CONSERVED INTEGRALS AND ENERGETIC FORCES

JAMES R. RICE

Harvard University, Cambridge, MA 02138, USA

ABSTRACT

Conserved integrals of the Eshelby type representing energetic forces on dislocations, inclusions, voids, cracks and the like are reviewed and related to invariant transformations. Applications are discussed based on path independence for 2D integrals of J and M type and on the Maxwell reciprocity satisfied by energetic forces. Such concepts have had wide use in crack mechanics to aid analysis of near tip fields and provide elegant short-cut solutions of boundary value problems. Here new applications of path independence to dislocations show that the M integral when centered on a dislocation line is equal to the prelogarithmic energy factor and, also, that simple expressions involving the factor result for the image force drawing a dislocation to a V-notch tip and to a bicrystal interface. A review of reciprocity for energetic forces reveals a wide range of applications to such topics as weight functions for elastic crack analysis, the structure of inelastic constitutive relations, and compliance methods in nonlinear fracture mechanics, and in a new application there is developed the full 2D interaction effects between a crack tip and a nearby dislocation in a crystal of general anisotropy.

1. INTRODUCTION

Eshelby (1,2) was the first to associate an energetic force on an elastic defect with a conserved integral of elastostatic field quantities over a surrounding surface (or contour for 2D fields). His concern was with "any source of internal stress" such as an inclusion or dislocation whose motion translated an incompatible and possibly singular field along with its core. The emergence of the same integral for the energy release rate in crack mechanics (3,4,5) together with the discovery of further non-translational conserved integrals (6,7) with energetic force interpretations (8,9) has now led to an enormous literature. I do not attempt to completely summarize it here but, rather,

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focus on two primary issues. The first relates to the fact that the integrals involved are of conservation type, i.e., independent of the surface chosen. It developed that this feature, in the form of 2D path independence, could be exploited to great advantage in the development of nonlinear crack mechanics (5,10,11,12) and allowed also elegant short-cut solutions to some elastic crack boundary value problems (10,13, 14). I present new applications of path independence here to an interpretation of the "M" integral and the determination of forces on dislocations near notch tips and grain interfaces. The second issue is that the energetic forces discussed relate to strain energy changes and must satisfy reciprocity relations of the Maxwell type in thermodynamics. This has proven to be the source of some remarkable and unexpected inter-relations which I summarize in the final section. The topics discussed there range from elastic weight functions to compliance testing and the structure of inealstic constitutive relations; they also include some new results on the energetic forces on dislocations near a crack tip, complementing those derived from path independent integrals.

The translational integrals are

$$J_\alpha = \int_\Gamma (W n_\alpha - n_\beta \sigma_{\beta\gamma} u_{\gamma,\alpha})\, ds \tag{1}$$

where Γ is some closed surface (or contour in 2D) and ds is an element of area (or arc length). Here $\underline{n}$ is the outer normal to Γ, $W = W(\underline{\nabla u})$ is the elastic strain energy density per unit volume of an unstressed reference configuration, $\underline{u}(\underline{x})$ is displacement of the particle which was at $\underline{x}$ in the reference configuration, and $\sigma_{\alpha\beta} = \partial W/\partial u_{\beta,\alpha}$. When the surface Γ surrounds a solute $\underline{J}$ is the drift force, and when Γ surrounds a straight dislocation line as a contour in a 2D field $\underline{J}$ is the combined glide/climb force on the dislocation (1,2). Similarly for planar crack growth in a 2D field J (= J_1, the component in the direction of crack growth) on any contour Γ surrounding the tip, starting on one traction free crack surface and ending on the other, is the Irwin energy release rate G (4,5), i.e., the energetic force conjugate to crack area.

Related to the force interpretations, $\underline{J} = \underline{0}$ when Γ is any closed surface (contour in 2D) surrounding homogeneous defect-free material. This property $\underline{J} = \underline{0}$ may be understood alternatively as a consequence of Noether's theorem (15) and follows from the fact that for translationally homogeneous materials the volume integral in the elastic variational principle,

$$\delta\left[\int_V W\, dV\right] + [\text{boundary terms}] = 0 \quad , \tag{2}$$

is invariant when we change positional coordinates and displacement from the unprimed to a primed set by

$$x'_\alpha = x_\alpha + \varepsilon_\alpha \qquad u'_\alpha(\underline{x}') = u_\alpha(\underline{x}) \tag{3}$$

where $\underline{\varepsilon}$ is arbitrary. The invariance discussed means that

$$\int_{V'} W[\underline{\nabla}\underline{u}'(\underline{x}')]\, dV' - \int_V W[\underline{\nabla}\underline{u}(\underline{x})]\, dV = 0 \tag{4}$$

Apparently Günther (6) first applied the Noether procedure to elastostatics and, in results discovered independently by Knowles and Sternberg (7), established the conservation integrals which result if invariance in the above sense applies for all or some of the transformations defined by

$$x'_\alpha = x_\alpha + \varepsilon_\alpha + (\underline{\omega} \times \underline{x})_\alpha + \gamma x_\alpha$$
$$u'_\alpha(\underline{x}') = u_\alpha(\underline{x}) + [\underline{\omega} \times \underline{u}(\underline{x})]_\alpha + [(m-n)/m]\gamma u_\alpha(\underline{x}) \tag{5}$$

where $\underline{\varepsilon}$, $\underline{\omega}$ and γ are infinitesimal. Here $\underline{\omega}$ refers to rotation and if there is invariance under the action of $\underline{\omega}$ there results the three conservation integrals customarily denoted by $\underline{L} = \underline{0}$. Of more interest to applications in the next section is that generated by self similar scale change by the factor γ. Then we will have invariance if the medium is homogeneous along rays from the coordinate origin and if W is homogeneous of degree m in $\underline{\nabla}\underline{u}$ (e.g., $m = 2$ for linear elasticity); n is the number of spatial dimensions of the "volume" denoted by V. For example the consequence for 2D linear elasticity is then that

$$M_o = \int_\Gamma x_\alpha (W n_\alpha - n_\beta \sigma_{\beta\gamma} u_{\gamma,\alpha})\, ds \tag{6}$$

satisfies $M_o = 0$ when Γ is a closed contour surrounding no singularity or defect. The subscript o (for coordinate origin) on M_o is to remind that the integral depends on the origin for $\underline{x}$. For example, if the integral is evaluated on a given path Γ in one case with origin at $\underline{x}_P$ and in another with origin at $\underline{x}_Q$, then

$$M_P = M_Q + (\underline{x}_Q - \underline{x}_P)_\alpha J_\alpha \quad . \tag{7}$$

Budiansky and Rice (8) showed that the integrals $\underline{L}$ and M, when taken on surfaces enclosing voids, could be interpreted as energetic forces: $\underline{L}$ is associated with erosion/addition of material so as to rotate the void boundary about the coordinate origin and M with self-similar erosion relative to the origin. Eshelby (13) noted the remarkable ability of path-independence of M to resolve some elastic crack boundary value problems, and such was pursued further by Freund (14), Ouchterlony (16,17) and Kubo (18) in application to various 2D point force (3D line force) loadings. The next section similarly makes use of M, but for dislocations.

