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COMMENTS ON "ON THE STABILITY OF SHEAR CRACKS AND THE CALCULATION OF COMPRESSIVE STRENGTH" BY J. K. DIENES

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Dienes [1983] has presented an evaluation of the Griffith crack instability criterion for a circular shear crack of radius c which is subject to a remotely applied shear stress σ, sustains frictional stress τ on its surfaces, and is assumed to grow in its own plane in a manner which retains the circular shape. Unfortunately, the instability condition which Dienes presents, namely,

$$(\sigma-\tau)(\sigma-3\tau) > \pi(2-\nu)\gamma\mu/2(1-\nu)c \quad (1)$$

(his equation (13), where ν is the Poisson ratio, μ is the shear modulus, and $2\pi c^2\gamma$ is the "surface energy" of the fracture) seems untenable. It implies that σ must exceed 3τ for the crack to grow, whereas elementary physical understanding of the problem convinces one that a crack of sufficiently low surface energy will grow if σ is only slightly in excess of τ. Thus one is inclined to seek some oversight in the calculation.

Although never enunciated explicitly in this form, the instability criterion employed by Dienes is

$$\delta W > \delta W_e + \delta W_f + \delta W_s \quad (2)$$

where δ denotes variation in a crack growth increment $\delta c (>0)$, δW is the work increment of externally applied loadings, $W_e \equiv U - U_{init}$ is the increase in elastic strain energy U of the body due to introduction of the crack (and U_{init} its initial value in the loaded but uncracked body), δW_f is the frictional dissipation increment due to sliding on the crack surfaces, and $W_s = 2\pi c^2\gamma$. The criterion is implemented with Segedin's [1950] solution for the slip displacement discontinuity $2w$ along the crack surface, where

$$w = [4(1-\nu)(\sigma-\tau)/\pi(2-\nu)\mu]\sqrt{c^2-x^2-y^2} \quad (3)$$

is given as Dienes' equation (6). Dienes does the instability calculation for fixed values of the externally applied loadings, commenting that the same result should apply for fixed displacements on a remote boundary.

A correct derivation of the instability criterion is given first. In the circumstances of fixed external loading, δW may be equated to the increment of a quantity W, where W is the work that the external loadings would do if the entire crack was introduced quasi-statically into a previously uncracked body in the presence of these fixed loadings (and with resistive stress τ on the crack surfaces). The change in elastic strain energy upon introduction of the crack is equal to the work W of external loads plus the (negative) work done on the crack surfaces where stress reduces from σ to τ in order for the slip $2w$ to develop:

$$W_e \equiv U-U_{init} = W - \int_{S_{cr}} \frac{1}{2}(\sigma+\tau)(2w)\,dA \quad (4)$$

The factor $(1/2)(\sigma+\tau)$ multiplying $2w$ appears because the body is assumed to be linearly elastic, and S_{cr} denotes the crack surface (which Dienes denotes as S in the body of his paper but as S_i in his Appendix A, in which S denotes the sum of both the crack surfaces S_i and the external surface S_o of the body). Since the frictional work increment is given by the variation of the quantity

$$W_f = \int_{S_{cr}} \tau(2w)\,dA \quad (5)$$

where we regard τ as fixed in calculating

$$\delta W_f = 2\int_{S_{cr}} \tau\,\delta w\,dA \quad (6)$$

(Dienes' equation (7)), we have

$$(W-W_f) - (U-U_{init}) \equiv (W-W_f) - W_e = \int_{S_{cr}} \frac{1}{2}(\sigma-\tau)(2w)\,dA = \frac{8}{3\mu}\frac{1-\nu}{2-\nu}c^3(\sigma-\tau)^2 \quad (7)$$

Hence the instability criterion, rearranged as $\delta W - \delta W_f - \delta W_e > \delta W_s$, is

$$\delta\left[\frac{8}{3\mu}\frac{1-\nu}{2-\nu}c^3(\sigma-\tau)^2\right] > \delta\left[2\pi c^2\gamma\right] \quad (8)$$

which implies that

$$(\sigma-\tau)^2 > \pi(2-\nu)\gamma\mu/2(1-\nu)c \quad (9)$$

Comparing with Dienes' result, quoted as (1) above, we see that his factor of 3 on the left should be replaced by unity. The corrected result (9) is in accord with the elementary requirement stated after (1).

Where did the Dienes calculation go wrong? In his Appendix A he derives, in a series of steps from (A1) to (A18) that seem correct and have some precedent in the crack mechanics literature [e.g., Bueckner, 1958; Sanders, 1960; Rice and Drucker,

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1967], the expression

$$U-U_{init} \equiv W_e = \frac{8}{3\mu}\frac{1-\nu}{2-\nu} c^3 (\sigma-\tau)^2 \qquad (10)$$

which he denotes as U^b-U^o in Appendix A and reports as his equation (12) for W_e in the body of his paper. Notice that (10) above has the same right-hand side as (7) above; this is as it should be because the Clapeyron relation (two times strain energy equals force times displacement) requires that

$$2U = (2U_{init}+W) - W_f \qquad (11)$$

(the term $2U_{init}$ equals the products of external loads with displacements that they would cause in absence of the crack and W equals their products with the additional displacements due to introduction of the crack, so that $(2U_{init}+W)$ is the proper external "force times displacement" term and $-W_f$ is the same type of term for displacements of the crack surface). Hence

$$W-W_f = 2(U-U_{init}) = 2W_e \qquad (12)$$

which rearranges to $(W-W_f) - W_e = W_e$ and explains why the right sides of (7) and (10) must agree.

