# ON THE THEORY OF PERFECTLY PLASTIC ANTI-PLANE STRAINING

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A general formulation of two-dimensional elastic-perfectly-plastic anti-plane straining is presented for materials with arbitrary anisotropic convex yield surfaces. Stress and strain distributions in plastic regions adjoining portions of the boundary are obtained directly in terms of the yield surface geometry. When specialized to the classical torsion problem, results lead directly to a generalization of the well-known plastic roof construction for limit loads. Examples of the determination of fully plastic stress distributions and corresponding limit torques are given for circular and rectangular shafts with various yield conditions. Another specialization is made to the contained plastic deformation created by longitudinal shearing of a body containing a sharp edge notch. Here the determination of the elastic-plastic boundary and strain distribution is reduced to a potential theory problem for a region in the stress plane bounded by straight line segments and a portion of the yield surface, and a membrane analogy is presented which allows effective visualization of the solution. A solution valid for small scale yielding near a crack is given in terms of a conformal transformation of the yield surface to a unit circle, and some specific examples are worked. Particular attention is given to single crystal type yield surfaces made up of straight line segments corresponding to discrete slip planes.

# 1. Introduction

Anti-plane strain is a possible 3D deformation mode for certain anisotropic prismatic bodies whose directionality of properties does not preclude states for which the only nonvanishing stresses and strains are the two shears associated with the face of a cross-section of the prism. These stresses and strains then vary only with position in the cross-section and not with distance in the length direction. Specifically, referring to Fig. 1 where a cross-section of the prism is shown and an (x, y) coordinate system attached, the



Fig. 1. Cross-section of infinite prismatic body; later used with reference to plastic region stress distribution.

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nonvanishing stresses are  $\tau_{xz} = \tau_{xz}(x, y)$  and  $\tau_{yz} = \tau_{yz}(x, y)$ . These may be viewed as components of a shear stress vector  $\boldsymbol{\tau}$ , with cartesian representation

$$\boldsymbol{\tau} = \tau_{xz} \boldsymbol{i}_x + \tau_{yz} \boldsymbol{i}_y. \tag{1}$$

Here  $(i_x, i_y)$  are unit vectors in the coordinate axes directions. The only stress equilibrium equation not identically satisfied by vanishing stresses is

$$\nabla \cdot \boldsymbol{\tau} = \boldsymbol{0},\tag{2}$$

where  $\nabla$  is the gradient operator and in cartesian form

$$\boldsymbol{\nabla} = \boldsymbol{i}_{x}\partial/\partial x + \boldsymbol{i}_{y}\partial/\partial y. \tag{3}$$

Deformations producing anti-plane straining are created by applying end couples to the prism, as in the classical torsion problem, resulting in an angle of twist per unit length,  $\mu$ , and/or by applying displacements or surface tractions along the surface of the prism in the direction z perpendicular to the cross-section. A sufficiently general form to assume for displacements under these conditions is

$$u = \mu zy, \quad v = -\mu zx, \quad w = w(x, y),$$
 (4)

where (u, v, w) pertain to the (x, y, z) directions respectively and the angle of twist per unit length,  $\mu$ , is taken positive for clockwise rotations of the prism cross-section shown in Fig. 1. Defining strains by

$$\gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$
 and  $\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$ ,

the strain vector  $\gamma$  is given by

$$\boldsymbol{\gamma} = \boldsymbol{\gamma}_{xz} \boldsymbol{i}_{x} + \boldsymbol{\gamma}_{yz} \boldsymbol{i}_{y} = \boldsymbol{\nabla} \boldsymbol{w} - \boldsymbol{\mu} \boldsymbol{k} \times \boldsymbol{r}, \tag{5}$$

where k is a unit vector perpendicular to the (x, y) cross-section plane, and

$$\boldsymbol{r} = \boldsymbol{x}\boldsymbol{i}_{\boldsymbol{x}} + \boldsymbol{y}\boldsymbol{i}_{\boldsymbol{y}} \tag{6}$$

is the position vector of a point in the cross-section. A strain compatibility equation results from noting that  $\nabla w$  is irrotational so that  $\nabla \cdot (\mathbf{k} \times \nabla w) = 0$  and thus

$$\nabla \cdot (\boldsymbol{k} \times \boldsymbol{\gamma}) = 2\mu. \tag{7}$$

Perfectly plastic stress-strain relations follow from splitting strains into a recoverable elastic part,  $\gamma^{e}$ , and a permanent plastic part,  $\gamma^{p}$ :

$$\mathbf{\gamma} = \mathbf{\gamma}^{\mathbf{e}} + \mathbf{\gamma}^{\mathbf{p}}.\tag{8}$$

The elastic strain is the usual Hookian function of stress, and to a degree of anisotropy for which anti-plane stresses create only anti-plane strains, the general relation is

$$\mathbf{\gamma}^{e} = \boldsymbol{\Gamma}^{*} \cdot \boldsymbol{\tau}, \tag{9}$$

where  $\Gamma^*$  is a two-by-two constant dyadic of elastic compliances. It has the representation

$$\Gamma^* = \Gamma_x i_x + \Gamma_y i_y, \tag{10}$$

where  $\Gamma_x$  and  $\Gamma_y$  are the elastic shear strain vectors created by a unit shear stress in the x and y directions, respectively. For an elastically isotropic material,  $\Gamma^* = I^*/G$ , where  $I^*$  is the unit dyadic and G the shear

modulus, (9) then becoming

$$\mathbf{\gamma}^{\mathbf{c}} = \mathbf{\tau} / G. \tag{11}$$

Perfectly plastic behavior according to accepted precepts requires the introduction of a convex yield surface in the two-dimensional  $(\tau_{xz}, \tau_{yz})$  stress plane such that the tip of the stress vector never penetrates this surface; a typical yield surface is shown in Fig. 2 where vectors **n** and **s** are defined at every point of the surface as the outward unit normal and the clockwise unit tangential vectors respectively. The same vectors may be represented in the (x, y) plane such that components in the  $\tau_{\alpha z}$  direction become components in the  $\alpha$  direction, where  $\alpha = x$  or y. When at a given point in the material the tip of the stress vector is on the yield surface, plastic strain increments may be nonzero and have the direction of the outward normal so that

$$\mathrm{d}\boldsymbol{\gamma}^{\mathbf{p}} = \mathrm{d}\boldsymbol{\Lambda}\boldsymbol{n},\tag{12}$$

where  $d\Lambda$  is a nonnegative scalar function of position which is determined from overall strain compatibility requirements. This is modified to the requirement that the plastic strain increment vector lie between outward normals at corners of the yield surface.

# 2. Stress distribution in plastic region

Consider a plastic region of the prism cross-section which is bounded by an elastic-plastic boundary, as shown by the dashed line in Fig. 1, and a portion of the prism boundary. Since the stress vector is on the yield surface at every point of the plastic region, the normal and tangential unit vectors of Fig. 2, when redrawn in the (x, y) plane, define perpendicular directions at each point of the plastic region of Fig. 1. Taking these as base vectors, the gradient operator is

$$\nabla = s\partial/\partial s + n\partial/\partial n, \tag{13}$$

where  $\partial/\partial s$  and  $\partial/\partial n$  are differential operators giving the rate of change of a quantity with respect to changes in physical coordinates in the s and n directions, respectively. The equilibrium equation (2) then



Fig. 2. Perfectly plastic yield surface.

becomes

$$\mathbf{s} \cdot \partial \boldsymbol{\tau} / \partial \boldsymbol{s} + \boldsymbol{n} \cdot \partial \boldsymbol{\tau} / \partial \boldsymbol{n} = 0. \tag{14}$$

But in the plastic region the stress vector  $\tau$  must everywhere be on the yield surface, so that any vector change,  $d\tau$ , of the stress vector, in going from point to point of the plastic region, must have a direction which is tangential to the yield surface. Thus  $n \cdot d\tau = 0$ , and consequently

$$\mathbf{n} \cdot \partial \tau / \partial \mathbf{n} = 0, \qquad \mathbf{n} \cdot \partial \tau / \partial s = 0, \tag{15}$$

a mathematical statement of the yield condition. From the first of these and equilibrium equation (14), we have

$$\mathbf{s} \cdot \partial \mathbf{\tau} / \partial s = 0. \tag{16}$$

Combining (16) with the last of (15), it is seen that the projection of  $\partial \tau / \partial s$  on the two mutually perpendicular directions, *n* and *s*, is zero, so that

$$\partial \tau / \partial s = 0. \tag{17}$$

Thus the stress vector is constant on s-lines (an s-line being defined as a line in the (x, y) plane having a tangent in the direction of s, the yield surface unit tangent vector, at each point of the line). Since s is fixed in direction when  $\tau$  is constant at a smooth point of the yield surface, the s-lines are straight. When  $\tau$  is at a vertex, s is not uniquely defined but there is seen to be no loss in generality if we choose s within the range of admissable tangent vectors at a vertex such that we define to be an s line is a straight line.

The construction of the stress distribution is illustrated in Figs. 1 and 2 when the portion of prismatic boundary which bounds the plastic region is traction free. At a given point P on the boundary, as in Fig. 1, the stress vector must then be tangential to the boundary. One goes to the yield surface of Fig. 2 and draws a line parallel to the tangent at point P of Fig. 1; this tangent line intersects the yield surface at two points, and one of these points (which may generally be chosen from a priori considerations, as for example, in the torsion problem by noting the direction of the angle of twist) corresponds to the stress,  $\tau_p$ , at point P on the boundary. Once  $\tau_p$  is known, the unit vectors n and s are determined at point P, and since  $\tau$  is constant on s-lines (which are, in turn, straight), the stress solution is  $\tau = \tau_p$  along the straight line emanating from point P and making an angle  $\theta$  with the x-axis which is equal to the angle  $\theta$  made, in Fig. 2, by the yield surface at all points of the plastic region. This procedure is recognized as a generalization to arbitrary convex yield surfaces of techniques recorded, e.g., by Prager and Hodge (1951) for the case of a circular yield surface corresponding to the Tresca or Mises criteria.