Before going on to specifics let us study the formal working of Noether's (15) procedure for elasticity, writing the transformations of Eqn (5) as

$$x'_\alpha = x_\alpha + y_\alpha(\underline{x}) \quad , \qquad u'_\alpha(\underline{x}') = u_\alpha(\underline{x}) + v_\alpha(\underline{x}) \quad , \tag{8}$$

where $\underline{y}$ and $\underline{v}$ are infinitesimal, and assuming that Eqn (4) holds to first order in them. Then note that

$$\int_{V'} W[\underline{\nabla}\underline{u}'(\underline{x}')]\ dV' = \int_V W[\underline{\nabla}\underline{u}'(\underline{x})]\ dV + \int_\Gamma n_\alpha y_\alpha W\ ds \tag{9}$$

and that for any variation $\delta\underline{u}$ from an elastic field

$$\int_V \delta W\ dV = \int_V \sigma_{\alpha\beta}\delta u_{\beta,\alpha}\ dV = \int_\Gamma n_\alpha \sigma_{\alpha\beta}\delta u_\beta\ ds \quad . \tag{10}$$

Thus, if we insert Eqn (9) into Eqn (4) and use the last expression with

$$\delta u_\beta = u'_\beta(\underline{x}) - u_\beta(\underline{x}) = v_\beta - y_\gamma u_{\beta,\gamma} \tag{11}$$

to deal with the difference between the remaining integrals over V, there results

$$\int_\Gamma [n_\alpha y_\alpha W + n_\alpha \sigma_{\alpha\beta}(v_\beta - y_\gamma u_{\beta,\gamma})]\ ds = 0 \quad . \tag{12}$$

This expresses the conserved integrals discussed; identifying $\underline{y}$ and $\underline{v}$ with the terms in Eqn (5) the integral can be rewritten

$$J_\beta \varepsilon_\beta + L_\beta \omega_\beta + M\gamma = 0 \quad , \tag{13}$$

implying that the integrals J_β, L_β and M thereby defined vanish when there is invariance in the sense of Eqn (4) relative to the respective coefficient ε_β, ω_β or γ. Perhaps the simplest context for realizing the energetic force interpretations of the integrals is to consider a traction-free void (8). When Γ surrounds the void the integral of Eqn (12) is conserved but not generally zero. If now we shrink Γ to the void surface and realize that the resulting integral is to first order the negative of the energy change when we erode a layer $n_\alpha y_\alpha$ from the void surface (see (19,20) for discussion of energy changes in such processes), then rewriting the integral as on the left side of Eqn (13) shows that $\underline{J}$, $\underline{L}$ and M are the energetic forces conjugate to translation $\underline{\varepsilon}$, rotation $\underline{\omega}$ and self-similar scaling γ.

2. DISLOCATIONS NEAR NOTCH TIPS AND INTERFACES

2.1 M_d IS THE DISLOCATION ENERGY FACTOR

Suppose that an indefinitely long straight dislocation line of Burgers vector $\underline{b}$ lies along the x_3 axis of a rectilinearly anisotropic solid (e.g., single crystal) which sustains a 2D deformation field of combined plane and anti-plane strain in the x_1x_2 plane. The dislocation line pierces the x_1x_2 plane at the point with coordinates c_1, c_2 (Fig. 1a). It is well known, and follows from dimensional considerations and linearity, that the stress field is of the form

$$\sigma_{\alpha\beta} = D_{\alpha\beta}(\theta)/r + \bar{\sigma}_{\alpha\beta}(x_1, x_2) \quad (14)$$

where r and θ are polar coordinates at the core site and $\bar{\sigma}_{\alpha\beta}$ is the combined non-singular result of applied loadings and image effects. With $\underline{h} = (-\sin\theta, \cos\theta, 0)$ denoting the unit vector in the direction of increasing θ, it is elementary to show from stress equilibrium equations that the functions $\underline{D}$ of θ are such that $h_\alpha(\theta) D_{\alpha\beta}(\theta)$ is is independent of θ. Further, if one introduces the positive definite symmetric prelogarithmic "energy factor" tensor $K_{\alpha\beta}$ of anisotropic elastic dislocation theory (21,22,23) it is easy to show that

$$h_\alpha(\theta) D_{\alpha\beta}(\theta) = 2K_{\beta\alpha} b_\alpha \quad . \quad (15)$$

This expression will be motivated by what follows; in it $\underline{b}$ is the Burgers vector of the dislocation. The energy factor arises further in, and derives its name from, the expression for the strain energy (per unit length in the x_3 direction) of the unloaded but dislocated body with core cut-off at r_o:

$$U = K_{\alpha\beta} b_\alpha b_\beta \, \ell n(L/r_o) + \text{terms which} \to 0 \text{ as } r_o \to 0 \quad . \quad (16)$$

In this expression L depends on the outer dimensions of the dislocated body. Barnett and Swanger (23) explain how to extract the tensor $\underline{K}$ numerically in terms of the elastic moduli of an anisotropic crystal. For the isotropic case

$$K_{11} = K_{22} = \mu/4\pi(1-\nu), \quad K_{33} = \mu/4\pi, \quad \text{other } K_{\alpha\beta} = 0 \quad . \quad (17)$$

The logarithmic dependence on r_o in Eqn (16) may be understood via the Clapeyron expression for U: introduce a cut along the ray $\theta = \text{constant}$ from $r = 0$ to $r = R(\theta)$, at the outer boundary, and displace its surfaces by $\underline{b}$ by introduction of tractions $h_\alpha \sigma_{\alpha\beta}$ on the cut; the work done is

$$U = \frac{1}{2} \int_{r_o}^{R(\theta)} (h_\alpha \sigma_{\alpha\beta}) b_\beta \, dr = \frac{1}{2} \int_{r_o}^{R} [(h_\alpha D_{\alpha\beta}/r) b_\beta + \ldots] \, dr$$

$$= K_{\alpha\beta} b_\alpha b_\beta \, \ell n \frac{R}{r_o} + \ldots \qquad (18)$$

where the dots denote bounded terms. Note that the independence of $h_\alpha D_{\alpha\beta}$ on θ is essential to the form of the final result.

Now, from Eshelby's work (1,2) we know that if we evaluate $\underline{J}$ on a circuit Γ (Fig. 1a) surrounding the dislocation we get the energetic force $\underline{f}$ on the dislocation, defined such that $\delta U = -f_\alpha \delta c_\alpha$ at fixed outer-boundary displacements and related to $\bar{\sigma}_{\alpha\beta}$ by the Peach-Koehler force expression. Thus, with such choice of Γ we have

$$J_\alpha = f_\alpha = e_{\alpha\lambda 3} \, \bar{\sigma}_{\lambda\beta}(c_1, c_2) b_\beta \qquad (\alpha = 1,2) \qquad (19)$$

where $\underline{e}$ is the alternating tensor.

