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# Elastic Wave Emission from Damage Processes

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The theory of elastic wave emission (i.e., acoustic emission; AE) from damage processes such as slip and microcracking is discussed. Analogous developments in the literature on earthquake seismology and dynamic dislocation theory are noted and utilized. A general representation of the displacement field of an AE event is given in terms of the double-couple response to a distribution of "moment density tensor" in the source region. Results are specialized to a point source model and to a general far-field analysis of outgoing elastic waves, and conditions for validity of such representations and their low-frequency specializations are noted. Emitted wave fields are compared for tensile opening and slip events, and procedures which might enable the approximate determination of the size or area increase of tensile microcracks are discussed.

**KEY WORDS:** Acoustic emission; slip; microcracking; deformations; displacement field; elastic waves; NDE.

## 1. INTRODUCTION

Acoustic emission (AE) is concerned with the detection of elastic waves generated by what might generically be termed "damage" processes in stressed solids. These processes may consist of various types of inelastic deformations (slip, twinning, phase transformations) and microcracking. In order to advance the opportunities for quantitative study of damage processes by AE, it is important to have available the relation of processes in the source region (e.g., the location, spatial extent, orientation, and time dependence of a slip or cracking event) to the resulting elastic wave field. The presentation of such relations, especially for microcracking and slip processes, is the concern of this paper.

The relation of source processes to elastic wave fields is, of course, a major concern of earthquake seismology. Accordingly, major references for the present study are provided by analyses of wave generation by seismic sources; this literature has been summarized recently in portions of a book on quantitative seismology by Aki and Richards[(2)] and in a review on the mechanics of earthquake rupture by Rice.[(21)] While no attempt is made here to comprehensively review the relevant literature, a brief summary of key papers should perhaps begin with the adaption to shear faults by Vvedenskaya[(26)] of Nabarro's[(19)] solution for a dynamically introduced dislocation, with DeHoop's[(7)] development of an elastodynamic representation theorem for radiation from surfaces of displacement discontinuity, and with the refinement of the theory of double-couple representations of seismic sources by Burridge and Knopoff.[(5)] Other notable contributions are the development of the seismic moment parameter by Maruyama[(17)] and Aki[(1)] for characterizing far-fields of sources, its generalization by Kostrov,[(12, 13)] who noted that a second-rank moment tensor characterized the far-field for general sources [see also ref. (3)], and the application of the general formulations to analyze radiation from spreading dislocations by Haskell[(9, 10)] and from spreading cracks by

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Richards[22, 23] and Madariaga.[14, 15] Brune et al.[4] have recently summarized work on inference of source size from high-frequency properties of far-field spectra.

These or analogous elastodynamic developments from crystal dislocation theory [e.g., the work of Mura[18]] have been used in developing the theory of AE. A noteworthy paper is that of Malén and Bolin,[16] and subsequent work has been reported by Simmons and Clough,[24, 25] Hsu et al.,[11] Pao,[20] and Wadley et al.[27] These works are focused on a point-source model and hence eliminate information on propagation through the source region (such information may be of limited relevance for typical materials, however, because of problems with high-frequency signal propagation through microscale heterogeneities such as grains). Also, the relations of source parameters such as microcrack size and orientation to properties of the emitted wave fields have not yet been very fully documented. The utility of results of this type is of course dependent on existing practical limitations on the detectability of AE signals, but it seems advisable to have available the principal features of results on how AE fields are related to processes of damage. For slip processes key results can be read off with little alteration from the seismic source literature, whereas for cracking processes some new results are derived here.

## 2. SOURCE REPRESENTATION

This section follows closely the presentation by Rice.[21] The equations governing displacements $\mathbf{u}$ of a continuum of mass density $\rho$ are

$$\partial\sigma_{\alpha\beta}/\partial x_\alpha + f_\beta = \rho\partial^2 u_\beta/\partial t^2, \tag{1}$$

and for linear elastic response, stress $\boldsymbol{\sigma}$ is related to strain $\boldsymbol{\varepsilon}$ by a modulus tensor $\mathbf{C}$ such that

$$\sigma_{\alpha\beta} = C_{\alpha\beta\gamma\delta}\varepsilon_{\gamma\delta},$$
$$\text{where } 2\varepsilon_{\gamma\delta} = \partial u_\gamma/\partial x_\delta + \partial u_\delta/\partial x_\gamma. \tag{2}$$

The tensor $\mathbf{C}$ is taken to be that for a homogeneous body or at least one with smoothly varying properties on the macroscale; microscale heterogeneities such as individual grains are ignored, which implies a limitation of results to wavelengths that are large compared to the grain size. The displacement field generated by an arbitrary distribution of body force $\mathbf{f}$ throughout some volume $V$ is written as

$$u_\nu(\mathbf{x}, t) = \int_{-\infty}^{t}\int_V G_{\nu\beta}(\mathbf{x}, \mathbf{x}', t-t') f_\beta(\mathbf{x}', t')\, d^3\mathbf{x}'\, dt', \tag{3}$$

which defines the Green's function $G_{\nu\beta}(\mathbf{x}, \mathbf{x}', t)$ for the medium.