A straightforward extension now allows treatment of the case where the surface traction is non-zero at point P. Suppose the shear stress on the prismatic surface is  $\tau^*$  at P; this means that  $\tau_p$  must have a component in the direction perpendicular to the boundary equal to  $\tau^*$ . The construction for locating the boundary stress  $\tau_p$  and angle  $\theta$  for the s-line along which  $\tau = \tau_p$  is shown in Fig. 3(a) and (b). Here the line parallel to the boundary tangent at point P is drawn not through the stress plane origin as in Fig. 2, but rather through the tip of a stress vector equal in magnitude to  $\tau^*$  and with the direction of the perpendicular to the boundary at P, and  $\tau_p$  is the vector sum of the prescribed  $\tau^*$  in the perpendicular direction and the component  $\tau^{**}$  in the tangential direction, the latter being found from the point of intersection of the tangent line and yield surface.

For later work it is convenient to prepresent the stress vector,  $\tau$ , at a particular material point in terms of components  $\tau_s$  and  $\tau_n$  in the directions of the s-line at that point and perpendicular to the s-line, respectively, so that

$$\tau_s = \mathbf{s} \cdot \boldsymbol{\tau}, \qquad \tau_n = \mathbf{n} \cdot \boldsymbol{\tau}. \tag{10}$$

(10)

In terms of the yield surface of Fig. 2, these are components in the directions locally tangential and normal to the surface, respectively, at the point where  $\tau$  contacts that surface.

#### 3. Warping displacement and strain distribution

When a surface traction,  $\tau^*$ , is prescribed at points of the portion of prismatic boundary bounding the plastic region, the direction of the stress vector at a given material point is not, in general, fixed in space. However, for a traction free boundary the stress vector at points of the plastic region will generally be fixed in space, as is clear from the discussion relating to Figs. 1 and 2, and thus proportional plastic flow ensues. Warping displacement and strain distributions are derived here for this latter case. We assume a smooth yield surface with a continuously turning tangent, and deal subsequently with the limit cases for which the surface has vertices and/or straight line segments.

As the yield surface normal, n, is constant at a given material point, (12) may be integrated to

$$\boldsymbol{\gamma}^{\mathbf{p}} = \boldsymbol{\Lambda} \boldsymbol{n}, \tag{19}$$

and the plastic strain is entirely in a direction perpendicular to the straight s-lines. Strain equations (5), (8), (9) and (19) then lead to

$$\gamma = \nabla w - \mu k \times r = \Gamma^* \cdot \tau + \Lambda n. \tag{20}$$

Substituting (20) into the strain compatibility equation (7), recognizing that  $k \times n = -s$ , employing the form of (13) for the gradient operator and the result  $\partial \tau / \partial s = 0$  from (17), and noting that straightness of the *s*-lines ensures that the unit vectors have zero derivatives with respect to *s*, the equation for the plastic strain magnitude  $\Lambda$  is

$$\frac{\partial \Lambda}{\partial s} + \Lambda \mathbf{n} \cdot \frac{\partial s}{\partial n} = -\mu + \mathbf{n} \cdot \left( \mathbf{k} \times \Gamma^* \cdot \frac{\partial \tau}{\partial n} \right) = -2\mu + s \cdot \Gamma^* \cdot \frac{\partial \tau}{\partial n}, \qquad (21)$$

where an identity involving the triple scalar product has been used in deriving the latter form. In the (s, n) coordinate system the elastic compliance dyadic is

$$\Gamma^* = \Gamma_s s + \Gamma_n n. \tag{22}$$



Fig. 3. Construction of stress solution when a surface traction  $\tau^*$  is specified at point P.

As *n* dotted with the derivatives of  $\tau$  is zero from the yield condition (15),

$$\Gamma^* \cdot \partial \tau / \partial n = \Gamma_s(s \cdot \partial \tau / \partial n).$$

Letting

$$\boldsymbol{\Gamma}_{s} = \boldsymbol{\Gamma}_{ss}\boldsymbol{s} + \boldsymbol{\Gamma}_{sn}\boldsymbol{n}, \tag{23}$$

where  $\Gamma_{ss}$  and  $\Gamma_{sn}$  are elastic strains in the s and n directions due to a unit stress in the s direction, the differential equation (21) for  $\Lambda$  becomes

$$\frac{\partial \Lambda}{\partial s} + \Lambda \mathbf{n} \cdot \frac{\partial s}{\partial n} = -2\mu + \Gamma_{ss} \left( \mathbf{s} \cdot \frac{\partial \tau}{\partial n} \right). \tag{24}$$

To establish the integral of (24), consider two s-lines infinitesimally close to one another and separated by an angle  $d\theta$ , as in Fig. 4, let I be their point of intersection, let  $\rho$  be the measure of distance from point I to some point Q on one of the s-lines, and let  $\rho_B$  be the distance from I to point B where the s-line intersects the elastic-plastic boundary. Then

$$\boldsymbol{n} \cdot \frac{\partial \boldsymbol{s}}{\partial \boldsymbol{n}} = \boldsymbol{n} \cdot \left(\frac{1}{\rho} \frac{\mathrm{d}\boldsymbol{s}}{\mathrm{d}\boldsymbol{\theta}}\right) = \boldsymbol{n} \cdot \left(\frac{1}{\rho} \boldsymbol{n}\right) = \frac{1}{\rho}.$$
(25)

Now, as in Fig. 2, let *l* be the measure of arc length on the yield surface, increasing in the contra-clockwise sense of traversal and starting from any convenient reference point. Then *l* has dimensions of stress, the parameters s, n,  $\tau$ ,  $\Gamma_{ss}$ , and the yield surface tangent angle  $\theta$  are all unique functions of *l*, and to each value of *l* there corresponds one *s*-line. Thus,

$$\mathbf{s} \cdot \frac{\partial \mathbf{\tau}}{\partial n} = \mathbf{s} \cdot \left(\frac{1}{\rho} \frac{\mathrm{d}\mathbf{\tau}}{\mathrm{d}\theta}\right) = \mathbf{s} \cdot \left(\frac{1}{\rho \theta'(l)} \frac{\mathrm{d}\mathbf{\tau}}{\mathrm{d}l}\right) = -\frac{1}{\rho \theta'(l)},\tag{26}$$



Fig. 4. For plastic region strain and displacement distribution.

since  $d\tau/dl = -s$  (here  $\theta'(l) = d\theta/dl$  is the curvature of the yield surface). Substituting (25), (26) into (24) and noting that  $\partial/\partial s = \partial/\partial \rho$ , we have

$$\frac{\partial \Lambda}{\partial \rho} + \frac{1}{\rho} \Lambda = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho \Lambda) = -2\mu - \frac{\Gamma_{ss}(l)}{\rho \theta'(l)}.$$
(27)

As  $\Gamma_{ss}$  and  $\theta'(l)$  are constant on an s-line, this may be integrated in  $\rho$ , and imposing the condition that  $\Lambda = 0$  when  $\rho = \rho_B$  since the plastic strain vanishes at the elastic-plastic boundary,  $\Lambda$  is determined as

$$\Lambda = \left(\rho_B - \rho\right) \left[ \frac{\Gamma_{ss}(l)}{\rho \theta'(l)} + \mu \left( \frac{\rho_B}{\rho} + 1 \right) \right].$$
(28)

It is noted that in passing from one s-line to another, not only does l vary but also the point from which  $\rho$  is measured.

Let the position of the elastic-plastic boundary be given by

$$\boldsymbol{r} = \boldsymbol{R}(1), \tag{29}$$

as in Fig. 4. Then if  $d\mathbf{R} = \mathbf{R}'(l) dl$  is the infinitesimal change in position vector of the elastic-plastic boundary in passing from one s-line to its neighbor,

$$\rho_B \,\mathrm{d}\theta = \mathbf{n} \cdot \mathrm{d}\mathbf{R} \qquad \rho_B = \mathbf{n} \cdot \mathbf{R}'(l)/\theta'(l). \tag{30}$$

Letting

$$s_{QB} = \rho_B - \rho \tag{31}$$

be the distance along the s-line from any point Q to the elastic-plastic boundary, as in Fig. 4, (28) may be written as

$$\Lambda = s_{QB} \left\{ \frac{\Gamma_{ss}(l) + \mu \left[ 2\mathbf{n} \cdot \mathbf{R}'(l) - s_{QB} \theta'(l) \right]}{\mathbf{n} \cdot \mathbf{R}'(l) - s_{QB} \theta'(l)} \right\}.$$
(32)

This form is now sufficient to deal, as a limit, with cases for which  $\theta'(l) = 0$ . Such corresponds to a yield surface containing a straight line segment ( $\theta = \text{constant}$ ), and then

$$\Lambda = s_{QB} \left\{ \Gamma_{ss}(l) / (\boldsymbol{n} \cdot \boldsymbol{R}'(l)) + 2\mu \right\}$$
(33)

in portions of the plastic region corresponding to straight line segments of the yield surface. Component forms of the strain vector are, from (20), (22) and (23)

$$\gamma_s = \Gamma_{ss}\tau_s + \Gamma_{ns}\tau_n, \qquad \gamma_n = \Gamma_{sn}\tau_s + \Gamma_{nn}\tau_n + \Lambda, \qquad (34)$$

where  $\Gamma_{ij}$  is the elastic shear strain in the *j*-direction due to a unit shear stress in the *i*-direction. For an elastically isotropic material,

$$\Gamma_{ss} = \Gamma_{nn} = 1/G, \qquad \Gamma_{ns} = \Gamma_{sn} = 0. \tag{35}$$