What is M for a dislocation? Plainly the result depends on the origin chosen for $\underline{x}$ in Eqn (6). Let us choose this origin at the dislocation line itself, denoting the corresponding M by M_d, and choose for path Γ a circle of radius r centered on the dislocation. Then

$$M_d = r^2 \int_0^{2\pi} [W(r,\theta) - n_\alpha(\theta) \sigma_{\alpha\beta}(r,\theta) \partial u_\beta(r,\theta)/\partial r] \, d\theta \qquad (20)$$

where here $\underline{n}$ denotes a radially directed unit vector. We note that M_d must be the same for any path Γ surrounding the dislocation and hence, in particular, this expression for M_d is independent of r. We can therefore evaluate it by letting $r \to 0$ and in that limit we see that only the $1/r$ singular stress terms in Eqn (14) can contribute to

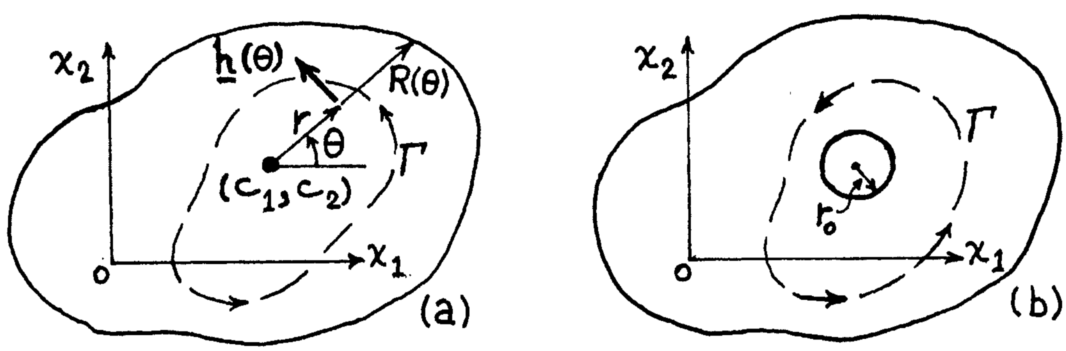

FIG. 1. (a) Dislocation line at c_1, c_2. (b) Traction-free core cylinder, radius r_o.

it; those terms and the corresponding $1/r$ singular terms in $u_{\alpha,\beta}$ cause each term within [...] above to behave as $1/r^2$ and hence to survive in the expression for M_d as $r \to 0$. The bounded terms $\bar{\sigma}_{\alpha\beta}$, $\bar{u}_{\alpha,\beta}$ do not so contribute, and hence we have reached the conclusion that M_d is a path invariant quantity associated with a dislocation and which can be evaluated solely in terms of the stress/deformation field which that dislocation would induce in an unbounded and otherwise unloaded body. Thus M_d can depend only on $\underline{b}$.

I will now show that M_d is, in fact, nothing other than the dislocation energy factor:

$$M_d = K_{\alpha\beta} b_\alpha b_\beta \quad . \tag{21}$$

We begin with a core model slightly different from that implied by Eqn (18) and, instead, regard the core radius r_o as the radius of a traction free circular cylindrical hole centered on the dislocation line (Fig. 1b). We shall always be interested in the case $r_o \lll L$, by which it will be understood that an intermediate range of r exists satisfying

$$r_o \ll r \ll L$$

within which the stress field is given by the first term of Eqn (14), $\sigma_{\alpha\beta} = D_{\alpha\beta}(\theta)/r$, with numerically negligible additional contributions from the core perturbations (whose effective size scale will be of order r_o; see below) and from $\bar{\sigma}_{\alpha\beta}$. This assures us that M_d has effectively the same numerical value (e.g., take the path Γ in the intermediate region) as for the coreless dislocation, and exactly so as $r_o \to 0$. However, M_d remains independent of path Γ and if we shrink Γ onto the core cylinder itself, on which tractions $n_\alpha \sigma_{\alpha\beta} = 0$, we have

$$M_d = r_o^2 \int_0^{2\pi} W(r_o,\theta)\, d\theta \quad . \tag{22}$$

Now, let us enlarge the core by eroding a layer of material of thickness δr_o from the core cylinder while the outer boundary of the dislocated body is held fixed. The strain energy change is minus the energy of the layer removed (19,20), and thus

$$\delta U = -\int_0^{2\pi} W(r_o,\theta)\,\delta r_o r_o \, d\theta = -M_d \,\delta r_o / r_o \quad . \tag{23}$$

Thus $r_o\, \partial U/\partial r_o = -M_d$, which is independent of r_o in the r_o range considered. If, however, we provisionally accept that for the present core model U will be given by an expression like that in Eqn (16), then $r_o\, \partial U/\partial r_o$ is seen to be nothing other than the energy factor, and Eqn (21) is proven. We have now only to understand that the same form for U as in Eqn (16), specifically the same energy factor coefficient, does indeed result for the present core model of a traction free cylinder as for model implied by Eqn (18). Let us note that the effect of introducing the cylindrical hole is that the solution for reverse tractions $n_\alpha D_{\alpha\beta}(\theta)/r_o$ on a cylinder of radius r_o must be added to the coreless solution of Eqn (14). Since these reverse tractions amount to zero net force on the cylinder, the stress field which they produce contains terms which, relative to boundary traction values at $r = r_o$, decay with r at least as fast as $(r_o/r)^2$; see (21). Thus the total stress field will have the form

$$\sigma_{\alpha\beta} = D_{\alpha\beta}(\theta)/r + \bar{\sigma}_{\alpha\beta}(x_1, x_2) + D^{(2)}_{\alpha\beta}(\theta)\, r_o/r^2 + D^{(3)}_{\alpha\beta}(\theta)\, r_o^2/r^3 + \ldots \qquad (24)$$

and the calculation of U given by the first equality in Eqn (18) is now exact for the cylindrical hole core model. The added terms in $\sigma_{\alpha\beta}$ above are readily seen to contribute a bounded amount to U, independently of r_o, and hence the structure of U will be precisely as in the last line of Eqn (18). Hence the provisional assumption above that the same energy factor applies for the cylindrical hole core model is seen to be correct, and we see therefore that M_d equals the energy factor.

2.2 ATTRACTION OF A DISLOCATION TO A NOTCH TIP

Fig. 2 shows a dislocation line at distance ρ from the tip of a V-notch in a bicrystal. We consider the case in which there is no externally applied loading, so that if we can evaluate the image forces on the dislocation we know how much it is attracted to the notch tip. When the notch has the form of a flat crack on a bicrystal boundary, this problem corresponds to a fundamental element of the Rice and Thomson (24) analysis of brittle versus ductile response, as that analysis would be extended to failure on grain interfaces. In keeping with this sort of application, we consider ρ to be small enough compared to overall notch and body dimensions that the bicrystal may be considered of infinite extent, with the V-notch extending to infinity.