Disturbances associated with the alteration of matter (which in degenerate cases may include slip and/or crack opening, and are collectively called damage here) can be regarded as being generated by a "transformation" strain $\boldsymbol{\varepsilon}^{\mathrm{T}}$. This is defined so that $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ satisfy

$$\sigma_{\alpha\beta} = C_{\alpha\beta\gamma\delta}\left(\varepsilon_{\gamma\delta} - \varepsilon_{\gamma\delta}{}^{\mathrm{T}}\right) \tag{4}$$

throughout the source region, where $\mathbf{C}$ is the same modulus tensor as existed before the damage process. In the special cases of concern here for which the source process involves the generation of a displacement discontinuity on a discrete surface $S$ (a surface of slip or cracking), but elastic behavior elsewhere, $\boldsymbol{\varepsilon}^{\mathrm{T}}$ is Dirac singular on $S$. In particular, if the sides of $S$ are labeled $+$ and $-$, and if $\mathbf{n}$ is the normal to $S$ directed from $-$ to $+$ and $\Delta\mathbf{u}$ is defined on $S$ as $\mathbf{u}^+ - \mathbf{u}^-$, then for any small volume $\delta V$ intersected by $\delta S$ of surface,

$$\int_{\delta V} \varepsilon_{\alpha\beta}{}^{\mathrm{T}} d^3x = \tfrac{1}{2}\left(n_\alpha \Delta u_\beta + n_\beta \Delta u_\alpha\right)\delta S. \tag{5}$$

Hence, for surface discontinuities, one writes

$$\varepsilon_{\alpha\beta}{}^{\mathrm{T}} = \tfrac{1}{2}\left(n_\alpha \Delta u_\beta + n_\beta \Delta u_\alpha\right)\delta_{\mathrm{D}}(S), \tag{6}$$

where $\delta_{\mathrm{D}}(S)$ is a surface Dirac function, converting any volume integral over a region intersected by some part of $S$ to a surface integral over that part of $S$.

By using Eq. (4) in Eq. (1) and identifying an effective body force, the displacement field generated by an arbitrary distribution of $\boldsymbol{\varepsilon}^{\mathrm{T}}$ within some region $V$ can be written as

$$u_\nu(\mathbf{x}, t) = \tfrac{1}{2}\int_V\int_{-\infty}^{t} H_{\nu\alpha\beta}(\mathbf{x}, \mathbf{x}', t-t')$$
$$\times m_{\alpha\beta}(\mathbf{x}', t')\, dt'\, d^3\mathbf{x}'. \tag{7}$$

Here $\mathbf{m}$ is the (seismic) moment density tensor, namely, the symmetric second-rank tensor defined by

$$m_{\alpha\beta} = C_{\alpha\beta\gamma\delta}\varepsilon_{\gamma\delta}{}^{\mathrm{T}} \equiv C_{\alpha\beta\gamma\delta}\varepsilon_{\gamma\delta} - \sigma_{\alpha\beta}, \tag{8}$$

and

$$H_{\nu\alpha\beta}(\mathbf{x},\mathbf{x}',t) = \partial G_{\nu\beta}(\mathbf{x},\mathbf{x}',t)/\partial x'_\alpha + \partial G_{\nu\alpha}(\mathbf{x},\mathbf{x}',t)/\partial x'_\beta \tag{9}$$

is the response $u_\nu(\mathbf{x}, t)$ to a "double couple without moment" exerted at $\mathbf{x}'$ at time $t=0$. Such a double couple is generated by a pair of impulsive force dipoles; e.g., $\partial G_{\nu\beta}/\partial x'_\alpha$ is the response $u_\nu$ generated, in the limit $h \to 0$, by a pair of oppositely directed point impulses of magnitude $1/h$, one acting in the negative $\beta$ direction at $\mathbf{x}'$ and the other in the positive $\beta$ direction at a point removed by distance $h$ in the $\alpha$ direction from $\mathbf{x}'$.

Defining the Fourier transform on time of a function $f(\mathbf{x}, t)$ by

$$\tilde{f}(\mathbf{x},\omega) = \int_{-\infty}^{+\infty} f(\mathbf{x},t)\, e^{-i\omega t}\, dt, \tag{10}$$

and recalling that time convolutions of functions transform to products, one has the frequency version of Eq. (7) as

$$\tilde{u}_\nu(\mathbf{x},\omega) = \tfrac{1}{2}\int_V \tilde{H}_{\nu\alpha\beta}(\mathbf{x},\mathbf{x}',\omega)\,\tilde{m}_{\alpha\beta}(\mathbf{x}',\omega)\, d^3\mathbf{x}'. \tag{11}$$

It is convenient in using this representation to observe that

$$\tilde{\dot{m}}_{\alpha\beta}(\mathbf{x},\omega) = i\omega\tilde{m}_{\alpha\beta}(\mathbf{x},\omega), \tag{12}$$

where $\dot{m}_{\alpha\beta}(\mathbf{x}, t)$ is the time rate of $m_{\alpha\beta}(\mathbf{x}, t)$, so that the expression can be written as

$$\tilde{u}_\nu(\mathbf{x},\omega) = \tfrac{1}{2}\int_V \left[(1/i\omega)\tilde{H}_{\nu\alpha\beta}(\mathbf{x},\mathbf{x}',\omega)\right]\tilde{\dot{m}}_{\alpha\beta}(\mathbf{x}',\omega)\, d^3\mathbf{x}'. \tag{13}$$

Here the quantity in brackets is the transform of

$$\int_{0-}^{t} H_{\nu\alpha\beta}(\mathbf{x},\mathbf{x}',\tau)\, d\tau,$$

which is the response to double couples without moments formed from forces which are suddenly applied and held constant for subsequent time, rather than from impulses. The representation of Eq. (13) is more convenient than Eq. (11), since the limit of the bracketed term in Eq. (13) defines a nonzero function of position in the limit $\omega \to 0$.