The warping displacement may be found from (20),

$$\nabla w = \Gamma^* \cdot \tau + \Lambda \mathbf{n} + \mu \mathbf{k} \times \mathbf{r},\tag{36}$$

as  $\Lambda$  has been chosen to satisfy compatibility. On the elastic-plastic boundry,  $\mathbf{r} = \mathbf{R}(l)$ ,  $\Lambda = 0$  and

$$\mathbf{d}w = \mathbf{d}\mathbf{R} \cdot \nabla w = \mathbf{d}\mathbf{R} \cdot \mathbf{\Gamma}^* \cdot \mathbf{\tau} + \mu \, \mathbf{d}\mathbf{R} \cdot (\mathbf{k} \times \mathbf{R}) = \mathbf{d}(\mathbf{R} \cdot \mathbf{\Gamma}^* \cdot \mathbf{\tau}) - \mathbf{R} \cdot \mathbf{\Gamma}^* \cdot \mathbf{d}\mathbf{\tau} + \mu \mathbf{k} \cdot (\mathbf{R} \times \mathbf{d}\mathbf{R}).$$
(37)

As  $\mathbf{n} \cdot d\mathbf{\tau} = 0$  from the yield condition, with the  $(s, \mathbf{n})$  coordinate system representation of  $\Gamma^*$ , (22),  $\Gamma^* \cdot d\mathbf{\tau} = \Gamma_s(s \cdot d\mathbf{\tau}) = -\Gamma_s dl$ . Then arbitrarily setting the displacement equal to  $w_0$  at point O on the boundary, (37) integrates to

$$w_{B} = w_{O} + \boldsymbol{R}_{B} \cdot \boldsymbol{\Gamma}^{*} \cdot \boldsymbol{\tau}_{B} - \boldsymbol{R}_{O} \cdot \boldsymbol{\Gamma}^{*} \cdot \boldsymbol{\tau}_{O} + \int_{l_{O}}^{l_{B}} \left\{ \left[ \boldsymbol{R}(l) \cdot \boldsymbol{\Gamma}_{s}(l) \right] + \mu \boldsymbol{k} \cdot \left[ \boldsymbol{R}(l) \times \boldsymbol{R}'(l) \right] \right\} \, \mathrm{d}l \tag{38}$$

for any point *B* on the elastic-plastic boundary, where  $l_0$  and  $l_B$  are the values of the yield surface arc length corresponding to *s*-lines through *O* and *B*. Here  $\mathbf{R} \cdot \mathbf{\Gamma}^* \cdot \boldsymbol{\tau} = \sum \sum R_j \Gamma_{ij} \tau_i = \mathbf{R} \cdot \boldsymbol{\tau}/G$  for an isotropic material. The displacement at any point *Q* inside the plastic region may be found from (36):

$$\frac{\partial w}{\partial s} = s \cdot \Gamma^* \cdot \tau + \mu s \cdot (\mathbf{k} \times \mathbf{r}) = \Gamma_{ss} \tau_s + \Gamma_{ns} \tau_n - \mu (\mathbf{n} \cdot \mathbf{r}).$$
(39)

Since  $(n \cdot r)$  is the perpendicular distance between the origin of coordinates and the s-line passing through the point with position vector r, as in Fig. 4, it is constant on an s-line as are the other terms on the right of (39). Thus w varies linearly with distance along an s-line, and integrating (39) from Q to the point B where the s-line through Q intersects the elastic-plastic boundary,

$$w_Q = w_B - s_{QB} \left[ \Gamma_{ss} \tau_s + \Gamma_{ns} \tau_n - \mu \left( \mathbf{n} \cdot \mathbf{r}_Q \right) \right], \tag{40}$$

with  $w_B$  found from (38). Note that (38) and (40) contain terms which remain well-defined when the yield surface contains straight line segments. Vertices may be treated as a limiting case for which  $\theta$  changes rapidly over a small range of l.

# 4. Longitudinal shearing of a sharp edge notch

Here a particular elastic-plastic boundary value problem is formulated in a general manner appropriate for arbitrary convex yield surfaces. Fig. 5 pictures a half space with a sharp edge notch, of depth a and notch angle  $2\alpha$ , perturbing a remotely applied uniform stress field  $\tau_{yz} = \tau$ ,  $\tau_{xz} = 0$ . This form of loading creates an anti-plane deformation known as longitudinal shear, with displacements w = w(x, y) perpendicular to the (x, y) plane; it is a special case of the general anti-plane deformation considered earlier when the angle of twist,  $\mu$ , is set equal to zero so that u = v = 0. If the notch tip was smooth instead of sharp, a particular s-line could be associated with each point of the tip boundary. As the tip is sharp, it is appropriate to view the plastic region stress distribution as a set of straight s-lines all emanating from the notch tip, with constant stress vectors corresponding to points on the yield surface of Fig. 2 so that a given s-line making an angle  $\theta$  with the x-axis carries a stress vector corresponding to a point on the yield surface where the unit tangent vector, s, makes the same angle  $\theta$  with the  $\tau_{xz}$ -axis. When the yield surface contains a straight line segment, the corresponding portion of the plastic region will be seen to collapse into a discrete line of slip displacement discontinuity along the associated s line, rather than being represented by the continuous fan of s-lines pictured in Fig. 5. Such is consistent with the fact that the fan construction just outlined gives a discontinuous  $\tau_s$  at the value of  $\theta$  corresponding to a flat segment.

Introducing polar coordinates  $(r, \theta)$  at the notch tip, the yield surface unit vectors (s, n), when drawn in the physical (x, y) plane, form unit vectors in the  $(r, \theta)$  directions, respectively, and  $\tau_s = \tau_{rz}$ ,  $\tau_n = \tau_{\theta z}$ ,  $\gamma_s = \gamma_{rz}$ ,  $\gamma_n = \gamma_{\theta z}$ . The position vector of the elastic-plastic boundary is

$$\boldsymbol{r} = \boldsymbol{R}(l) = \boldsymbol{R}(l)\boldsymbol{s},\tag{41}$$

with the polar coordinate r = R(l) at the point of boundary corresponding to arc length l on the yield

surface. Eq. (38) for displacements on the boundary becomes

$$w_{B} = w_{O} + R_{B} \left( s_{B} \cdot \boldsymbol{\Gamma}^{*} \cdot \boldsymbol{\tau}_{B} \right) - R_{O} \left( s_{O} \cdot \boldsymbol{\Gamma}^{*} \cdot \boldsymbol{\tau}_{0} \right) + \int_{l_{O}}^{l_{B}} R(l) \Gamma_{ss}(l) \, \mathrm{d}l, \tag{42}$$

with  $s \cdot \Gamma^* \cdot \tau = s \cdot \tau/G = \tau_s/G = \tau_{rz}/G$  and  $\Gamma_{ss} = 1/G$  for an elastically isotropic material, and (38) for displacements inside the plastic region becomes.

$$w_Q = w_B - (R_B - r_Q) [\Gamma_{ss} \tau_{rz} + \Gamma_{ns} \tau_{\theta z}]$$
(43)

since  $s_{QB} = (R_B - r_Q)$ . Strain components are given from (34) and (28) as

$$\gamma_{rz} = \Gamma_{ss}\tau_{rz} + \Gamma_{ns}\tau_{\theta z}, \qquad \gamma_{\theta z} = \Gamma_{sn}\tau_{rz} + \Gamma_{nn}\tau_{\theta z} + \left(\frac{R_{B}}{r} - 1\right)\frac{\Gamma_{ss}}{\theta'(l)}.$$
(44)

The latter expression loses meaning for straight line segments on yield surfaces, but already portends the discrete slip lines of that case announced earlier. In (44), r and  $R_B$  are measured along the same *s*-lines as in Fig. 5.

The direction of the stress vector at typical points of the physical plane is shown by arrows drawn in Fig. 5. As in previous work based on 'hodograph' style transformations in anti-plane strain (McClintock and Hult, 1956; Koskinen, 1963; Neuber, 1961; Rice, 1966, 1967a), the elastic region may be seen to map into a region of the stress plane of Fig. 6 bounded by a portion of the yield surface and straight line segments. Corresponding points are labelled with the same capital leters. Along CD and FE of Fig. 5 the stress



Fig. 5. Edge notch loaded in shear; direction of stress vector shown at typical points.

vectors make the constant angles  $\pi - \alpha$  and  $\alpha$  with the horizontal; the corners at C and F force the stress vector to vanish there and yield occurs at points D and E. Thus points C and F map into the origin of the stress plane of Fig. 6, D and E are on the yield surface, and CD and FE make angles  $\pi - \alpha$  and  $\alpha$  with the  $\tau_{xz}$ -axis. The elastic-plastic boundary of Fig. 5 maps into the portion of yield surface between D and E in Fig. 6. The stress vector is vertical on CA and FH of Fig. 5, and the stress is  $\tau_{yz} = \tau$  at A and H as these points are at infinity in the physical plane. Thus CA and FH map into the portion of the  $\tau_{yz}$ -axis between 0 and  $\tau$  in Fig. 6, CA and FH being identified with left and right sides of the  $\tau_{yz}$ -axis as  $\tau_{xz}$  is negative near CA and positive near FH.