Two contours Γ and Γ' are shown and it is evident the integral M_d, corresponding to the coordinate origin at the notch tip, must be the same for each contour. We can see, however, that $M_o = 0$ on contour Γ': The integrand of M_o vanishes identically on the traction free notch surfaces ($\underline{n}$ is perpendicular to $\underline{x}$ there). The stress singularity at a V-notch tip is less strong than $1/r$ (and never stronger than $1/\sqrt{r}$), where here r means distance from the tip, and hence by shrinking the radius of the small circular arc at the tip to zero, there is no contribution to M_o from that part of Γ' either. Finally, by letting the radius of the large arc expand to infinity and

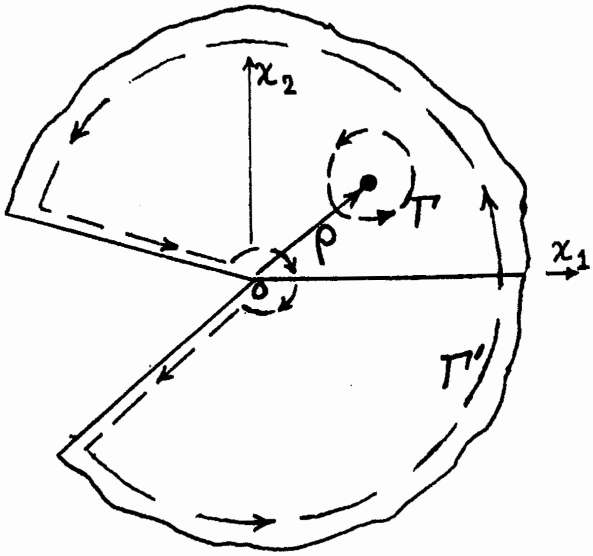

FIG. 2. Dislocation near notch tip.

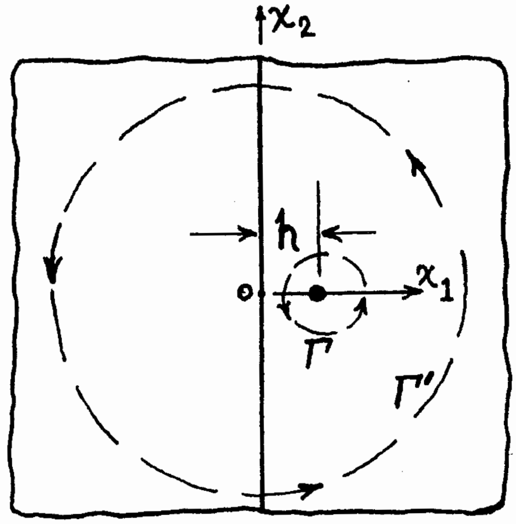

FIG. 3. Dislocation near bicrystal interface.

recognizing that, owing to the traction free notch, the stress field falls off at large r more rapidly than $1/r$, there is again seen to be no contribution to M_o.

The conclusion is that $M_o = 0$ on Γ', and thus on Γ also. However, by application of Eqn (7), this means that

$$(M_o)_\Gamma = c_1(J_1)_\Gamma + c_2(J_2)_\Gamma + (M_d)_\Gamma = 0 \quad . \tag{25}$$

Thus, using Eqns (21) and (19), we see that the radial component of "image" force attracting the dislocation into the tip is

$$f_\rho \equiv (c_1 f_1 + c_2 f_2)/\rho = -K_{\alpha\beta} b_\alpha b_\beta/\rho \ , \tag{26}$$

where $\underline{K}$ is evaluated for the crystal in which the dislocation resides. This is a remarkable result; the force component f_ρ depends on neither the angular range of the V-notch nor on the properties of the second material constituting the bicrystal.

Special versions of Eqn (26) have appeared previously. Rice and Thomson (24) derived by a novel method an expression which can be put in the same form for the force on a dislocation in a radially gliding orientation near the tip of a crack in an isotropic and homogeneous solid. Also, Asaro (25) subsequently showed by conventional 2D anisotropic elasticity methods that Eqn (26) applies for a dislocation near the tip of a crack in a homogeneous anisotropic solid, a problem addressed earlier by Atkinson (26).

The generality of Eqn (26) suggests that there must be another approach to understanding it and, in hindsight, I suggest the following. Let us adopt a model of a dislocation with a finite core size r_o, as discussed above, but assume that $r_o <<< \rho$. The energy can then be written as in Eqn (16) and we ask, then, on what can L depend? Dimensional analysis suggests that L must be of the form $\rho H(\phi)$ where H is a function of the angle ϕ which the line of length ρ makes with the x_1 axis, and also of the angles defining the V-notch and the ratios of moduli of the two crystals. Thus

$$f_\rho = -\frac{\partial U}{\partial \rho} = -\frac{\partial}{\partial \rho}\left[K_{\alpha\beta}b_\alpha b_\beta \,\ell n\, \frac{\rho H(\phi)}{r_o}\right] = -K_{\alpha\beta}b_\alpha b_\beta/\rho . \quad (27)$$

The argument is, perhaps, less pleasing than that based on the conservation integrals, but it is always worthwhile to examine problems from different viewpoints and it would be interesting to know if some of the applications of M to point force loads on cracks (13,14,16,18) could be approached in a similar fashion.

2.3 DISLOCATION NEAR AN INTERFACE

Fig. 3 shows a bicrystal with a line dislocation at distance h from the boundary. We shall use conservation integrals to derive the same expression for the image force on such a dislocation as presented by Barnett and Lothe (27). We assume that h is small compared to crystal dimensions so that the bicrystal can be assumed to be infinite.

Again the value of M_o, based on a coordinate origin on the interface, is the same for path Γ' as for Γ. We know from Eqn (7) that

$$(M_o)_\Gamma = (M_d)_\Gamma + (J_1)_\Gamma \, h \quad . \quad (28)$$

The value of M_o on path Γ' is the same for any such contour encircling the dislocation and hence, for a large circle whose radius increases to infinity. In that case the leading far field stress term, which decays as $1/r$ and is the only term contributing to M_o as $r \to \infty$, must coincide with the solution for a dislocation on the interface. Hence

$$(M_o)_{\Gamma'} = M_d^{int} \quad (29)$$

where the superscript "int" refers to the interface dislocation of the same Burgers vector. Thus, writing $(J_1)_\Gamma = f$ for the horizontal (and only non-zero) component of force on the dislocation, we have

$$f = -(K_{\alpha\beta} - K^{int}_{\alpha\beta}) b_\alpha b_\beta / h \quad , \tag{30}$$

which is the Barnett and Lothe result.

3. FORCES AND RECIPROCITY

It is, I think, not well recognized how powerful are the notions that energetic forces can be associated with defect motion or other forms of structural rearrangement within a solid and that these forces can be related to local fields at the defect site. For example, within linear elasticity the Peach-Koehler force f on a dislocation is given in terms of stress at the dislocation site by Eqn (19) and the Irwin crack extension force, for growth involving a single mode of crack tip deformation, by

$$G = k^2/M \tag{31}$$

where k is the crack tip stress intensity factor for that mode and M is an appropriate modulus or combination of moduli. To see the strength of these ideas, a few short developments are given here. All but the last two subsections presume 2D deformation fields.