## 3. POINT-SOURCE MODEL

Let the origin of coordinates be chosen somewhere in or very near to the source region. If the spatial extent of the source is small compared to the distance $r_0(=|\mathbf{x}|)$ to the receiver position, Eq. (13) is suggestive of a "point-source" model in which the resulting field is written as

$$\tilde{u}_\nu(\mathbf{x},\omega) \simeq \tfrac{1}{2}\left[(1/i\omega)\tilde{H}_{\nu\alpha\beta}(\mathbf{x},o,\omega)\right]\tilde{\dot{M}}_{\alpha\beta}(\omega), \tag{14}$$

where

$$M_{\alpha\beta}(t) = \int_V m_{\alpha\beta}(\mathbf{x},t)\, d^3\mathbf{x} = \int_V C_{\alpha\beta\gamma\delta}\varepsilon_{\gamma\delta}{}^{\mathrm{T}}(\mathbf{x},t)\, d^3\mathbf{x} \tag{15}$$

for general sources or, in the case of planar discontinuities,

$$M_{\alpha\beta}(t) = \int_S C_{\alpha\beta\gamma\delta} n_\gamma(\mathbf{x})\Delta u_\delta(\mathbf{x},t)\, d^2\mathbf{x}. \tag{16}$$

This quantity $M_{\alpha\beta}$ is referred to as the (seismic) moment tensor of the source.

It is important to have a clear understanding of the approximation involved in Eq. (14). Obviously, it is assumed that $r_0/a \gg 1$, where $a$ is some typical radius of the source region. There is, however, also a frequency restriction involved in using Eq. (14), in that wavelengths involved must be large compared to $a$. This seems not always to be well understood but will be clearer when specific forms of Eq. (7) or Eq. (13) for an unbounded isotropic solid are considered. Essentially, the high-frequency portions of the radiated signal are sensitive to the spatial distribution of origins in the source zone from which disturbances emanate. These origins have a different set of propagation times to a receiver at, say, reception point $r_0\boldsymbol{\gamma}'$

than at another reception point, $r_0\gamma''$. Hence, high-frequency information, involving periods comparable to differences in these propagation times, is different for the receiver at $r_0\gamma'$ than at $r_0\gamma''$. Looking ahead to the isotropic results, the essential requirement for independence of the receiver orientation, and hence validity of Eq. (14), is that for disturbances spreading at speed $c$, the frequency $\omega$ be low enough that

$$\int_V \tilde{\dot{m}}_{\alpha\beta}(\mathbf{x}', \omega) e^{i\gamma\cdot\mathbf{x}'\omega/c} d^3\mathbf{x}'$$

$$\simeq \int_V \tilde{\dot{m}}_{\alpha\beta}(\mathbf{x}', \omega)\, d^3\mathbf{x}' \equiv \tilde{\dot{M}}_{\alpha\beta}(\omega). \quad (17)$$

This condition will be met for all $\gamma$ if $e^{i\omega a/c} \simeq 1$ or $\omega a/c \ll 1$. The corresponding approximation in the time domain is that

$$\int_V \dot{m}_{\alpha\beta}(\mathbf{x}', t+\gamma\cdot\mathbf{x}'/c)\, d^3\mathbf{x}'$$

$$\simeq \int_V \dot{m}_{\alpha\beta}(\mathbf{x}', t)\, d^3\mathbf{x}' \equiv \dot{M}_{\alpha\beta}(t), \quad (18)$$

which means that time differences should not be resolved finer than periods of order $a/c$.

Suppose that transformations $\boldsymbol{\varepsilon}^{\mathrm{T}}$, or displacement discontinuities $\Delta\mathbf{u}$, in the source region have effectively reached their long-time (static) values at a time $t_{\mathrm{r}}$. Then for sufficiently low $\omega$, such that $e^{i\omega t_{\mathrm{r}}} \simeq 1$ (i.e., $\omega \ll 1/t_{\mathrm{r}}$),

$$\tilde{\dot{M}}_{\alpha\beta}(\omega) \equiv \int_0^{\infty} \dot{M}_{\alpha\beta}(t) e^{-i\omega t}\, dt$$

$$\simeq \int_0^{t_{\mathrm{r}}} \dot{M}_{\alpha\beta}(t)\, dt = M_{\alpha\beta}(t_{\mathrm{r}}). \quad (19)$$

Hence for frequencies at which $e^{i\omega t_{\mathrm{r}}} \simeq 1$, the point-source model reduces to

$$\tilde{u}_{\nu}(\mathbf{x}, \omega) = \tfrac{1}{2}\left[(1/i\omega)\tilde{H}_{\nu\alpha\beta}(\mathbf{x}, o, \omega)\right] M_{\alpha\beta}(t_{\mathrm{r}}). \quad (20)$$

This shows that it is the moment $M_{\alpha\beta}(t_{\mathrm{r}})$ at the completion of the source process which governs the low-frequency range of elastic wave emission.

From the results presented it is clear that at low frequencies (in the sense $e^{i\omega a/c} \simeq 1$), the basic observable quantities for a small AE source are the six functions $M_{\alpha\beta}(t)$ of Eqs. (15) and (16), low-pass filtered in the sense of Eqs. (17) and (18). Further, at what are generally yet lower frequencies (i.e., $e^{i\omega t_{\mathrm{r}}} \simeq 1$), the observable quantities reduce to the six constants $M_{\alpha\beta}(t_{\mathrm{r}})$ corresponding to the static state in the source region at completion of the damage event.

Aki and Richards[2] use the representation of Eq. (14) and solve for the Rayleigh (and Love) wave radiation from an arbitrary point source in a half-space (Chap. 7). This solution may have relevance to particular experimental situations for detection of AE signals.