The procedure of solution involves solving for physical coordinates in the elastic region as functions of stresses:  $x = x(\tau_{xz}, \tau_{yz})$ ,  $y = y(\tau_{xz}, \tau_{yz})$ . Equilibrium (2) and compatibility (7) equations are, in cartesian form,

$$\partial \tau_{xz} / \partial x + \partial \tau_{yz} / \partial y = 0, \qquad \partial \gamma_{xz} / \partial y - \partial \gamma_{yz} / \partial x = 0.$$
 (45)

Now, by noting that

$$dx = \frac{\partial x}{\partial \tau_{xz}} \left( \frac{\partial \tau_{xz}}{\partial x} dx + \frac{\partial \tau_{xz}}{\partial y} dy \right) + \frac{\partial x}{\partial \tau_{yz}} \left( \frac{\partial \tau_{yz}}{\partial x} dx + \frac{\partial \tau_{yz}}{\partial y} dy \right),$$
(46)

with a similar equation for dy, and equating coefficients of dx and dy one has a system of four linear equations which, when solved, give partial derivatives of  $(\tau_{xz}, \tau_{yz})$  with respect to (x, y) in terms of partial derivatives of (x, y) with respect to  $(\tau_{xz}, \tau_{yz})$ . Solving these and substituting into the equilibrium equation, one finds as in the references cited that equilibrium is equivalent to

$$\partial x / \partial \tau_{xz} + \partial y / \partial \tau_{yz} = 0. \tag{47}$$

Similarly, working in terms of strain, compatibility is equivalent to

$$\partial x / \partial \gamma_{yz} - \partial y / \partial \gamma_{xz} = 0. \tag{48}$$

Equilibrium (47) is identically satisfied by writing the position vector  $\mathbf{r} = \mathbf{x}\mathbf{i}_x + y\mathbf{i}_y$  as

$$\mathbf{r} = -\mathbf{k} \times \nabla_{\tau} \phi \qquad x = +\partial \phi / \partial \tau_{yz} \qquad y = -\partial \phi / \partial \tau_{xz}, \tag{49}$$



Fig. 6. Stress plane map of elastic region of physical plane.

where  $\phi = \phi(\tau)$  is a scalar function of the stress components and  $\nabla_{\tau}$  is the gradient operator in the stress plane:

$$\nabla_{\tau} = i_x \partial / \partial \tau_{xz} + i_y \partial / \partial \tau_{yz}. \tag{50}$$

Let the inversion of the elastic stress strain relation  $\gamma = \Gamma^* \cdot \tau$  be  $\tau = T^* \cdot \gamma$ , or

$$\tau_{xz} = T_{xx}\gamma_{xz} + T_{yx}\gamma_{yz}, \qquad \tau_{yz} = T_{xy}\gamma_{xz} + T_{yy}\gamma_{yz}, \tag{51}$$

with elastic modulus  $T_{ij}$  being the shear stress in the *j*-direction due to a unit elastic shear strain in the *i*-direction. Then with the aid of (49,51), the compatibility equation (48) determines  $\phi$  as satisfying

$$T_{xx}\frac{\partial^2 \phi}{\partial \tau_{xz}^2} + \left(T_{xy} + T_{yx}\right)\frac{\partial^2 \phi}{\partial \tau_{xz}\partial \tau_{yz}} + T_{yy}\frac{\partial^2 \phi}{\partial \tau_{yz}^2} = 0.$$
(52)

For an elastically isotropic material,  $T_{xx} = T_{yy} = G$  and  $T_{xy} = T_{yx} = 0$  so that (52) for  $\phi$  becomes

$$\nabla_{\tau}^2 \phi = \partial^2 \phi / \partial \tau_{xz}^2 + \partial^2 \phi / \partial \tau_{yz}^2 = 0.$$
(53)

Eq. (52), or (53) for the isotropic case, must be satisfied in the region of Fig. 6 corresponding to the stress plane map of the elastic region of the physical plane of Fig. 5. The solution is completely determined once boundary conditions are prescribed. On the elastic-plastic boundary DE, as s-lines are radial, one has

$$0 = \mathbf{n} \cdot \mathbf{r} = -\mathbf{n} \cdot (\mathbf{k} \times \nabla_{\tau} \phi) = -\mathbf{s} \cdot \nabla_{\tau} \phi = + \frac{\partial \phi}{\partial l} \quad \text{on } DE,$$
(54)

where *l* is the measure of yield surface arc length. Along the notch surfaces, CD and FE, the stress vector is collinear with the position vector. Thus, extending the definition of unit vector (s, n) and arc length *l* to the inclined straight lines of Fig. 6, one again has

$$0 = \mathbf{n} \cdot \mathbf{r} = +\partial \phi / \partial l \quad \text{on } CD \text{ and } FE.$$
(55)

On lines CA and FH one has x = -a, and, from (49),

$$-a = \partial \phi / \partial \tau_{yz}$$
 on CA and FH. (56)

Now arbitrarily setting  $\phi = 0$  at some point on the elastic-plastic boundary and integrating the last three equations, one obtains for boundary conditions that  $\phi$  vanishes everywhere along the yield surface and inclined lines of Fig. 6, but along the  $\tau_{yz}$ -axis  $\phi$  varies linearly with stress at a slope of -a for  $0 < \tau_{yz} < \tau$ :

$$\phi = 0 \qquad \text{on } DE, CD \text{ and } FE, \phi = -a\tau_{yz} \qquad \text{on } CA \text{ and } FH \qquad (0 < \tau_{yz} < \tau).$$
(57)

The position of the elastic-plastic boundary r = R(l) is of primary interest. Once it is known the entire plastic region stress and displacement field is readily computed from earlier results. Since r = R(l)s on the boundary,  $R(l) = s \cdot r$  and from (49) the position of the elastic-plastic boundary is

$$R(l) = -s \cdot (\mathbf{k} \times \nabla_{\tau} \phi) = \mathbf{n} \cdot \nabla_{\tau} \phi, \qquad R(l) = \partial \phi / \partial n.$$
(58)

Thus the distance to the elastic-plastic boundary along an s-line corresponding to yield surface arc length l and angle  $\theta(l)$  is the normal derivative of the function  $\phi$  at the point on the yield surface of Fig. 6 having the appropriate arc length.

A membrane analogy may be established which permits a simple visualization of the solution. For the elastically isotropic case the Laplacian equation (53) is that for a stretched membrane under zero transverse

pressure if  $\phi$  is taken to be the membrane deflection and  $(\tau_{x_2}, \tau_{y_2})$  the physical coordinates of points in the membrane surface. The membrane is placed over a hole cut out of a flat sheet to the dimensions of the boundary of Fig. 6 as made up of a portion of the yield surface and the two inclined lines, so that the membrane deflection,  $\phi$ , vanishes on this boundary. The membrane is then loaded by laying a thin straight wire of length equal to the remotely applied stress,  $\tau$ , on the membrane along the  $\tau_{y_2}$ -axis from  $\tau_{y_2} = 0$  to  $\tau_{y_2} = \tau$  and giving the wire a slope equal to the notch depth, a, so as to simulate boundary conditions of Fig. 6. According to (49), physical coordinates in the elastic region are equal to slope components of the membrane, and in particular, by (58), the distance R(l) to the elastic-plastic boundary along any particular *s*-line is simply the membrane slope at the corresponding point on the yield surface. This analogy can be extended at once to the general elastically anisotropic case, since a linear transformation of coordinates in (52) may be found which reduces it to a Laplacian equation.

The elastic-plastic field corresponding to a yield surface with straight line segments may now be addressed; such a surface appears in Fig. 7(a) and the corresponding plastic region is shown qualitatively in Fig. 7(b). Along straight line segments the angle  $\theta$  between the yield surface unit tangent vector and  $\tau_{x}$ -axis is constant and thus portions of the elastic-plastic boundary of Fig. 7(b) corresponding to stresses on a given straight line segment of the yield surface form a straight line in the physical (x, y) plane having the same angle  $\theta$  with the x-axis. The distance R(1) from the notch tip of a particular point on the elastic-plastic boundary corresponding to a point on the yield surface of arc length / is again the normal derivative of  $\phi$  at that point. From the membrane analogy it is clear that the slope vanishes at sharp corners like A, B and D, and thus these map to x = y = 0 (the notch tip). At a slightly rounded corner such as C a continuous fan of s-lines is created and the yielding is smeared out as illustrated. From (43) it is clear that the displacement is discontinuous in crossing a straight line portion of the elastic-plastic boundary, and thus straight line segments on the yield surface cause discrete slip lines emanating from the notch tip. The maximum depth of penetration of these slip lines into the material is found by locating points such as K, L. M where the normal derivative  $\partial \phi / \partial n$  attains a local maximum on each of the yield surface straight line segments; the corresponding slip line lengths are equal to  $R(l) = \partial \phi / \partial n$  at these points. The results summarized here are seen to result alternatively as limits of the general solution outlined earlier when a yield surface with a continuously turning tangent degenerates into one with straight line segments and vertices.

For a notched single crystal (or at the single grain scale in a polycrystal), a straight line segment yield surface with vertices is appropriate, with the orientation and normal component of stress on each segment corresponding to the orientation of a particular slip plane and the resolved shear stress required for slip. If similar results accrue for Mode I tensile loadings as developed here for anti-plane shear loadings, then the observed inhomogeneous geometries of notch tip yield in crystals (e.g., Tetelman (1963)), sometimes



Fig. 7. Geometry of notch tip yielding for straight line segment yield surface.

involving bands of concentrated shear emanating from an arrested crack tip, may perhaps be regarded as being compatible with continuum plasticity theory when an appropriate single crystal type yield surface is employed in that theory. Also, it may be noted that for anti-plane shear of a single crystal, the same solution as obtained through methods presented here would result if continuous arrays of screw dislocations were placed in an elastic body along lines emanating from the notch tip in directions collinear with slip directions, with the dislocation density and slip line length chosen to give bounded stress at slip line tips and a shear stress on the slip line equal to the resolved stress required for slip. This, of course, simply defines the well-known Bilby–Cottrell–Swinden (1963) procedure, if applied to a sufficient variety of appropriately placed line plastic zones. In such a case the BCS procedure duplicates the exact continuum plasticity solution.