3.1 WEIGHT FUNCTIONS

Here is presented a brief synopsis of my interpretation (28) of the Bueckner (29) weight function theory. Let a 2D elastic body contain a crack of length a (Fig. 4, ignoring the dislocation) and be loaded by two systems of applied force, one with intensity measured by Q_1 and the other by Q_2 (only one load system is illustrated in the figure). Both systems induce the same loading mode

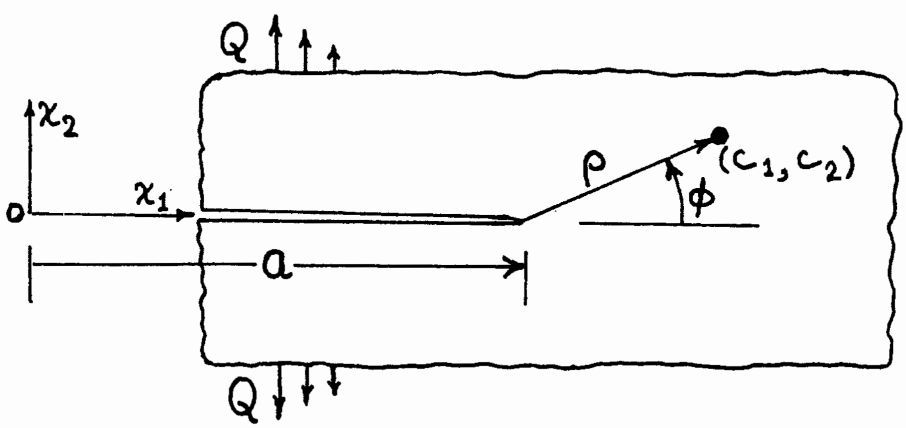

FIG. 4. Solid with loadings proportional to Q (generalized force) containing crack of length a and dislocation at c_1, c_2.

at the crack tip, say, mode I (tension). The Q's may be regarded as generalized forces; with them we may associate generalized displacements q_1 and q_2 which will be defined by differently weighted averages of the displacement field in the body such that $Q_1\delta q_1 + Q_2\delta q_2$ denotes the work of applied loading per unit thickness of the 2D body. Then the energy U per unit thickness satisfies

$$\delta U = Q_1\delta q_1 + Q_2\delta q_2 - (k^2/M)\delta a \quad , \tag{32}$$

where the meaning of G as an energetic force and its expression by Eqn (31) have been used. It is understood that k is homogeneously linear in Q_1 and Q_2.

We suppose that we know the complete solution for the displacement throughout the body and the stress intensity factor when load system 1 is applied, but nothing about the solution for system 2. In particular this means that we know q_2 and k as functions of a and (linearly) of Q_1 when $Q_2 = 0$. The power of such information is shown by use of the reciprocal relations that follow from δU being an exact differential. First introducing the Legendre transform of Eqn (32) to

$$\delta(U - Q_1 q_1 - Q_2 q_2) = -q_1\delta Q_1 - q_2\delta Q_2 - (k^2/M)\delta a \quad , \tag{33}$$

we find the particularly useful reciprocal relation

$$\partial q_2(Q_1,Q_2,a)/\partial a = 2[k(Q_1,Q_2,a)/M]\,\partial k(Q_1,Q_2,a)/\partial Q_2 \quad . \tag{34}$$

Setting $Q_2 = 0$, it is seen that all terms in this equation except $\partial k/\partial Q_2$ are known and thus we find that we can solve for $\partial k/\partial Q_2$ (which is independent of the Q's since k is linear in them).

Thus we can write the stress intensity induced by non-zero load system 2 as

$$Q_2\ \partial k/\partial Q_2 = Q_2[M(\partial q_2/\partial a)/2k]_{Q_2=0} \quad . \tag{35}$$

But we observe that the bracket on the right cannot possibly depend on the nature of loading system "1". It is then a small step to understand that the "weight function", namely

$$h_\alpha(x_1,x_2,a) = M[\partial u_\alpha(x_1,x_2,a)/\partial a]/2k(a) \quad , \tag{36}$$

computed from the displacement field u_α and stress intensity k due to any particular loading, is universal. It is the same (within unessential rigid motions) for all possible loadings. Once determined from the solution for one particular loading, we find k for any second loading by summing the scalar product of $\underline{h}$ with the applied forces of that loading. Once k is thereby determined for the second loading, we may find $\underline{u}$ for that loading by multiplying through by $k(a)$ in Eqn (36) and then integrating on a. The procedure gives the complete displacement field $\underline{u}$ for the second loading if it is known before crack introduction. This is a powerful and widely used method.

3.2 FORCES ON DISLOCATIONS

Let Q and q be a conjugate generalized force and displacement set acting on a 2D elastic body containing a line dislocation at c_1, c_2 (Fig. 4, now ignoring the crack). Then

$$\delta U = Q\,\delta q - f_1 \delta c_1 - f_2 \delta c_2 \tag{37}$$

with $f_1 = \bar{\sigma}_{2\beta} b_\beta$, $f_2 = -\bar{\sigma}_{1\beta} b_\beta$. After a Legendre transformation on q, similar to Eqn (33), we can read off the reciprocal relations

$$\begin{aligned} \partial q(Q,c_1,c_2)/\partial c_\alpha &= \partial f_\alpha(Q,c_1,c_2)/\partial Q \quad , \\ \partial f_1(Q,c_1,c_2)/\partial c_2 &= \partial f_2(Q,c_1,c_2)/\partial c_1 \quad . \end{aligned} \tag{38}$$

Using the expressions for the f's just given, the last relation requires that

$$(\partial\bar{\sigma}_{1\beta}/\partial c_1 + \partial\bar{\sigma}_{2\beta}/\partial c_2) b_\beta = 0 \quad . \tag{39}$$

But $\bar{\sigma}_{\alpha\beta}$ consists additively of the stress field induced by load Q, say $\hat{\sigma}_{\alpha\beta}$, plus the image field which is independent of Q. Thus if Eqn (39) is to be satisfied for arbitrary dislocation vectors it is seen to be necessary that

$$\partial\hat{\sigma}_{1\beta}/\partial c_1 + \partial\hat{\sigma}_{2\beta}/\partial c_2 = 0 \quad . \tag{40}$$

That is, the existence of δU as a perfect differential in conjunction with the formulae for force on a dislocation requires that the stress field due to loading Q satisfy the equations of equilibrium. This is an unexpected connection.

We see also from the first of Eqns (38) that the effect of change of dislocation position on the displacement is given by

$$\partial q(Q,c_1,c_2)/\partial c_\alpha = e_{\alpha\lambda 3}\hat{\sigma}_{\lambda\beta}(c_1,c_2) b_\beta / Q \quad , \tag{41}$$

which is closely related to Eqn (60) to follow.

3.3 DISLOCATION NEAR A CRACK TIP

Consider a geometry like that in Fig. 2 with the V-notch collapsed to a flat crack and, for simplicity, assume that the material is homogeneous. Fig. 4 shows the configuration. Crack length is denoted by a and position of the near tip dislocation by (c_1, c_2), all measured from a fixed origin not at the crack tip. The cracked body with near tip dislocation is loaded by three independent loading systems of strength Q_1, Q_2, Q_3 which are now understood to be chosen so that any combination of opening, in-plane sliding or anti-plane sliding can be induced near the tip.