Another approach for which the point-source model is useful is for an examination of the excitation of normal modes of the body in which the AE event takes place. The solution can be read off from Aki and Richards[2] (Chap. 8). Let $u^{(k)}(\mathbf{x})$, $k=1,2,\ldots,$ denote the normal modes and suppose these are normalized so that

$$\int_{\text{body}} \rho(\mathbf{x}) u_{\alpha}^{(j)}(\mathbf{x}) u_{\alpha}^{(k)}(\mathbf{x})\, d^3\mathbf{x} = \delta_{jk}. \quad (21)$$

Then the solution for a step-function point source $M_{\alpha\beta}$ [this corresponds to using $M_{\alpha\beta}(t_{\mathrm{r}})$ and ignoring information in oscillation frequencies higher than those for which $e^{i\omega t_{\mathrm{r}}} \simeq 1$] applied at $t=0$ is

$$\mathbf{u}(\mathbf{x}, t) = \sum_k M_{\alpha\beta} \varepsilon_{\alpha\beta}{}^{(k)} \mathbf{u}^{(k)}(\mathbf{x}) \times \{1-\cos\omega^{(k)} t\}/\{\omega^{(k)}\}^2, \quad (22)$$

where $\varepsilon_{\alpha\beta}{}^{(k)}$ is the strain in the $k$th mode at the place of the point source. For linear damping without coupling between modes, the time-dependent terms become (approximately, for light damping)

$$1-\exp\left(-\omega^{(k)} t/2Q^{(k)}\right)\cos\omega^{(k)} t,$$

where $Q^{(k)}$ is the quality factor for damping in the $k$th mode. This form shows also how the long-time static field evolves.

## 4. FAR-FIELD RESULTS FOR ISOTROPIC HOMOGENEOUS BODIES

Here results are given for the displacement field at points distant from the source in an otherwise isotropic and homogeneous material. The results presented are for outgoing waves from the source region, calculated as if the source resided in an unbounded body. They apply until wave reflections from boundaries intervene, but do not include the reflection and wave-guide effects associated with finite regions.

In this case $\mathbf{u}$ can be written as a sum of dilational ($\mathbf{u}^d$) and shear ($\mathbf{u}^s$) contributions,

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{u}^d(\mathbf{x}, t) + \mathbf{u}^s(\mathbf{x}, t). \tag{23}$$

These are given by [e.g., ref. (21) or equivalent expressions in ref. (2)]

$$\begin{aligned} u_\nu^d(\mathbf{x}, t) &= \partial^3 \mu_{\alpha\beta}(\mathbf{x}, t;\, c_d)/\partial x_\nu \partial x_\alpha \partial x_\beta, \\ u_\nu^s(\mathbf{x}, t) &= \nabla^2[\partial \mu_{\nu\beta}(\mathbf{x}, t;\, c_s)/\partial x_\beta] \\ &\quad - \partial^3 \mu_{\alpha\beta}(\mathbf{x}, t;\, c_s)/\partial x_\nu \partial x_\alpha \partial x_\beta, \end{aligned} \tag{24}$$

where $\mu_{\alpha\beta}(\mathbf{x}, t;\, c)$ is defined by

$$\mu_{\alpha\beta}(\mathbf{x}, t;\, c) = \int_V \int_{-\infty}^{t} \frac{\{|\mathbf{x}-\mathbf{x}'| - c(t-t')\}\, U\{c(t-t') - |\mathbf{x}-\mathbf{x}'|\}}{4\pi\rho c|\mathbf{x}-\mathbf{x}'|} \times m_{\alpha\beta}(\mathbf{x}', t')\, dt'\, d^3\mathbf{x}' \tag{25}$$

and where $U\{\ldots\}$ is the unit step function; here $c_d$ is the dilational (or longitudinal) wave speed and $c_s$ is the shear (or transverse) wave speed.

Corresponding results may also be given in the frequency domain by writing $\tilde{\mathbf{u}}$ in place of $\mathbf{u}$ and $\tilde{\boldsymbol{\mu}}$ in place of $\boldsymbol{\mu}$ in Eqs. (24). The expression for $\tilde{\boldsymbol{\mu}}(\mathbf{x}, \omega;\, c)$ is

$$\tilde{\mu}_{\alpha\beta}(\mathbf{x}, \omega;\, c) = \int_V \frac{\exp\{-i\omega|\mathbf{x}-\mathbf{x}'|/c\}}{4\pi\rho\omega^2|\mathbf{x}-\mathbf{x}'|} \times \tilde{m}_{\alpha\beta}(\mathbf{x}', \omega)\, d^3\mathbf{x}'. \tag{26}$$

To obtain the "far-field" displacement distribution, it is now assumed that the receiver point at $x = r_0\gamma$ is far enough removed that

$$r_0 \gg a \qquad \text{and} \qquad r_0 \gg c/\omega. \tag{27}$$

The far-field approximation should not be confused with the point-source approximation, which assumes that

$$r_0 \gg a \qquad \text{and} \qquad c/\omega \gg a, \tag{28}$$

although it is obvious from these inequalities that for sufficiently small $a/r_0$, there will be a range of (low) frequencies where both sets of inequalities are satisfied simultaneously. The calculation of the far-field is simplest in the frequency domain and begins with the recognition that

$$|\mathbf{x}-\mathbf{x}'| = (r_0 - \boldsymbol{\gamma}\cdot\mathbf{x}')[1 + O(\boldsymbol{\gamma}\cdot\mathbf{x}'/r_0)] \tag{29}$$

and that [see Eqs. (24) and (26)]

$$\begin{aligned} &\partial^3[\exp\{-i\omega|\mathbf{x}-\mathbf{x}'|/c\}/|\mathbf{x}-\mathbf{x}'|]/\partial x_\nu \partial x_\alpha \partial x_\beta \\ &\quad = (i\omega^3/r_0 c^3)\gamma_\nu\gamma_\alpha\gamma_\beta \exp\{-i\omega r_0/c + i\omega\boldsymbol{\gamma}\cdot\mathbf{x}'/c\} \\ &\quad\quad \times [1 + O(\boldsymbol{\gamma}\cdot\mathbf{x}'/r_0,\, c/\omega r_0)]. \end{aligned} \tag{30}$$

