_{\alpha q}(x,z-z')$ appropriate to the present case, we have

$$h_{\alpha j}(\bar{x},\bar{y},\bar{z}) = -\int_{-\infty}^{0}\int_{-\infty}^{+\infty}\Sigma_{yqj}^{\text{Kelv.}}(x-\bar{x},-\bar{y},z-\bar{z})k_{\alpha q}(x,z)\,dz\,dx. \qquad (85)$$

The integrations implied are somewhat formidable and the further development of this topic is left to future work. However, apart from the inconvenient form of the $h_{\alpha j}$, the way is now available for the half-plane crack under general loadings to find stress intensities from equation (72), to find first-order changes in the displacement field for arbitrary deviations of the tip from straightness by equation (71), and to use the simpler equations (73) and (75) as a starting point in finding associated first-order changes in the stress-intensity factors along the crack front.

Discussion

Regarding possible applications for calculations of the type given here for tensile cracks or as outlined in the last section for general loadings, the following might be mentioned:

1. At a crustal tectonic scale it seems appropriate in some circumstances to regard the shallow seismogenic crust along plate margins as locked between great earthquakes while, in accommodation of plate motion, creep slippage occurs continuously below on the downward extension of the margin into the lower crust and mantle (e.g., see Li and Rice, 1983*a,b*, and references therein). Roughly speaking this geometry describes a shear crack which, late in the earthquake cycle, may show some preinstability advance into the locked shallow crust. Thus one would like to know what deformation at the Earth's surface would be associated with such crack advance and how measurements of surface deformation could constrain its form. Equation (71), with the mode I part

deleted for slipping cracks, provides a manner of addressing such issues. Clearly, the shear loaded half-plane crack, for which specific solutions are discussed here, should be modified to have a traction-free surface along, e.g., a plane of form x = constant to represent more accurately the Earth's surface above a strike-slip margin. Such questions are conventionally addressed in geodetic modeling by assuming a distribution of sub-surface slip rather than by assuming alteration of the boundary between a locked and slipping region, although calculations of the latter type have recently been done by Tse et al. (1984) based on approximate "line spring" modeling.

2. As suggested to the author in different contexts by B. Budiansky and E. Vanmarcke, an important problem in fracture theory is that of how a crack advances through a material with randomly variable fracture resistance. The problem arises for both tensile and tectonic shear cracking. To the extent that such crack advance can be modeled as the elastic-brittle growth of a planar crack whose front is only slightly deviated from straightness by the random highs and lows of fracture resistance, the techniques discussed here for relating local K to crack front location may be applied as a basis for analyzing growth. The problem would seem to have attributes similar to that of the failure of a long earth embankment of randomly variable properties as treated by Vanmarcke (1977). A localized weak spot may engender a local, short wavelength advance of the crack, but such will not necessarily go very far because the K at the advanced tip will generally be reduced according to the analysis here. On the other hand, long wavelength advances of the crack may correspond to effectively sampling the fracture resistance over a wide zone and are not sensitive to locally weakened zones. Thus, as is the case in Vanmarcke's (1977) analysis of embankment failure, one expects an intermediate wavelength to be most critical. There is also the prospect in such studies of quantifying the random small ruptures which precede overall failure.

3. As emphasized by Meade and Keer (1984*b*), crack-front segmentation is observed in laboratory study of brittle materials under mode 3 or combined mode 1 and mode 3 loading. They observe that modes 2 and 3 are coupled in the sense that when the crack front protrudes ahead locally in the plane $y=0$ a mode 2 intensity is induced which has different signs on opposite sides of the localized protrusion. They suggest that the induced mode 2 causes deviations from planarity of opposite sense (up versus down in y direction) on the two sides of the protrusion during tensile crack advance, leading to nonplanar segmentation. The revised basis for calculating stress intensities along a nonstraight crack front set out here may allow further evaluation of the proposed mechanism.

4. An approach to assessing whether a given solid will fail by atomistically brittle versus ductile mechanisms is based on addressing the following question (Rice and Thomson, 1974; Mason, 1979): As loading is applied to an initially sharp crack will one first reach conditions for tensile separation of lattice planes or, instead, will dislocation lines first be nucleated from the tip and blunt-out the stress concentration? Thus far the dislocation nucleation problem has been addressed beginning with an exact elastic analysis for the "image force" pulling a straight dislocation line, lying parallel to the tip, back into the tip (Rice and Thomson, 1974; Asaro, 1975; Rice, 1984). Then the analysis is adapted in a more-or-less ad hoc manner to deal with discrete atomistic and three-dimensional aspects of the problem (e.g., the dislocation line is expected to nucleate as a loop). However, the elastic field of an arbitrary three-dimensional dislocation loop can be represented in terms of the Green's function for the dislocated body. Since the Green's function for the half-plane crack is given here, it is possible in principle to represent the elastic field of a dislocation loop near a crack tip in terms of the present results, although it is far from clear that it will be feasible to carry through the calculations implied. Ideally, one would like to calculate the loop shape allowing easiest nucleation.

Acknowledgment

I am grateful for discussions on the topic of this paper with B. Budiansky, H. J. Gao, J. W. Hutchinson, L. M. Keer, P. Matagna, D. M. Parks, and E. Vanmarcke. Work in this area is supported by the NSF Materials Research Laboratory at Harvard, by the NSF Division of Earth Sciences, and the USGS Earthquake Hazards Reduction Program.

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