We have

$$\delta U = Q_\alpha \delta q_\alpha - G\delta a - f_\alpha \delta c_\alpha \tag{42}$$

(summation convention, $\alpha = 1,2,3$ in the first product and $\alpha = 1,2$ in the last) where f_α is given in terms of $\bar{\sigma}_{\alpha\beta}$ as before, Eqn (19), and where it is known from anisotropic crack mechanics as presented by Stroh (22) and Barnett and Asaro (30) that

$$G = (8\pi)^{-1} K^{-1}_{\alpha\beta} k_\alpha k_\beta \tag{43}$$

with

$$k_\alpha = \lim_{x_1 \to a+} [(2\pi)^{1/2} (x_1 - a)^{1/2} \sigma_{2\alpha}(x_1, x_2 = 0)] \tag{44}$$

defining the stress intensity factors; $\underline{K}^{-1}$ is the inverse of the energy factor tensor. Let us suppose, reasonably, that we know the stress intensity factors and stress fields $\sigma_{\alpha\beta}$ induced in the cracked body by each of the loadings Q_1, Q_2, Q_3 acting singly, in the absence of any dislocation. We wish to find the effect of the dislocation on the stress intensity factors (hence on G) and also find the "image" contribution to the forces $\underline{f}$ themselves; the radial component f_ρ is given already by Eqn (26) but that will be rederived here. This is a generalization to anisotropic media and to fuller determination of dislocation forces of a problem posed and solved on the basis of reciprocity by Rice and Thomson (24). I shall outline its solution here and then proceed to give full details when the dislocation is very close to the crack tip.

Regard Q_1, Q_2, Q_3, a, c_1, c_2 as the independent variables in all differentiations. Then reciprocity requires that

$$\begin{aligned} \partial G/\partial c_\mu &= (4\pi)^{-1} K^{-1}_{\alpha\beta}\, k_\alpha\, \partial k_\beta/\partial c_\mu \\ &= \partial f_\mu/\partial a = e_{\mu\lambda 3} b_\beta \partial\bar{\sigma}_{\beta\lambda}/\partial a\ . \end{aligned} \tag{45}$$

Recognizing that $\partial k_\beta/\partial c_\mu$ is independent of the Q's and that $\partial k_\alpha/\partial Q_\gamma$ (denoted $R_{\gamma\alpha}$ below) are known functions of a only, whereas $\partial\bar{\sigma}_{\beta\lambda}/\partial Q_\gamma$ are known functions of a, c_1 and c_2 (the same as if the dislocation were absent), we have after differentiation with respect to Q_γ and multiplication by KR^{-1} that

$$\partial k_\beta/\partial c_\mu = 4\pi K_{\beta\alpha} R^{-1}_{\alpha\gamma} e_{\mu\lambda 3} b_\delta [\partial(\partial\bar{\sigma}_{\delta\lambda}/\partial Q_\gamma)/\partial a] \quad . \tag{46}$$

The right hand side is a known function of a, c_1 and c_2 and is independent of the Q's. Hence, by integration beginning at values of c_1, c_2 large enough that the k's vanish, we can get the dependence of k_β on a, c_1, c_2 when all the Q's vanish. If this solution for the k's is now substituted into the previous equation, we have an expression for $\partial f_\mu/\partial a$ when all the Q's vanish, from which we can solve for f_μ by an integration beginning at such large negative a that the f's effectively vanish.

In this way all the desired information can in principle be extracted and, curiously, such is done without posing any standard elastic boundary value problem for a dislocation in a cracked elastic solid.

The details are now presented for the case of a dislocation which is much closer to the crack tip than distances like overall crack length and body dimensions. In that case it suffices to regard the crack as semi-infinite and to treat all parameters denoting interaction between the crack and dislocation (e.g., like the k's and f's when $\underline{Q}=0$) as functions of $c_1 - a$ and c_2 only. We can then further regard $R_{\alpha\gamma}$ as changing sufficiently little with a over distance scales of interest to be taken as effectively constant. For the stress field induced in the region of interest near the crack when the body is loaded butthere is no dislocation, it suffices to represent the stress state only by the standard leading crack tip singular term

$$\bar{\sigma}_{\delta\lambda}(c_1,c_2;a) = k_\alpha C^\alpha_{\delta\lambda}(\phi)/\sqrt{2\pi\rho} \tag{47}$$

where $\rho^2 = (c_1-a)^2 + c_2^2$, $\tan\phi = c_2/(c_1-a)$. This represents the well understood inverse square root singular crack tip field for the anisotropic solid considered (22,30). Consistently with the definition of stress intensity factors, the functions $\underline{C}$ giving the angular dependence of the crack tip field are normalized such that $C^\alpha_{\lambda 2}(0) = \delta_{\alpha\lambda}$ (Kronecker delta). Now, since $k_\alpha = Q_\gamma R_{\gamma\alpha}$ when there is no dislocation, we see that then

$$R^{-1}_{\alpha\gamma}(\partial\bar{\sigma}_{\delta\lambda}/\partial Q_\gamma) = C^\alpha_{\delta\lambda}(\phi)/\sqrt{2\pi\rho} \tag{48}$$

and, of course, the equality remains valid even when the dislocation is present for reasons already discussed. Now set $\mu=1$ in Eqn (46) and observe that $\partial/\partial a = -\partial/\partial c_1$. Then upon integration in c_1 we obtain

$$k_\beta = -(2\pi)^{3/2} K_{\beta\alpha} b_\delta C^\alpha_{\delta 2}(\phi)/\sqrt{\rho} \tag{49}$$

for the stress intensity factors induced by a dislocation $\underline{b}$ at position ρ, ϕ near the crack tip in an unloaded body. We can now insert this expression into Eqn (45), still presuming $\underset{\sim}{Q} = 0$, set $\mu = 1$, use $\partial/\partial a = -\partial/\partial c_1$, and integrate to get

$$\begin{aligned} f_1 &= -G = -(8\pi)^{-1} K^{-1}_{\alpha\beta} k_\alpha k_\beta \\ &= -K_{\alpha\beta} b_\lambda b_\mu C^\alpha_{\lambda 2}(\phi) C^\beta_{\mu 2}(\phi)/\rho \end{aligned} \tag{50}$$

for the crack energy release rate and parallel component of image force in the unloaded solid. The corresponding results can then be written at once for the loaded solid since we know the results for forces and stress intensity factors due to the loadings $\underset{\sim}{Q}$.