However, in his (A21), Dienes writes $\delta W = 2\delta W_e$, whereas we see from (12) above that instead one should write $\delta W - \delta W_f = 2\delta W_e$. The mistake is in Dienes' (A19),

$$\delta W = \delta c \int_S \sigma_{ij} \partial u_i/\partial c \, n_j dS$$

$$= \delta [\int_S \sigma_{ij} u_i n_j dS] \qquad (13)$$

(the latter since $\sigma_{ij}n_j$ is constant on S), followed by his (A20) implying that

$$\delta W_e = \delta [\frac{1}{2}\int_S \sigma_{ij} u_i n_j dS] \qquad (14)$$

and leading him to conclude that $\delta W = 2\delta W_e$. The problem is that (13) is a proper expression of the work of external loadings only when S denotes the external surface of the body, whereas (14) is a proper expression of the Clapeyron relation only when S denotes (as elsewhere in his Appendix A) the sum of the external surface S_o plus the surfaces S_i of the crack. If we adopt the latter meaning of S, so that (14) is correct, then the correct form of (13) is

$$\delta W = \delta [\int_{S_o} \sigma_{ij} u_i n_j dS]$$

$$= \delta [\int_S \sigma_{ij} u_i n_j dS] + \delta W_f \qquad (15)$$

where S_o denotes the external surface and we recognize that $-\delta W_f$ is the corresponding work integral carried out over the part S_i of S representing the crack surfaces. Thus (15), which is a corrected reformulation of Dienes' (A19), in combination with (14) verifies that

$$\delta W - \delta W_f = 2\delta W_e \qquad (16)$$

as already stated above in (12). When (16) and (10) for W_e are used to evaluate the instability criterion, one gets (9). When Dienes' incorrect (A21), $\delta W = 2\delta W_e$, and (10) for W_e are used, together with the expression for δW_f calculated by Dienes as his (10), to evaluate the instability criterion one gets the incorrect result given as (1).

Three further comments seem to be in order. First, the extension of elastic-brittle crack mechanics to cases in which uniform frictional resistive stress acts on the crack surfaces has been considered previously by Palmer and Rice [1973] as a limiting case of what has come to be known as slip-weakening models for fracture extension. That work, further reviewed and developed by Rice [1980], shows that the same energy release rate G per unit new crack area as would be associated with loading $\sigma-\tau$ on a stress-free crack is what should be equated to the critical energy release rate (here taken as 2γ) for fracture advance. The result is equally obvious from the facts that the stress intensity factors of the crack border elastic singularity are proportional to $\sigma-\tau$ and that the energy release rate G is expressible in terms of stress intensity factors. Such considerations are consistent with (9) but not with (1).

Second, the local energy release rate G is not constant around the front of a shear-loaded circular crack, and thus the assumption of crack advance in a continuing circular shape is not strictly compatible with the assumption of a constant critical fracture energy, 2γ. Rather, the slip distribution described by (3) causes simultaneous mode II and mode III shear conditions at each point. For shear loading in the x direction on a circular crack lying in the xy plane the mode II and mode III stress intensity factors are extracted in a standard way [e.g., Paris and Sih, 1965] from the form of the slip distribution near the tip and are

$$K_{II} = \frac{4}{(2-\nu)\pi} (\sigma-\tau)\sqrt{\pi c}\,\frac{x}{c}$$

$$K_{III} = -\frac{4(1-\nu)}{(2-\nu)\pi} (\sigma-\tau)\sqrt{\pi c}\,\frac{y}{c} \qquad (17)$$

where $x^2+y^2 = c^2$ (the latter corrects equation (5.13) of Rice [1980] where the $1-\nu$ factor has been deleted). Thus the local energy release rate, calculated from the well-known Irwin relation [Irwin, 1957; Paris and Sih, 1965], is

$$G = \frac{1-\nu}{2\mu}K_{II}^2 + \frac{1}{2\mu}K_{III}^2$$

$$= \frac{8(1-\nu)c}{\pi\mu(2-\nu)^2} (\sigma-\tau)^2 (1-\nu\frac{y^2}{c^2}) \qquad (18)$$

which differs from constancy due to the last term, varying from 1 to $1-\nu$ as one moves around the crack. The instability criterion reported as (9) is reproduced by setting the average value of G around the crack equal to 2γ, which is consistent with the Dienes assumption that the crack advances in a circular shape. On the other hand, the crack extension criterion (local $G = 2\gamma$) is first met at segments of crack front along the x axis when

$$(\sigma-\tau)^2 = \pi(2-\nu)^2\gamma\mu/4(1-\nu)c \tag{19}$$

which is smaller than the right side of (9) by the factor $(1-\nu/2)$, and crack extension into a noncircular shape first commences at those segments.

Third, it might be noted that the relation between elastic fields in cracked bodies and energy changes in crack extension is succinctly summarized once and for all in the Irwin relation between G and the stress intensity factors, reproduced here as the first part of (18) (and to which $(1-\nu)K_I^2/2\mu$ can be added to include mode I tension also). This relation would seem to cover all cases and certainly suffices for the problem under discussion, as the previous paragraph shows. There is no need to begin ab initio in each case with elaborate energy calculations as presented by Dienes and discussed earlier here. Besides, the history of the subject shows that it is quite easy to go wrong in those calculations, in a distinguished tradition which (as Dienes mentions) began with Griffith.

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