The far-field approximation is such that the bracketed terms in Eqs. (29) and (30) reduce to unity and, after a little calculation, one has results in the frequency domain in the form

$$\begin{aligned} \tilde{u}_\nu^d(\gamma r_0, \omega) &= \frac{e^{-i\omega r_0/c_d}}{4\pi\rho c_d^3 r_0}\gamma_\nu\gamma_\alpha\gamma_\beta \times \int_V \tilde{m}_{\alpha\beta}(\mathbf{x}', \omega)\, e^{i\omega\boldsymbol{\gamma}\cdot\mathbf{x}'/c_d} d^3\mathbf{x}', \\ \tilde{u}_\nu^s(\gamma r_0, \omega) &= \frac{e^{-i\omega r_0/c_s}}{4\pi\rho c_s^3 r_0}\{\tfrac{1}{2}(\delta_{\nu\alpha}\gamma_\beta + \delta_{\nu\beta}\gamma_\alpha) - \gamma_\nu\gamma_\alpha\gamma_\beta\} \times \int_V \tilde{m}_{\alpha\beta}(\mathbf{x}', \omega)\, e^{i\omega\boldsymbol{\gamma}\cdot\mathbf{x}'/c_s} d^3\mathbf{x}'. \end{aligned} \tag{31}$$

Corresponding results in the time domain can be written at once by observing that

$$e^{-i\omega r_0/c} \int_V \tilde{m}_{\alpha\beta}(\mathbf{x}', \omega)\, e^{i\omega\boldsymbol{\gamma}\cdot\mathbf{x}'/c} d^3\mathbf{x}' \tag{32}$$

is the transform of

$$\int_V \dot{m}_{\alpha\beta}(\mathbf{x}', t - r_0/c + \boldsymbol{\gamma}\cdot\mathbf{x}'/c)\, d^3\mathbf{x}'; \tag{33}$$

the latter form has an obvious interpretation in terms of a retardation of time between the source process and its reception.

The expressions in Eqs. (31), transformed to the time domain as in Eqs. (32) and (33), make evident the origin of conditions (17) and (18) as requirements for validity of a point-source model.

For isotropic materials

$$m_{\alpha\beta}=C_{\alpha\beta\gamma\delta}\varepsilon_{\gamma\delta}{}^{T}=\Lambda\delta_{\alpha\beta}\varepsilon_{\gamma\gamma}{}^{T}+2G\varepsilon_{\alpha\beta}{}^{T},$$
$$=\rho\left(c_d{}^2-2c_s{}^2\right)\delta_{\alpha\beta}\varepsilon_{\gamma\gamma}{}^{T}+2\rho c_s{}^2\varepsilon_{\alpha\beta}{}^{T}, \quad (34)$$

where $\Lambda$ and $G$ are the Lamé elastic constants. Hence for the particular case of a surface $S$ with normal $\mathbf{n}$ across which velocity discontinuities $\Delta\dot{\mathbf{u}}$ occur,

$$\int_V \tilde{m}_{\alpha\beta}(\mathbf{x},\omega)e^{i\omega\boldsymbol{\gamma}\cdot\mathbf{x}/c}d^3\mathbf{x}$$
$$=\int_S\left\{\rho\left(c_d{}^2-2c_s{}^2\right)\delta_{\alpha\beta}n_\gamma(\mathbf{x})\Delta\tilde{\dot{u}}_\gamma(\mathbf{x},\omega)\right.$$
$$\left.+\rho c_s{}^2\left[n_\alpha(\mathbf{x})\Delta\tilde{\dot{u}}_\beta(\mathbf{x},\omega)+n_\beta(\mathbf{x})\Delta\tilde{\dot{u}}_\alpha(\mathbf{x},\omega)\right]\right\}$$
$$\times e^{i\omega\boldsymbol{\gamma}\cdot\mathbf{x}/c}d^2\mathbf{x} \quad (35)$$

## 5. RESULTS FOR PLANAR SURFACES OF TENSILE OPENING AND SLIP

A general displacement discontinuity can be resolved into an opening displacement and into two tangential slip displacements. These cases are considered separately here for the special case in which $S$ is a flat surface ($\mathbf{n}$ constant).

Observing that for a displacement discontinuity $\Delta\mathbf{u}$ in the direction of any *fixed* unit vector $\mathbf{v}$, one may write

$$\Delta\mathbf{u}(\mathbf{x},t)=\mathbf{v}\Delta u(\mathbf{x},t), \quad (36)$$

where $\Delta u$ is the magnitude of the discontinuity, Eqs. (31)–(33) and (35) lead to

$$\mathbf{u}^{d}(\boldsymbol{\gamma}r_0,t)=\frac{\boldsymbol{\gamma}\Omega(t;\,c_d)}{4\pi r_0c_d}$$
$$\times\left\{\mathbf{n}\cdot\mathbf{v}+2\left(c_s^2/c_d^2\right)(\mathbf{n}\cdot\boldsymbol{\gamma}\,\boldsymbol{\gamma}\cdot\mathbf{v}-\mathbf{n}\cdot\mathbf{v})\right\},$$
$$\mathbf{u}^{s}(\boldsymbol{\gamma}r_0,t)=\frac{\Omega(t;\,c_s)}{4\pi r_0c_s}$$
$$\times\{(\boldsymbol{\gamma}\cdot\mathbf{v})\mathbf{n}+(\boldsymbol{\gamma}\cdot\mathbf{n})\mathbf{v}-2(\mathbf{n}\cdot\boldsymbol{\gamma}\,\boldsymbol{\gamma}\cdot\mathbf{v})\boldsymbol{\gamma}\}, \quad (37)$$

where

$$\Omega(t;\,c)=\int_S\Delta\dot{u}(\mathbf{x},t-r_0/c+\boldsymbol{\gamma}\cdot\mathbf{x}/c)\,d^2\mathbf{x}. \quad (38)$$