The expression for f_1 enables calculation of both components f_1 and f_2 of image force, since we know f_ρ from Eqn (26). However, for completeness, f_ρ is derived here in the following steps. Observe that ρf_1 is independent of ρ, so that

$$0 = f_1 + \rho \partial f_1/\partial \rho = f_1 + (c_1 - a)\partial f_1/\partial c_1 + c_2 \partial f_1/\partial c_2 \quad , \tag{51}$$

and rewrite the last term using the second reciprocal relation of Eqns (38). Hence

$$\begin{aligned} 0 &= f_1 + (c_1 - a)\partial f_1/\partial c_1 + c_2 \partial f_2/\partial c_1 \\ &= \partial[(c_1 - a) f_1 + c_2 f_2]/\partial c_1 \end{aligned} \tag{52}$$

so that the bracketed term, which equals ρf_ρ, is independent of c_1. Since by dimensional considerations that term is at most a function of ϕ, this means that it is constant. We can evaluate the constant by setting $\phi = 0$, along which line f_ρ must coincide with f_1 as given in Eqn (50). Thus there results

$$f_\rho = f_1(\rho, \phi{=}0) = -K_{\alpha\beta} b_\alpha b_\beta/\rho \tag{53}$$

since $C^\alpha_{\lambda 2}(0) = \delta_{\alpha\lambda}$ in Eqn (50) with $\phi = 0$.

The fact derived above that $G + f_1 = 0$ for the unloaded but dislocated solid follows also from the J_1 integral taken on an outer contour surrounding the dislocation and crack tip. $J_1 = 0$ on that contour (evident by expanding it to large radius), but J_1 gets contributions $G + f_1$ when the contour is wrapped, respectively, around the crack tip and dislocation. The relation follows also from invariance of the total elastic energy when δa and δc_1 are changed equally in Eqn (42) as written for the unloaded solid.

The angular dependence $C^{\lambda}_{\alpha\beta}(\phi)$ of the crack tip stress $\sigma_{\alpha\beta}$ for mode λ was introduced in the previous discussion, and the dependence $D_{\alpha\beta}(\theta)$ for the dislocation stress field was introduced in Section 2.1, Eqn (14). Since, as Bilby and Eshelby (31) emphasized, cracks can be represented as continuous arrays of dislocations, these fields are not independent. I leave it for the interested reader to verify that if we write the singular dislocation stress field as

$$\sigma_{\alpha\beta} = b_{\mu}D^{\mu}_{\alpha\beta}(\theta)/r \quad , \tag{54}$$

then

$$C^{\lambda}_{\alpha\beta}(\phi) = \frac{1}{2\pi} K^{-1}_{\lambda\mu} \int_{0}^{\phi} [\sin\theta\sin(\phi-\theta)]^{-1/2} D^{\mu}_{\alpha\beta}(\theta)\, d\theta \tag{55}$$

for $\phi > 0$. A minus sign should precede the integral for $\phi < 0$. It helps in getting started to note that associated with the singular crack tip stress field, the displacement discontinuity $\Delta\underline{u}$ between crack surfaces is

$$\Delta u_{\alpha} = (1/\pi)\; K^{-1}_{\alpha\beta} k_{\beta} \sqrt{\rho/2\pi} \tag{56}$$

at distance ρ from the tip, which is necessary for consistency between Eqns (43) and (44).

3.4 STRUCTURE OF INELASTIC CONSTITUTIVE RELATIONS

Following earlier work (32,33) we may represent an increment of inelastic deformation within a 3D macroscopic sample of material by a set of local incremental variables $d\xi$, marking the advance of dislocation loops or more macroscopic measures of crystalline shear, microcracks, phase boundaries, and the like. Eshelby (1,2,20), Rice (33) and others have shown how to identify energetic forces with these and we may write

$$\delta\Phi = S_{\alpha\beta}\delta E_{\alpha\beta} - \langle F\delta\xi\rangle \tag{57}$$

where $\underline{S}$ and $\underline{E}$ are work conjugate macroscopic stress and strain tensors, Φ the strain energy per unit reference volume, and $Fd\xi$ denotes the inner product of all structural rearrangements $d\xi$ with their conjugate energetic forces F, and $\langle \ldots \rangle$ denotes the average of these products over a unit reference volume. Specific forms of $\langle Fd\xi\rangle$ for crystalline slip within grains of a polycrystalline aggregate are given in (32,33).

For example, if inelasticity results in a particular case from the glide motion of dislocation lines within a sample of material, then

$$<Fd\xi> = \frac{1}{V} \int_L f\delta c \, ds \tag{58}$$

where L denotes all such dislocation lines within a representative sample of volume V, δc is the incremental glide of the dislocation line normal to itself and f is the force resolved in the direction of glide. It is well known that within the conventional linear elastic treatment of dislocations, f may be written as the sum of two terms. The first represents the force associated with the same dislocation array in an unloaded solid ($\underline{S} = \underline{0}$); it depends on Peach-Koehler like contributions from the internal stresses due to other segments of dislocation line within the sample and also, at points where a dislocation line has curvature, on the core size. The second term is linear in the applied stress $\underline{S}$ and given by a Peach-Koehler term based on that part $\hat{\underline{\sigma}}$ of the local stress field due to application of $\underline{S}$ at a fixed dislocation configuration.

In plastic constitutive studies one often averages out discrete dislocations, representing their effects as amounts of shear γ^λ on each of the possible slip systems $\lambda = 1,2,3,4,...$ within the crystalline element at a local point of, say, a polycrystalline aggregate. In that case

$$<F\delta\xi> = \frac{1}{V} \int_V \left(\sum_\lambda \tau^\lambda \delta\gamma^\lambda \right) dV \quad , \tag{59}$$

with the summation extending over all slip systems at each point. In that case the thermodynamic "forces" τ^λ are, when the elasticity is treated as linear, the sum of those "forces" in the plastically sheared but unloaded solid plus the shear stress $\hat{\tau}^\lambda$ resolved onto slip system λ due to the local stress $\hat{\underline{\sigma}}$ created by application of $\underline{S}$ at fixed plastic shears (33,34). The τ's have also been specified precisely for finite elastic distortions (32,35). Other representations of structural rearrangement can be considered (33). For example, in deformation due to advance of a phase interface, Eshelby's (20) representation of force in terms of a jump in the integrand for $\underline{J}$ at the interface effectively defines F.

Now, any infinitesimal change $d\underline{E}$ in $\underline{E}$ may be regarded as the sum of that due to a change $d\underline{S}$ in $\underline{S}$ at fixed structural arrangement plus that due to a change $d\xi$ in structural arrangement at fixed $\underline{S}$. We may define the latter increment as the plastic part $d^p\underline{E}$ of a general deformation increment. It represents the residual strain in an infinitesimal load-unload cycle, providing that the unloading is elastic (here taken to mean at fixed structural arrangement). There follows then the reciprocal relation

$$d^pE_{\alpha\beta} = \left\langle \frac{\partial F}{\partial S_{\alpha\beta}} \, d\xi \right\rangle \tag{60}$$

where the derivatives of energetic forces with respect to macroscopic stress are taken at fixed structural arrangement. This represents an extension of Maxwell reciprocity to cases where some of the variables $d\xi$ have only a significance as incremental quantities, there being no definable variable ξ of which $d\xi$ may be said to represent an infinitesimal increase; see (32,33,36) for derivation.

Eqn (60) provides a formalism for relating structural rearrangement to macroscopic strain. In general the summing over sites and volume averaging implied cannot be carried out exactly but there is a literature on approximate and numerical ways of achieving this in the case of crystalline slip; see (33) for a summary.