When a circular Tresca-Mises yield surface  $\tau_{xz}^2 + \tau_{yz}^2 = \tau_0^2$  is employed the present general formulation reduces to that developed by McClintock and Hult (1956), and extended by Koskinen (1963) and Rice (1966) to cases of edge notches in bodies of finite, rather than semi-infinite, width. The latter formulations are readily generalized to the case of arbitrary convex yield surfaces. Assuming the x-direction width to be b, the modification amounts simply to prescribing  $\phi$  along an additional portion  $\tau < \tau_{yz} < \tau'$  of the  $\tau_{yz}$ -axis of Fig. 6 as  $\phi = -a\tau + (b-a)(\tau_{yz} - \tau)$ ;  $\tau'$  is an unknown parameter found from the condition that the derivatives of  $\phi$  be bounded at  $\tau_{yz} = \tau'$ ,  $\tau_{xz} = 0$ , as in the solution for the isotropic case.

#### 5. Small scale yielding near a longitudinally sheared crack

A plastic region small compared to geometric dimensions (that is, notch depth) will result when the remotely applied stress,  $\tau$ , is far below the limit stress for the notched plane of Fig. 5; this state of deformation is called small scale yielding. formally, as  $\tau \to 0$  and notch depth  $a \to \infty$  in such a way that the plastically yielded region is of finite extent, the length,  $\tau$ , of the line  $0 < \tau_{yz} < \tau$ , along which  $\phi = -a\tau_{yz}$ , shrinks to zero but the slope, -a, approaches infinity. Thus one anticipates that for small scale yielding the appropriate stress plane map of the elastic region is that of Fig. 6, but with the line along which  $\phi$  is prescribed replaced by a singularity at the origin of the stress plane. Here, for simplicity, we deal with the case of a crack (notch angle  $\alpha = 0$ ) in an elastically isotropic material. A general small scale yielding solution, valid for an arbitrary convex yield surface, is then presented in terms of the conformal mapping of the interior of the upper portion ( $\tau_{yz} \ge 0$ ) of the yield surface, and its mirror image across the  $\tau_{xz}$ -axis, into a unit circle.

Isotropic elastic solutions for symmetrically loaded longitudinal shear cracks are well known to always result in stress distributions of the form

$$\tau_{yz} + i\tau_{xz} = \frac{K}{(2\pi)^{1/2} (x + iy)^{1/2}} + O(x + iy)^{1/2},$$
(59)

where the crack tip is at x = y = 0, the crack line is on the negative x-axis, the branch cut of  $(x + iy)^{1/2}$  is along the crack line, and  $O(x + iy)^{1/2}$  represent other nonsingular terms, all vanishing at the crack tip, which are required for a complete solution. Here K is the Mode III stress intensity factor. For the configuration of Fig. 5 with  $\alpha = 0$ , and also for a crack of length 2a in an infinite body under remote stress  $\tau$ ,

$$K = \tau \left( \pi a \right)^{1/2}. \tag{60}$$

When the yield zone is small compared to all geometric dimensions (more precisely, in the limit as the ratio of yield zone dimensions to planar geometric dimensions approaches zero), the elastic-plastic solution still perturbs the elastic solution (59) substantially in the plastic region. But at distances away from the crack tip which are large compared to yield zone dimensions, the elastic solution is little perturbed and the stresses are given by the singular term (59) in the elastic solution. Thus the plastic region is immersed in the surrounding elastic stress singularity, and the situation is indistinguishable from that of a semi-infinite crack on the negative x-axis with the asymptotic boundary conditions imposed that

$$\tau_{yz} + i\tau_{xz} \to \frac{K}{(2\pi)^{1/2}(x+iy)^{1/2}} \quad \text{as } |x+iy| \to \infty,$$
(61)

where K is chosen as appropriate for the particular geometry and method of loading. This is, in essence, a boundary layer approach employed for analogous problems by Rice (1966, 1967) and now widely used to model nonelastic crack tip phenomena. Recalling (49), the asymptotic boundary condition (61) is equivalent to requiring that

$$x + i y = \frac{\partial \phi}{\partial \tau_{yz}} - i \frac{\partial \phi}{\partial \tau_{xz}} \rightarrow \frac{K^2}{2\pi (\tau_{yz} + i \tau_{xz})^2} \quad \text{as } \tau_{yz} + i \tau_{xz} \rightarrow 0,$$
(62)

so that  $\phi = \phi(\tau_{xz}, \tau_{yz})$  has a singularity of prescribed form at the origin of the stress plane.

According to (53),  $\phi$  is a harmonic function of stresses for the elastically isotropic material and may thus be represented as the imaginary part of an analytic function  $g(\tau_{xz} + i\tau_{yz})$ ,

$$\phi = \operatorname{Im}\left\{ g\left(\tau_{xz} + \mathrm{i}\tau_{yz}\right) \right\},\tag{63}$$

and upon differentiation and use of (49), physical coordinates of points in the elastic region are given by

$$x - \mathbf{i} \, y = \frac{\partial \phi}{\partial \tau_{yz}} + \mathbf{i} \frac{\partial \phi}{\partial \tau_{xz}} = g' \big( \tau_{xz} + \mathbf{i} \tau_{yz} \big). \tag{64}$$

Now taking the complex conjugate of (62), the asymptotic boundary condition becomes

$$g'(\tau_{xz} + i\tau_{yz}) \rightarrow -\frac{K^2}{2\pi(\tau_{xz} + i\tau_{yz})^2} \quad \text{as } \tau_{xz} + i\tau_{yz} \rightarrow 0, \tag{65}$$

and integrating this,

$$g(\tau_{xz} + i\tau_{yz}) \rightarrow \frac{K^2}{2\pi(\tau_{xz} + i\tau_{yz})} \quad \text{as } \tau_{xz} + i\tau_{yz} \rightarrow 0.$$
(66)

As mentioned earlier, it is appropriate to view the crack as semi-infinite as in Fig. 8(a). The stress plane map of the elastic region into the upper half of the yield surface is shown in Fig. 8(b); this mapping, the location of corresponding points, and the imposition of boundary conditions  $\phi = 0$  on the yield surface and  $\tau_{xz}$ -axis follows from the same reasoning involved in the construction of Fig. 6. Prescription of  $\phi$  on a portion of the  $\tau_{yz}$ -axis of Fig. 6 is now replaced by the prescription of a singularity (66) at the origin of Fig. 8(b).

Let  $\Omega = \Omega(\tau_{xz} + i\tau_{yz})$  conformally map the interior of the upper portion  $(\tau_{yz} > 0)$  of the yield surface onto the upper half  $(\text{Im}\{\Omega\} > 0)$  of a unit circle in the complex  $\Omega$  plane, as shown in Fig. 8(b) and (c). The mapping is uniquely determined by the specification of three points, here chosen as (i)  $\Omega(0) = 0$  (that is, the origin of the stress plane is mapped into the origin of the  $\Omega$  plane), (ii) point E in Fig. 8(b) where the yield surface crosses the positive  $\tau_{xz}$ -axis is mapped into  $\Omega = 1$ , and (iii) point D of Fig. 8(b) where the yield surface crosses the negative  $\tau_{xz}$ -axis is mapped into  $\Omega = -1$ . These imply, incidentally, that  $\Omega(\tau_{xz})$  is real, and thus

$$\Omega(\tau_{xz} + i\tau_{yz}) = \overline{\Omega}(\tau_{xz} + i\tau_{yz}), \tag{67}$$

the bar indicating a complex conjugation. So far the function  $\Omega = \Omega(\tau_{xz} + i\tau_{yz})$  is defined only for  $(\tau_{xz}, \tau_{yz})$  in the interior of the upper portion  $(\tau_{yz} > 0)$  of the yield surface. Now introducing the mirror image of this upper portion with respect to the  $\tau_{xz}$ -axis, as shown by the dashed line in Fig. 8(b), (67) permits the analytic continuation of  $\Omega$  into the image region with

$$\Omega = \Omega \left( \tau_{xz} + i \tau_{yz} \right) = \overline{\Omega \left( \tau_{xz} - i \tau_{yz} \right)},\tag{68}$$

when  $\tau_{yz} < 0$ . That is, for any point  $(\tau_{xz}, \tau_{yz})$  in the image region  $(\tau_{yz} < 0)$ , the corresponding value of  $\Omega$  is the conjugate of the value of  $\Omega$  corresponding to the conjugate point  $(\tau_{xz}, -\tau_{yz})$  in the upper portion of the stress plane. Thus the mapping function introduced here is identical to the function which conformally maps the interior of the upper portion of the yield surface and its mirror image onto a complete unit circle, with the same three points specified as above. The latter mapping is generally more directly determined.

The solution for  $g(\tau_{xz} + i\tau_{yz})$  of (63) may now be written by inspection in terms of  $\Omega(\tau_{xz} + i\tau_{yz})$  as

$$g(\tau_{xz} + i\tau_{yz}) = \frac{\Omega'(0)K^2}{2\pi} \left\{ \frac{1}{\Omega(\tau_{xz} + i\tau_{yz})} + \Omega(\tau_{xz} + i\tau_{yz}) \right\},\tag{69}$$





Fig. 8. Small scale yielding near a crack tip.

where  $\Omega'(0)$ , the derivative of the mapping function evaluated at the origin of the stress plane, is real and of necessity nonzero as the transformation is conformal. To see that this is indeed the correct solution note that g is real on the  $\tau_{xz}$ -axis as  $\Omega$  is real there, and that g is real on the upper portion of the yield surface as this maps onto the unit circle where  $|\Omega| = 1$  and thus  $1/\Omega$  equals the conjugate of  $\Omega$ . Since, by (63),  $\phi$  is the imaginary part of g, the condition  $\phi = 0$  everywhere on the boundary is satisfied. Also, since  $\Omega(0) = 0$ , a Taylor expansion of terms in (69) results in

$$g(\tau_{xz} + i\tau_{yz}) = \frac{\Omega'(0)K^2}{2\pi} \left\{ \left( \Omega'(0)(\tau_{xz} + i\tau_{yz}) + \frac{1}{2}\Omega''(0)(\tau_{xz} - i\tau_{yz})^2 + \cdots \right)^{-1} + \Omega'(0)(\tau_{xz} + i\tau_{yz}) + \frac{1}{2}\Omega''(0)(\tau_{xz} + i\tau_{yz})^2 + \cdots \right\}$$
$$\to \frac{K^2}{2\pi} \frac{1}{\tau_{xz} + i\tau_{yz}} \quad \text{as } \tau_{xz} + i\tau_{yz} \to 0,$$
(70)

so that the singularity of (66), as required by the asymptotic boundary conditions, is obtained.