Corresponding results in the frequency domain are obtained by observing that the transform of $\Omega(t;\,c)$ is

$$\tilde{\Omega}(\omega;\,c)=e^{-i\omega r_0/c}\int_S\Delta\tilde{\dot{u}}(\mathbf{x},\omega)e^{i\boldsymbol{\gamma}\cdot\mathbf{x}\omega/c}d^2\mathbf{x}. \quad (39)$$

One may observe that $\mathbf{u}^d$ has the direction of $\boldsymbol{\gamma}$ and that $\mathbf{u}^s$ is perpendicular to $\boldsymbol{\gamma}$.

To examine specific cases, suppose with reference to Fig. 1 that the axes are chosen so that $x_2$ has the direction of $\mathbf{n}$, i.e., perpendicular to $S$, and that $x_1$, $x_3$ lie in the plane of $S$. Then for angles $\phi,\theta$ as shown,

$$\gamma_1=\sin\phi\cos\theta,\qquad \gamma_2=\cos\phi,\qquad \gamma_3=\sin\phi\sin\theta. \quad (40)$$

*Tensile Opening.* For this case $\mathbf{v}=\mathbf{n}$, i.e., $\mathbf{v}$ has the direction of $x_2$. Hence

$$\mathbf{u}^{d}(r_0\boldsymbol{\gamma},t)=\frac{\boldsymbol{\gamma}\Omega(t;\,c_d)}{4\pi r_0c_d}\left\{1-\left(2c_s^2/c_d^2\right)\sin\phi\right\},$$
$$\mathbf{u}^{s}(r_0\boldsymbol{\gamma},t)=\frac{\boldsymbol{\lambda}\Omega(t;\,c_s)}{4\pi r_0c_s}\sin2\phi, \quad (41)$$

where

$$\boldsymbol{\lambda}=(\mathbf{n}-\boldsymbol{\gamma}\cos\phi)/\sin\phi \quad (42)$$

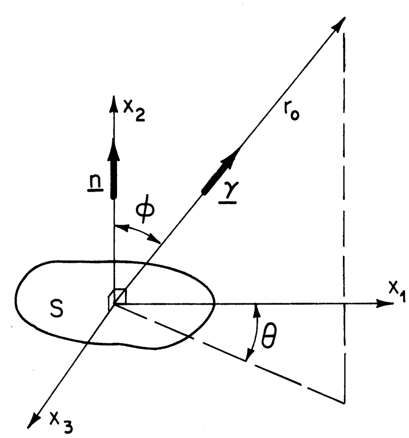


**Fig. 1.** A planar surface S of tensile opening and/or slip; notation for far-field analysis.

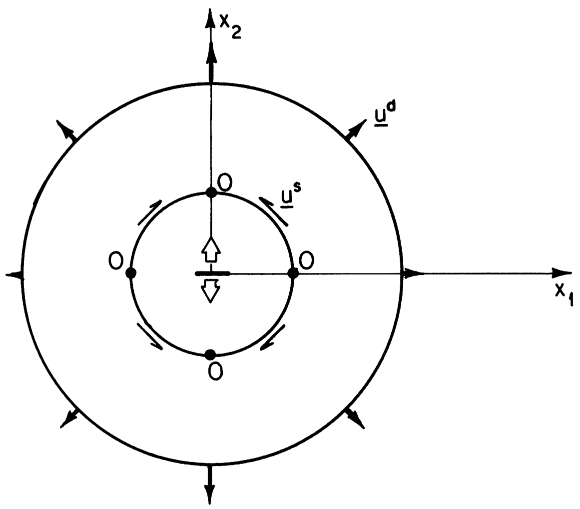


**Fig. 2.** Far-field dilational (d) and shear (s) displacement patterns for tensile opening.

is a unit vector which is tangent to great circles in the plane of $\mathbf{n}$ and $\gamma$, and which always has a positive projection onto the direction of $\mathbf{n}$ (i.e., $\boldsymbol{\lambda}$ is a unit vector in the direction of decreasing $\phi$). Since $2c_s^2/c_d^2=2G/(\Lambda+2G)$, the orientation term in the expression for $\mathbf{u}^d$ is always positive.

Note that not all of the orientation dependence is displayed explicitly in the above formulas; $\Omega$ as defined in Eqs. (38) and (39) also depends on $\gamma$, although this orientation dependence can be neglected in the low-frequency limit as discussed in connection with Eqs. (17) and (18).

The radiation patterns of Eqs. (41) are shown in Fig. 2. Zeros on the s-wave pattern denote nodes; there are no nodes on the d pattern. Both patterns shown are rotationally symmetric about the $x_2$ axis.

*Slip.* Now let $\mathbf{v}$ be in the $x_1$ direction. Eqs. (37) lead in this case to

$$\mathbf{u}^d(\gamma r_0,t)=\frac{\gamma c_s^2\Omega(t;c_d)}{4\pi r_0 c_d^3}\sin 2\phi\cos\theta,$$

$$\mathbf{u}^s(\gamma r_0,t)=\frac{\Omega(t;c_s)}{4\pi r_0 c_s}(-\cos 2\phi\cos\theta\boldsymbol{\lambda}+\cos\phi\sin\theta\boldsymbol{\mu}), \tag{43}$$

where $\boldsymbol{\lambda}$ is defined above and $\boldsymbol{\mu}$ is a unit vector in the direction of decreasing $\theta$.