In the specific models for plastic flow noted earlier, we evidently get $d^pE_{\alpha\beta}$ by replacing f by $\partial f/\partial S_{\alpha\beta}$ and τ^λ by $\partial\tau^\lambda/\partial S_{\alpha\beta}$ on the right sides of Eqns (58) and (59), the derivatives on $\underline{S}$ being taken at fixed dislocation or plastic shear configuration and hence involving only that part of f or τ^λ traceable to the Peach-Koehler force or resolved shear stress, respectively, due to the local stress field $\hat{\underline{\sigma}}$ induced by $\underline{S}$. When the material sample is elastically homogeneous, e.g., a single crystal, $\hat{\underline{\sigma}} = \underline{S}$ and the expressions for plastic strain increment reduce to those widely quoted in the literature but, as is less widely realized, are not precisely true for an elastically heterogeneous solid such as a polycrystal (34).

Eqn (60) also leads to an interesting attribute of macroscopic constitutive relations in cases for which structural rearrangements can be characterized locally by scalar variables (like increments of shear on active crystallographic slip systems) whose rate of advance $d\xi/dt \equiv r$ is stress state dependent only via a dependence of each $d\xi/dt$ on the conjugate F (e.g., shear rate related to the resolved shear stress on a slip system). Such encompasses the accepted Schmid resolved shear stress phenomenology for crystal plasticity (32,33,34,35,36). In all such cases a macroscopic flow potential is readily shown to exist. It is defined by

$$\Omega = \left\langle \int^F r(F)\, dF \right\rangle , \tag{61}$$

and the plastic part of the instantaneous macroscopic strain rate is related to it by

$$d^pE_{\alpha\beta}/dt = \partial\Omega(\underline{S}, \text{structural arrangement})/\partial S_{\alpha\beta} . \tag{62}$$

For a more extended discussion the reader is referred to (32,34,37).

3.5 CRACKS AND COMPLIANCE

Consider now a tensile loaded non-linear elastic solid containing a 3D planar crack so that only mode I is involved. Let Q and

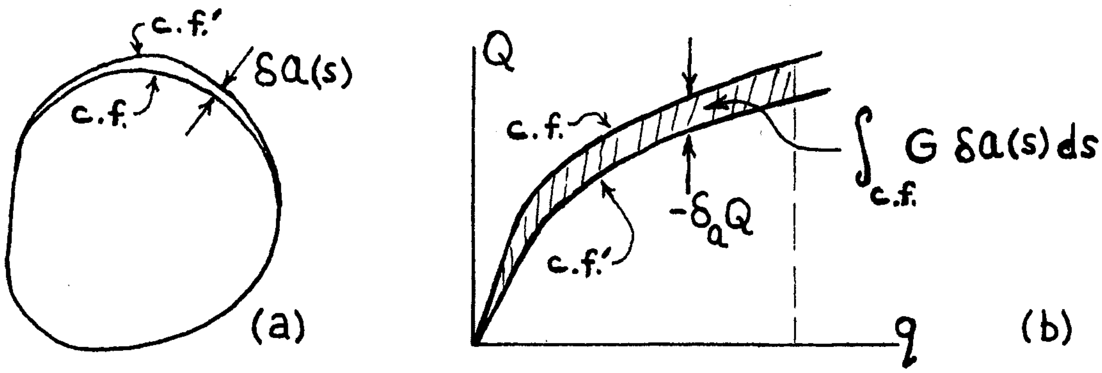

FIG. 5. (a) Planar tensile crack; c.f. =initial position of crack front, c.f.' =position after crack advance by $\delta a(s)$, s is arc length along c.f. (b) Alteration of force (Q) versus displacement (q) diagram.

q be conjugate force and displacement. Then, if "c.f." denotes crack front, Fig. 5a, ds arc length along it, and $\delta a(s)$ the advance of the crack normal to itself (all measured relative to the undeformed state of the cracked body), then G may be defined locally along the crack front at any location s by requiring that

$$\delta U = Q\delta q - \int_{c.f.} G\,\delta a\,ds \tag{63}$$

be valid to first order in $\delta a(s)$ for arbitrary distributions of crack advance along c.f. It then follows by similar reciprocity arguments as those leading to Eqn (60) that the change $\delta_a Q$ in Q when crack advance δa occurs at fixed q is

$$\delta_a Q = -\int_{c.f.} \frac{\partial G(q,s;c.f.)}{\partial q}\,\delta a(s)\,ds \quad . \tag{64}$$

Here, to confirm the notation, if c.f.' denotes the alteration of c.f. after crack advance by $\delta a(s)$ along c.f., then

$$\delta_a Q = Q(q,c.f.') - Q(q,c.f.) \quad . \tag{65}$$

Thus if we determine complete load versus displacement relations for two crack front positions c.f.' and c.f., $\delta_a Q$ is determined as a function of Q and

$$\int_{c.f.} G\,\delta a\,ds = -\int_0^q \delta_a Q\,dq = \int_0^Q \delta_a q\,dQ \tag{66}$$

where

$$\delta_a q = q(Q,c.f.') - q(Q,c.f.) \quad . \tag{67}$$

The integration region is shown in Fig. 5b.

These expressions summarize the Irwin relations between compliance changes and G. They are widely used in elastic (possibly non-linear) fracture mechanics where they are perhaps more familiar in 2D form (with $Q\delta q$ redefined as work per unit thickness) as

$$G = -\int_0^q \frac{\partial Q(q,a)}{\partial a} dq = \int_0^Q \frac{\partial q(Q,a)}{\partial a} dQ \quad . \tag{68}$$

The same relations between compliance changes with crack size and G are also widely used in a version of elastic-plastic fracture mechanics based on plastic "deformation" theory (38,39,40,41). There G is understood not to be interpretable in terms of energy changes in crack growth under applied load, but rather only as the crack tip J integral for a loaded but nongrowing crack, and it is understood that all force versus displacement relations used in Eqns (66,68) are to be generated by loading from an unstressed state with the considered crack configuration. Eshelby (20) put it nicely when he suggested that all works well in this procedure "... if we do not call the material's bluff by unloading...", although perhaps it is more a matter of the material calling our bluff!

In any event the approach to elastic-plastic fracture as outlined enables J to be inferred from load versus displacement data as obtained in the laboratory or estimated from simple calculations (40,41). The integral itself, no longer valid as an energy release rate, is instead interpreted as a parameter characterizing the intensity of the near tip deformation field (10); this is sensible since the path Γ for evaluation of J can be taken arbitrarily close to the tip. Thus, as long as there is a one-parameter form for the very near tip deformation field at a tensile crack tip (e.g., the "HRR" field (11,12)), we can use J as the parameter, and as long as that field dominates over size scales large enough to envelop that of fracture micro-mechanisms at the tip, we can use J in the "resistance curve" sense to correlate the early increments of ductile crack advance and sometimes to predict crack instability (41,42,43). There is now a substantial literature on this topic and also its extension to viscoplastic crack analysis, contained in large part in the Special Technical Publication series of ASTM over the last decade, and it is not further summarized here.

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