Let us now discuss the solution just constructed. Differentiating (69), according to (64) physical coordinates and stresses in the elastic region are represented by

$$x - i y = \frac{\Omega'(0) K^2}{2\pi} \Omega'(\tau_{xz} + i \tau_{yz}) \left\{ 1 - \frac{1}{\Omega^2(\tau_{xz} + i \tau_{yz})} \right\}.$$
 (71)

For  $(\tau_{xz}, \tau_{yz})$  on the yield surface,  $\Omega$  is on the unit circle and may be represented as

$$\Omega(\tau_{xz} + i\tau_{yz}) = e^{i[\pi/2 + \psi(I)]} = i e^{i\psi(I)},$$
(72)

where the function  $\psi(l)$  of yield surface arc length, l, is the angle between the Im{ $\Omega$ }-axis and the corresponding point on the unit circle, as shown in Fig. 8(c). Also, as clear from Fig. 8(b),

$$d(\tau_{xz} + i\tau_{yz}) = -e^{i\theta} dl$$
(73)

on the yield surface, so that

$$\Omega'(\tau_{xz} + i\tau_{yz}) = \frac{d[i e^{i\psi(l)}]}{(-e^{i\theta} dl)} = e^{-i\theta}e^{i\psi(l)}\psi'(l).$$
(74)

Since  $x - iy = R(l)e^{-i\theta}$  on the elastic-plastic boundary, the distance R(l) from the crack tip to a point on the elastic-plastic boundary corresponding to yield surface arc length l is given by (71)-(74) as

$$R(l) = \frac{\Omega'(0)K^2}{\pi} \psi'(l) \cos[\psi(l)].$$
(75)

Displacement at any point B on the elastic-plastic boundary is, from (42) as specialized to elastic isotropy,

$$w_{B} = w_{O} + \frac{1}{G} \left\{ R_{B} \tau_{rzB} - R_{O} \tau_{rzO} + \frac{\Omega'(0) K^{2}}{\pi} (\sin \psi_{B} - \sin \psi_{O}) \right\},$$
(76)

with subscripts B and O referring to values of R(l),  $\tau_{rz}$  and  $\psi(l)$  as evaluated at those points. Identifying B and O with points D and E of Fig. 8(a),  $\psi_B = \frac{1}{2}\pi$  and  $\psi_0 = -\frac{1}{2}\pi$ . Then  $w_B - w_O = \Delta w$  is the displacement discontinuity occurring at the crack tip, i.e., the crack tip shear displacement, and as  $R_B = R_O = 0$  for B and O at the crack tip,

$$\Delta w = 2\Omega'(0)K^2/(\pi G). \tag{77}$$

Remarkably, the entire influence of the anisotropic yield condition is summarized here in terms of the derivative of the mapping function at the origin.

# 6. Some examples of small scale yielding near cracks

The Tresca and Mises yield criteria lead to the circular elastic shaped region

$$\tau_{xz}^2 + \tau_{yz}^2 < \tau_0^2 \tag{78}$$

as shown in Fig. 9(a). The required mapping function carrying the upper part of the yield surface onto the upper part of the unit circle is

$$\Omega(\tau_{xz} + i\tau_{yz}) = (\tau_{xz} + i\tau_{yz})/\tau_0.$$
<sup>(79)</sup>

In this case,  $\psi = \theta = l/\tau_0$ . The stress solution in the plastic region is

$$\tau_{\theta_z} = \tau_0, \qquad \tau_{rz} = 0, \tag{80}$$

since the stress vector at points on the yield surface has no component in the tangential direction. From (75), the distance to the elastic-plastic boundary is

$$R = \left(\frac{K^2}{\pi \tau_0^2}\right) \cos\theta \tag{81}$$

since  $\psi'(l) = 1/\tau_0$ , and this is the equation of a circle as in Fig. 9(b) with diameter equal to the maximum plastic zone dimension

$$R_{\max} = (1/\pi) \left( \frac{K^2}{\tau_0^2} \right) \approx 0.318 \frac{K^2}{\tau_0^2}.$$
(82)

Taking the displacement to be zero at point O where the elastic-plastic boundary crosses the x-axis, displacements at other points of the elastic-plastic boundary are, from (76),

$$w = \left(\frac{K^2}{\pi G \tau_0}\right) \sin \theta, \tag{83}$$

and since  $\gamma_{rz} = \frac{\partial w}{\partial r} = 0$  from (44), this equation also gives displacements at all interior points of the plastic region. The nonvanishing strain component,  $\gamma_{\theta z} = (1/r)(\frac{\partial w}{\partial \theta})$ , is thus

$$\gamma_{\theta z} = \left( K^2 / \pi G \tau_0 \right) (\cos \theta) / r, \tag{84}$$

and the total displacement discontinuity occurring at the crack tip is

$$\Delta w = 2K^2 / \pi G \tau_0. \tag{85}$$



Fig. 9. Small scale yielding near crack; circular yield surface.

This solution was first given by Hult and McClintock (1956).

As another example which leads to a discrete slip line plasticity solution, rather than a continuous field of plastic strain as above, consider the diamond shaped yield surface of Fig. 10(a). Here plastic flow ensues when the resolved shear stress in directions at plus or minus 45° with the x-axis equals the flow stress  $\tau_0$ , so that the elastic region is defined by

$$\frac{1}{2}|\tau_{xz} + \tau_{yz}|\sqrt{2} < \tau_0 \quad \text{and} \quad \frac{1}{2}|\tau_{xz} - \tau_{yz}|\sqrt{2} < \tau_0.$$
(86)

The appropriate mapping function transforming the upper portion of the diamond yield surface onto a unit semi-circle is (e.g., Moretti (1964))

$$\Omega(\tau_{xz} + i\tau_{yz}) = \operatorname{sn} \zeta / (\sqrt{2} \, \operatorname{dn} \zeta), \qquad \zeta = \lambda (\tau_{xz} + i\tau_{yz}) / (\sqrt{2} \, \tau_0). \tag{87}$$

Here  $\lambda$  is the complete elliptic integral of modulus  $\frac{1}{2}$ ,

$$\lambda = \int_0^1 \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-\frac{1}{2}t^2)}} \approx 1.8541,\tag{88}$$

and sn  $\zeta$ , cn  $\zeta$  and dn  $\zeta$  are Jacobian elliptic functions of modulus  $\frac{1}{2}$ , sn  $\zeta$  being the analytic continuation of the function sn u defined for real u and  $|u| \leq \lambda$  by

$$u = \int_0^{\operatorname{sn} u} \frac{\mathrm{d}t}{\sqrt{(1 - t^2)(1 - \frac{1}{2}t^2)}},$$
(89)

and with

cn 
$$\zeta = (1 - \mathrm{sn}^2 \zeta)^{1/2}$$
, dn  $\zeta = (1 - \frac{1}{2} \mathrm{sn}^2 \zeta)^{1/2}$ , (90)

branches being chosen so that cn 0 = 1 and dn 0 = 1. The resulting plastic slip lines are shown in Fig. 10(b); the upper slip line corresponds to the upper left quadrant of the yield surface. Measuring yield surface arc length *l* from point *O* as in Fig. 10(a), one may show, employing a variety of formulae for elliptic functions, that  $\psi(l)$  is given, for l > 0, by

$$\psi(l) = \arcsin\left(\frac{1 - \operatorname{cn}(\lambda l/\tau_0)}{2 \operatorname{dn}(\lambda l/\tau_0)}\right).$$
(91)



Fig. 10. Small scale yielding near crack; diamond shaped yield surface.

Thus (75) gives for distance to points on the elastic-plastic boundary (that is, on the upper slip line of Fig. 10(b))

$$R(l) = \frac{\lambda^2 K^2}{8\pi\tau_0^2} \frac{\mathrm{sn}(\lambda l/\tau_0) [1 + \mathrm{cn}(\lambda l/\tau_0)]}{\mathrm{dn}^2(\lambda l/\tau_0)}.$$
(92)

Setting R'(l) = 0, one finds the maximum distance occurs when  $cn(\lambda l/\tau_0) = 2 - \sqrt{3}$ , and the slip line length is

$$R_{\max} = \frac{\lambda^2}{8\pi} (\sqrt{3} - 1) \sqrt{\frac{3\sqrt{3}}{2(2 - \sqrt{3})}} \frac{K^2}{\tau_0^2} \approx 0.322 \frac{K^2}{\tau_0^2}.$$
(93)

Setting the displacement to zero at point O of Fig. 10(b), displacements of points along the slip line are, from (76) after noting (92) and that  $\tau_{rz} = \tau_0 l$ ,

$$w = \frac{\lambda K^2}{4\pi G \tau_0 \mathrm{dn}(\lambda l/\tau_0)} \left\{ 1 - \mathrm{cn}(\lambda l/\tau_0) + \frac{1}{2}\lambda(1 - l/\tau_0) \frac{\mathrm{sn}(\lambda l/\tau_0)[1 + \mathrm{cn}(\lambda l/\tau_0)]}{\mathrm{dn}(\lambda l/\tau_0)} \right\}.$$
 (94)