The resulting radiation patterns are shown in Fig. 3; both contain nodes. As is well known for this case, the slip plane orientation cannot be determined uniquely from the far-field radiation patterns, but can only be reduced to two candidate directions at 90° with one another.

*Moments.* For the tensile opening,

$$M_{22}(t)=(\Lambda+2G)M_{33}(t)/\Lambda=(\Lambda+2G)M_{11}(t)/\Lambda$$

$$=(\Lambda+2G)\int_S \Delta u(\mathbf{x},t)\,d^2\mathbf{x} \tag{44}$$

are the only nonvanishing components of moment and, in circumstances for which the orientation-dependent parts of $\Omega$, Eq. (38), can be neglected (low-frequency limit),

$$\Omega(t;c)=\dot{M}_{22}(t-r_0/c)/(\Lambda+2G). \tag{45}$$

Similarly, for the slip case,

$$M_{12}(t)=M_{21}(t)=G\int_S \Delta u(\mathbf{x},t)\,d^2\mathbf{x} \tag{46}$$

are the nonvanishing components of moment and, in the same circumstances as above,

$$\Omega(t;c)=\dot{M}_{12}(t-r_0/c)/G. \tag{47}$$

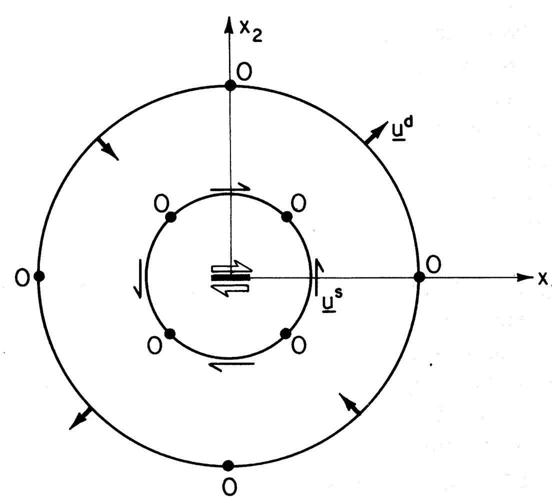


**Fig. 3.** Far-field patterns for slip.

*Comparison of Tensile Opening Versus Slip.* The main comparison of these two damage modes follows from a comparison of Figs. 2 and 3. The tensile opening radiation patterns are axisymmetric [at least to the neglect of "propagation" or "directivity" effects which occur at sufficiently high frequencies for the $\boldsymbol{\gamma}$ dependence of $\Omega$, Eqs. (38) and (39), to be nonnegligible] and thereby enable a unique determination of the *orientation* of the surface $S$ on which the opening occurs. However, no information is retained on the *shape* of $S$, at least in the low-frequency range. For slip events the radiation patterns are not axisymmetric and divide the unit sphere into four sectors formed by great circles connecting diametrically opposite nodes in Fig. 3. These sectors are differently located on the unit sphere for d versus s radiation. As commented, the patterns do not uniquely determine the slip plane normal, but only two directions, of which either one may be the normal and the other the slip direction.

The ratios of the maximum amplitudes of d and s waves are different for tensile opening versus slip. In the opening case this ratio is (Eq. (41)]

$$|\mathbf{u}^{\mathrm{d}}|_{\max}/|\mathbf{u}^{\mathrm{s}}|_{\max} = c_{\mathrm{s}}/c_{\mathrm{d}} \simeq 0.58, \qquad (48)$$

whereas for slip [Eq. (43)]

$$|\mathbf{u}^{\mathrm{d}}|_{\max}/|\mathbf{u}^{\mathrm{s}}|_{\max} = (c_{\mathrm{s}}/c_{\mathrm{d}})^3 \simeq 0.19. \qquad (49)$$

The numbers are for the case $\Lambda = G$, for which the Poisson ratio $\nu = \frac{1}{4}$.

*Propagation in the Source Region.* In principle, the high-frequency portions of the radiated field contain details of the space–time distribution of the damage event throughout the source region. The frequency range for which such effects are observable is that for which $\omega a/c$ is of order unity or larger. There may in some materials be a reasonable range of frequencies between $\omega a/c = 1$ and $\omega d/c = 1$, where $d$ is grain size and the latter equation refers to a cutoff frequency at which heterogeneities at the grain size make signal interpretation impossible. Obviously, for use of this range in quantitative AE, it is necessary that $a \gg d$. This condition will frequently not be met for microcracking processes.

The high-frequency spectrum of slip events and its relation to propagation processes in the source region is discussed by Aki and Richards[2] (Chaps. 14 and 15) as well as by Das and Aki,[6] Madariaga,[14,15] Rice,[21] and Richards.[22, 23] Such results are not pursued here except to note that from Eqs. (37), (41), and (43), when transformed to the frequency domain, the frequency content of the far-field radiation is determined by $\tilde{\Omega}(\omega;\, c)$ of Eq. (39). This expression can be rewritten as

$$\tilde{\Omega}(\omega;\, c) = e^{-i\omega r_0/c}\int_{-\infty}^{+\infty}\int_S \times \Delta\dot{u}(\mathbf{x}, t)\, e^{-i\omega t - i\mathbf{k}\cdot\mathbf{x}}\, d^2\mathbf{x}\, dt, \qquad (50)$$

where $\mathbf{k} = -\boldsymbol{\gamma}\omega/c$. The integrals define the full space–time Fourier transform of the velocity discontinuity, and observation of the far-field at orientation $\boldsymbol{\gamma}$ corresponds to sampling this space–time transform along the ray $\mathbf{k} = -\boldsymbol{\gamma}\omega/c$ in $\mathbf{k}-\omega$ space. [Since $|\boldsymbol{\gamma}| = 1$, this sampling, even if carried out for all orientations, cannot even in principle enable one to fully reconstruct the function $\Delta\dot{u}(\mathbf{x}, t)$. The spatial structure along $S$ of a given frequency component of $\Delta\dot{u}$ is resolvable only over a range of wave numbers $k_1, k_3$ that are smaller in magnitude than $\omega/c$; information involving shorter wavelengths is evidently not transmitted to the far-field.]