The total displacement discontinuity occurring at the crack tip is

$$\Delta w = \lambda K^2 / \pi G \tau_0. \tag{95}$$

Considering again a yield surface leading to a discrete slip line solution, Fig. 11(a) represents a material with infinite shear strength in the x-direction but which yields under a resolved shear stress  $\tau_0$  when the elastic range

$$|\tau_{yz}| < \tau_0 \tag{96}$$

is exceeded. This would model a zinc single crystal with an appropriately oriented basal slip plane. The mapping function is

$$\Omega(\tau_{xz} + i\tau_{yz}) = \tanh\left\{\frac{\pi}{4\tau_0}(\tau_{xz} + i\tau_{yz})\right\},\tag{97}$$

leading through (75), after noting that yield surface arc length  $l = -\tau_{xz}$ , to

$$R(l) = R(\tau_{xz}) = \frac{\pi K^2}{8\tau_0^2} \operatorname{sech}^2\left(\frac{\pi \tau_{xz}}{2\tau_0}\right),\tag{98}$$



Fig. 11. Small scale yielding near crack; one set of slip planes.

for the distance from the crack tip to a point on the slip line carrying stress  $\tau_{xz}$ . The slip line length is

$$R_{\max} = \frac{1}{8}\pi K^2 / \tau_0^2 \approx 0.392 K^2 / \tau_0^2.$$
<sup>(99)</sup>

Noting that  $\tau_{rz} = \tau_{xz}$  and setting w = 0 at point O of Fig. 11(b), displacements are given by

$$w = -\frac{K^2}{4G\tau_0} \left\{ \tanh\left(\frac{\pi\tau_{xz}}{2\tau_0}\right) + \left(\frac{\pi\tau_{xz}}{2\tau_0}\right) \operatorname{sech}^2\left(\frac{\pi\tau_{xz}}{2\tau_0}\right) \right\},\tag{100}$$

with the crack tip displacement discontinuity

$$\Delta w = K^2 / (2G\tau_0). \tag{101}$$

Of course, this solution corresponds to the well-known Dugdale-Barenblatt-Bilby/Cottrell/Swinden model for Mode III yielding.

As a final example, when the single slip direction yield surface of Fig. 11(a) is rotated through 90° as in Fig. 12(a), so that the elastic range is

$$|\tau_{xz}| < \tau_0, \tag{102}$$

the mapping function is

$$\Omega(\tau_{xz} + i\tau_{yz}) = \tan\left\{\frac{\pi}{4\tau_0}(\tau_{xz} + i\tau_{yz})\right\}.$$
(103)

As in Fig. 12(a), yield surface arc length  $l = -\tau_{yz}$  on the left straight line segment corresponding to the upper slip line of Fig. 12(b), and from (75), for  $\tau_{yz} > 0$ .

$$R(l) = R(\tau_{yz}) = \frac{\pi K^2}{8\tau_0^2} \sinh\left(\frac{\pi \tau_{yz}}{2\tau_0}\right) \operatorname{sech}^2\left(\frac{\pi \tau_{yz}}{2\tau_0}\right).$$
(104)

The maximum occurs at  $\sinh(\pi \tau_{yz}/2\tau_0) = 1$ , so that the slip line length is

$$R_{\max} = \frac{1}{16} \pi K^2 / \tau_0^2 \approx 0.196 K^2 / \tau_0^2.$$
(105)

Displacements on the upper slip line are, from (76) with w = 0 at point O,

$$w = \frac{K^2}{4G\tau_0}\operatorname{sech}\left(\frac{\pi\tau_{yz}}{2\tau_0}\right)\left\{1 + \left(\frac{\pi\tau_{yz}}{2\tau_0}\right) \operatorname{tanh}\left(\frac{\pi\tau_{yz}}{2\tau_0}\right)\right\},\tag{106}$$



Fig. 12. Small scale yielding near crack; one set of slip planes.

and the crack tip displacement discontinuity is

$$\Delta w = K^2 / (2G\tau_0). \tag{107}$$

It is interesting to note that gross features of the plastic region solution, such as the maximum plastic zone dimension and notch tip displacement discontinuity, are relatively insensitive to the choice of yield surface. However, the shape of the plastic region is strongly dependent on the shape of the yield surface: curved yield surfaces as appropriate for a polycrystalline aggregate relax high notch tip stresses through continuous fields of plastic strains; straight line yield surfaces as appropriate for single crystals, or on the single grain scale, relax stresses through discontinuities in the form of discrete slip lines. These results may have important implications for the application of mechanics to the microstructural deformation of materials and it is important to determine the extent to which such observations carry over to the case of plane strain with single crystal yield surfaces.

#### 7. Limits loads in torsion

As another application of the general results for integrals of the stress equilibrium equations as constrained by an arbitrary convex yield surface, the construction of limit torques corresponding to the fully plastic torsion of a prismatic shaft is discussed here. It is convenient to phrase results in terms of a stress function,  $\chi$ , as in previous elastic and plastic treatments, such that the stress equilibrium equation (2) is identically satisfied by writing

$$\boldsymbol{\tau} = \boldsymbol{k} \times \boldsymbol{\nabla} \boldsymbol{\chi},\tag{108}$$

where k is again perpendicular to the (x, y) plane of the cross-section. Some well-known formulae involving this stress function are now reviewed. Boundary conditions of traction free lateral surfaces of the prism require that the stress vector,  $\tau$ , be tangential to the boundary. Since an increment, dr, of position vector taken along the boundary is also tangential, the boundary condition is

$$0 = \mathbf{k} \cdot [\mathbf{d}\mathbf{r} \times \boldsymbol{\tau}] = k \cdot [\mathbf{d}\mathbf{r} \times (\mathbf{k} \times \nabla \chi)] = \mathbf{d}\mathbf{r} \cdot \nabla \chi = \mathbf{d}\chi.$$
(109)

Thus the stress function,  $\chi$ , is constant on the boundary, and to simplify later results, we take  $\chi = 0$  on the boundary. (The case of multiply connected regions, for which  $\chi$  must be given a different constant value on each bounding contour, is not considered here.) The stress vector generally is discontinuous along certain arcs in the fully plastic solution. Let L be such an arc and let the superscripts + and - refer to values on different sides of L. Action-reaction requires the components of the stress vectors normal to the line of discontinuity be continuous. Letting N be the normal to L,

$$N \cdot \tau^+ = N \cdot \tau^- \,. \tag{110}$$

Thus on L,

$$0 = \mathbf{N} \cdot [\mathbf{k} \times \nabla(\chi^+ - \chi^-)] = (\mathbf{N} \times \mathbf{k}) \cdot \nabla(\chi^+ - \chi^-) = \frac{\partial}{\partial L} (\chi^+ - \chi^-), \qquad (111)$$

since  $N \times k$  is tangential to L; here  $\partial/\partial L$  denotes a differentiation with respect to arc length along L. Clearly, every arc of discontinuity must either end at the boundary or at an interior point of the cross-section. In either case  $\chi^+ = \chi^-$  at the origin of the discontinuity, and integrating (111) one has  $\chi^+ = \chi^-$  everywhere along L. Thus equilibrium requires that  $\chi$  be continuous, and arcs of discontinuity may be located by imposing this condition of continuity. Measuring the torque, M, on the prism as positive clockwise with reference to Fig. 1, one has, with  $\int_A \cdots dA$  denoting an interaction over the prism cross-section,

$$\boldsymbol{M} = -\boldsymbol{k} \cdot \int_{A} \boldsymbol{r} \times \boldsymbol{\tau} \, \mathrm{d}\boldsymbol{A} = -\int_{A} \boldsymbol{r} \cdot \boldsymbol{\nabla} \boldsymbol{\chi} \, \mathrm{d}\boldsymbol{A} = 2 \int_{A} \boldsymbol{\chi} \, \mathrm{d}\boldsymbol{A}$$
(112)

by a standard argument based on the Green-Gauss theorem and the fact that  $\chi = 0$  on the boundary.

There remains only the problem of determining  $\chi$  as a function of position in the cross-section. Recall, from the discussion in connection with Figs. 1 and 2, that the stress distribution in plastic regions adjacent to a traction-free boundary is found by first locating the point on the yield surface for which the stress vector is tangential to the boundary at a given boundary point, then by drawing a straight line from that boundary point in the direction of the yield surface unit tangent vector, s, and finally by setting the stress vector equal to a constant along such an *s*-line. Employing the *s*-lines as natural coordinates,

$$\tau_n = \mathbf{n} \cdot \boldsymbol{\tau} = \mathbf{s} \cdot \boldsymbol{\nabla} \boldsymbol{\chi} = \partial \boldsymbol{\chi} / \partial \boldsymbol{s}, \tag{113}$$

so that, along an s-line, the stress function has constant slope equal to  $\tau_n$ , the component of the stress vector in the direction normal to the yield surface. Integrating subject to the condition  $\chi = 0$  on the boundary, and understanding s to now denote distance from the boundary to the point under consideration as measured along an s-line,

$$\chi = \tau_n s \tag{114}$$

in the region between the boundary and the nearest arc of discontinuity.

Now it is seen that the well-known plastic roof analogy for fully plastic torsion (Prager and Hodge, 1951), as formulated for circular yield surfaces of the Tresca or Mises type, carries over to the general case.

Viewing  $\chi$  as the height of a roof which is constructed over the cross-section such that the roof touches the boundary in satisfaction of the boundary condition  $\chi = 0$ , the limit torque is, by (112), twice the volume between the roof and cross-section. The roof is constructed on straight timbers which emanate from each boundary point in the direction of the local s-line and with a slope equal, by (113), to the corresponding component of the stress vector in the direction of the yield surface unit normal. The location of juncations of different timbers is determined, as  $\chi$  is continuous, by the condition that there be no gaps in the roof.

# 8. Limit torques-some examples

As a first example, consider a circular shaft of radius c with a rectangular yield surface requiring a resolved shear stress  $\tau_0$  for yielding in the x-direction and  $\tau_0 \tan \beta$  for the y-direction, as in Fig. 13. We assume  $\beta \leq \frac{1}{4}\pi$  so the y-direction is weakest. Symmetry requires consideration only of the quadrant x > 0, y > 0 corresponding to the yield surface quadrant  $\tau_{xz} > 0$ ,  $\tau_{yz} < 0$ . The straight line segments AB and BC separate regimes I and II of the yield surface. In the corresponding regime I of the (x, y) plane all s-lines are parallel and in the y-direction, with  $\tau_n = \tau_0$ ; in regime II all s-lines are again parallel but in the x-direction, with  $\tau_n = \tau_0$  tan  $\beta$ . Thus, by (114),

and

$$\chi = \tau_0 \left[ \left( c^2 - x^2 \right)^{1/2} - y \right] \qquad \text{in regime I}, \tag{115}$$

$$\chi = \tau_0 \tan \beta \left[ \left( c^2 - y^2 \right)^{1/2} - x \right] \quad \text{in regime II.}$$
(116)

Equating these, the arc of discontinuity separating the two regimes is given by

$$y = x \sin 2\beta + (c^2 - x^2)^{1/2} \cos 2\beta, \tag{117}$$

which intersects the boundary at point *B*,  $(c \sin \beta, c \cos \beta)$ , and the y-axis at  $(0, c \cos 2\beta)$ . Another discontinuity occurs on the y-axis between the origin and  $(0, c \cos 2\beta)$ . The figures were actually drawn for  $\beta = 30^{\circ}$ . With y = f(x) denoting (117), symmetry and (112) lead to

$$M = 8\tau_0 \int_0^c \frac{\sin\beta}{f(x)} \int_{f(x)}^{(c^2 - x^2)^{1/2}} \left[ (c^2 - x^2)^{1/2} - y \right] dy dx$$
  
+  $8\tau_0 \tan\beta \int_0^c \frac{\sin\beta}{f_0^{0}} \int_0^{f(x)} \left[ (c^2 - y^2)^{1/2} - x \right] dy dx$   
+  $8\tau_0 \tan\beta \int_c^c \frac{\beta}{\sin\beta} \int_0^{(c^2 - x^2)^{1/2}} \left[ (c^2 - y^2)^{1/2} - x \right] dy dx$  (118)

for the limit torque. After integrating and considerable algebraic rearrangement, one obtains

$$M = 4\tau_0 c^3 \{ \tan \beta \left[ (2\beta - \frac{1}{2}\sin 4\beta)(1 - \sin \beta) + 4\sin^4 \beta \cos \beta + \frac{2}{3} \right] - \frac{4}{3}\sin^3 \beta \left[ \cos^2 2\beta + 2\cos \beta (1 - \cos^3 \beta) + 2\sin \beta (1 - \sin^3 \beta) \right] \}.$$
 (119)

For  $\beta = 45^{\circ}$ , a square yield surface, this gives

$$M = \frac{1}{3}\tau_0 c^3 \sqrt{2} \left[ 2(5 - 2\sqrt{2}) + 3\pi(\sqrt{2} - 1) \right] \approx 2.24\tau_0 c^3.$$
(120)

If one lets  $\tau_0 \to \infty$  and  $\beta \to 0$  in such a way that the minimum shear strength,  $\tau_m = \tau_0 \tan \beta$ , remains finite, the limit torque is

$$M = \frac{8}{3}\tau_{\rm m}c^3 \approx 2.66\tau_{\rm m}c^3.$$
(121)

For comparison a Tresca or Mises circular yield surface of radius  $\tau_m$  leads to (Prager and Hodge, 1951)

$$M = \frac{2}{3}\pi\tau_{\rm m}c^3 \approx 2.10\tau_{\rm m}c^3.$$
(122)

The last two equations and the requirement of convexity of the yield surface lead to absolute bounds on the limit load of a circular shaft for arbitrary, but symmetric, anisotropic yield surfaces (defining a symmetric yield surface by the property that if  $\tau$  is on the yield surface, so also is  $-\tau$ ). Fig. 14 pictures such a yield



Fig. 13. Circular shaft, rectangular yield condition.

surface. Let  $\tau_m$  be the minimum magnitude of all the stress vectors,  $\tau$ , which touch the yield surface. Then a circular yield surface of radius  $\tau_m$  lies completely inside the symmetric yield surface, and thus the lower bound (statical) theorem of limit analysis (Drucker, 1960) assures that the limit torque is not smaller than that given by (122). The fact that  $\tau_m$  is the minimum stress vector magnitude and convexity of the yield surface forces the surface to be smooth and tangent to the circle at the points of contact. Convexity further requires that a straight line tangent to a yield surface never passes inside the surface. Thus the yield surface formed by two infinite straight line segments, tangent to the circle at the points of contact, must enclose the symmetric yield surface so that (121) provides an upper bound to the limit torque. Therefore the limit torque of a circular shaft of plastically symmetric material is always bounded by

2.10 
$$\tau_{\rm m}c^3 \approx \frac{2}{3}\pi\tau_{\rm m}c^3 \leqslant M \leqslant \frac{8}{3}\tau_{\rm m}c^3 \approx 2.66\tau_{\rm m}c^3$$
, (123)

where  $\tau_{\rm m}$  is the minimum shear strength.

As a second example for the construction of limit torques, consider a rectangular cross-section, of dimensions 2*a* in the *x*-direction and 2*b* in the *y*-direction, and an arbitrary yield surface, as in Fig. 15. The stress vector is parallel to the straight line boundaries and thus constant along each line, so that the solution is simply four plastic regimes, I, II, III, IV, of constant stresses corresponding to the four points of the yield surface intersected by the  $(\tau_{xz}, \tau_{yz})$  axes. There is no need to introduce the *s*-lines for this case; if the subscript *u* denotes one of the four regimes, the stress function in regime *u* is

$$\chi = \tau_{\mu} L, \tag{124}$$

where L is the perpendicular distance from the boundary into a given regime. The maximum perpendicular distance a given regime penetrates is  $L_{\mu}$ . Continuity of  $\chi$  then requires that  $\tau_{\mu}L_{\mu}$  be constant, or

$$\tau_{\rm I} L_{\rm I} = \tau_{\rm II} L_{\rm II} = \tau_{\rm II} L_{\rm III} = \tau_{\rm IV} L_{\rm IV}. \tag{123}$$

If a is sufficiently large compared to b so that regimes I and III touch, as in Fig. 15,

$$L_1 + L_{111} = 2b.$$
 (126)



Fig. 14. Symmetric yield surface and bounding surfaces.

Solving for  $L_u$ ,

$$L_{\rm I} = \frac{2b\tau_{\rm III}}{\tau_{\rm I} + \tau_{\rm III}}, \qquad L_{\rm III} = \frac{2b\tau_{\rm I}}{\tau_{\rm I} + \tau_{\rm III}}, \qquad L_{\rm II} = \frac{2b\tau_{\rm I}\tau_{\rm III}}{\tau_{\rm II}(\tau_{\rm I} + \tau_{\rm III})}, \qquad L_{\rm IV} = \frac{2b\tau_{\rm I}\tau_{\rm III}}{\tau_{\rm IV}(\tau_{\rm I} + \tau_{\rm III})}.$$
(127)

The limit torque is

$$M = 2 \sum_{u=1}^{IV} \int_{A_u} \tau_u L \, \mathrm{d}A = 2 \sum_{u=1}^{IV} \tau_u \int_{A_u} L \, \mathrm{d}a = 2 \sum_{u=1}^{IV} \tau_u \widetilde{L}_u A_u,$$
(128)

where  $A_u$  is the area of regime u and  $\overline{L}_u$  is the perpendicular distance from the boundary to the centroid of regime u. Carrying out the calculations,

$$M = 8b^{2} \frac{\tau_{1} \tau_{111}}{\tau_{1} + \tau_{111}} \left[ a - \frac{1}{3}b \frac{\tau_{1} \tau_{111}}{\tau_{11} \tau_{1V}} \frac{\tau_{11} + \tau_{1V}}{\tau_{1} + \tau_{111}} \right].$$
 (129)

This solution is valid for  $L_{II} + L_{IV} \leq 2a$ , or for

$$\frac{a}{b} \ge \frac{\tau_{\rm I} \tau_{\rm II}}{\tau_{\rm II} \tau_{\rm IV}} \frac{\tau_{\rm II} + \tau_{\rm IV}}{\tau_{\rm I} + \tau_{\rm I} + \tau_{\rm III}};$$
(130)

otherwise one simply interchanges a and b,  $\tau_{I}$  and  $\tau_{II}$ , and  $\tau_{II}$  and  $\tau_{IV}$  in (129). The formulation of the limit torque is equally straightforward for other cross-sections with convex straight line segment boundaries.

# Postscript

The original manuscript for this paper was written in early 1966 and issued at that time as Technical Report NSF GK-286/2 at Brown University but was never published. Some of the results for cracks in anisotropic solids were summarized briefly in a paper written shortly thereafter (Rice, 1967b). The present paper differs from the original only by minor shortening and rewording. The work was supported in 1966 by a NSF Research Initiation Grant and its present refurbishing by the NSF Materials Research Laboratory at Harvard University. (Owing to progress on other fronts, my work on 1966 turned away from



fig. 15. Rectangular cross-section, arbitrary yield surface.

the anti-plane shear modelling of yield and fracture. However, the present work has remained of interest for the generality of its treatment of anisotropy, has been referred to on occasion in the published literature, and has proven basic for some recent work on crack plasticity in ductile crystals, leading to the decision to seek publication at this time.)

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