## 6. CRACKS

Suppose that a crack exists along a planar surface $A_0$ and, in a damage event, spreads to area $A(=A_0 + \Delta A)$ in the same plane. As a special case, $A_0$ may vanish. For simplicity, it is assumed that the crack plane is perpendicular to the tensile direction so that $\Delta\mathbf{u}$ consists only of tensile opening; more complicated cases of tensile opening in combination with shear are deferred to later treatment.

From earlier discussions it is clear that at sufficiently low frequencies $e^{i\omega t_r} \simeq 1$, the approximation of Eq. (19) applies, and [Eqs. (44), (39), and (45)]

$$\tilde{M}_{22}(\omega) \simeq M_{22}(t_{\mathrm{r}}) \simeq (\Lambda + 2G)\, e^{i\omega r_0/c}\tilde{\Omega}(\omega;\, c) \simeq (\Lambda + 2G)\Delta V, \qquad (51)$$

where $\Delta V$ is the change in volume displaced by the crack surfaces in going from one statical configuration to the other:

$$\Delta V = \int_A (\Delta u)_A^{\mathrm{static}} dA - \int_{A_0} (\Delta u)_{A_0}^{\mathrm{static}} dA_0. \qquad (52)$$

Hence it is the change in crack volume which is the basic observable quantity in the low-frequency limit, as noted by Wadley et. al.,[27] at least in the present case when the crack opens parallel to the tensile direction. In particular, the low-frequency amplitude spectra of the outgoing d and s waves are [Eqs. (41), transformed]

$$|\tilde{\mathbf{u}}^{\mathrm{d}}(\gamma r_0, \omega)| = (\Delta V/4\pi r_0 c_{\mathrm{d}})\left\{1 - 2(c_{\mathrm{s}}^2/c_{\mathrm{d}}^2)\sin^2\phi\right\},$$

$$|\tilde{\mathbf{u}}^{\mathrm{s}}(\gamma r_0, \omega)| = (\Delta V/4\pi r_0 c_{\mathrm{s}})|\sin 2\phi|, \qquad (53)$$

and the general point-source result for this case, Eq. (20), reduces to

$$\tilde{u}_\nu(\mathbf{x}, \omega) = \tfrac{1}{2}\Delta V\{(1/i\omega)[(\Lambda + 2G)H_{\nu 22}(\mathbf{x}, o, \omega) + \Lambda H_{\nu 11}(\mathbf{x}, o, \omega) + \Lambda H_{\nu 33}(\mathbf{x}, o, \omega)]\}. \qquad (54)$$

In view of the potential observability of $\Delta V$, it is useful to have estimates which give its quantitative relation to crack size or crack growth. The volume opening $V$ of a disk-shaped crack of radius $a$ subjected to tensile stress $\sigma$ can be calculated from the expression for the static crack surface opening [e.g., ref. (8)],

$$\Delta u = \{4\sigma(1-\nu)/\pi G\}(a^2 - r^2)^{\frac{1}{2}} \qquad (55)$$

where $r$ is the distance from the crack center, and hence

$$V = 2\pi\int_0^a \Delta u r\, dr = 8(1-\nu)\sigma a^3/3G. \qquad (56)$$

For new crack formation [$A_0 = 0$ in Eq. (52)], $\Delta V = V$ and hence an equivalent disk-shaped crack radius $a$ can be associated with an AE event. (The corresponding procedure for a slip event is less successful, since it cannot then be assumed that the drop $\sigma$ in shear stress transmitted across the slip surface is equal to the applied shear stress.)

Another perspective on the problem is provided by the recognition that $\sigma\Delta V/2$, with $\Delta V$ defined by Eq. (52), is the static energy reduction due to introducing a crack in a loaded elastic body (this energy loss goes partly to wave emission and partly to the work of fracture). However, the energy loss can also be calculated by growing the crack from the initial configuration ($A_0$) to the final configuration ($A$). A given infinitesimal step of this growth involves crack advance by amounts $\delta a(s)$ along the crack front, $\Gamma$, parameterized by arc length $s$ along it. Hence

$$\tfrac{1}{2}\sigma\delta V = \int_\Gamma g\delta a(s)\, ds, \qquad (57)$$

where $g$ is Irwin's elastic energy release rate, related to the stress intensity factor $K$ by

$$g = (1-\nu)K^2/2G. \qquad (58)$$

One may therefore write

$$\Delta V = 2\bar{g}\Delta A/\sigma, \qquad (59)$$

where $\bar{g}$ is the average $g$ that would be encountered in *statically* enlarging the crack area by $\Delta A$.

In some cases it would seem appropriate to assume that $\bar{g}$ is less than or equal to the macroscopic crack toughness $g_{\mathrm{Ic}}$ for the material (e.g., the microcracking takes place in the weaker regions of a material with statistically variable properties). In such case the $\Delta V$ inferred from the AE event provides a lower bound to the change in crack area, in the form

$$\Delta A \geqslant \sigma\Delta V/2g_{\mathrm{Ic}}. \qquad (60)$$

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