Int. J. Solids Structures Vol. 16, pp. 597–605
Pergamon Press Ltd., 1980. Printed in Great Britain

A NOTE ON SOME FEATURES OF THE THEORY OF LOCALIZATION OF DEFORMATION

J. R. RICE
Division of Engineering, Brown University, Providence, RI 02912, U.S.A.

and

J. W. RUDNICKI
Department of Theoretical and Applied Mechanics, University of Illinois, Urbana, IL 91125, U.S.A.

(*Received* 18 *June* 1979; *received for publication* 8 *September* 1979)

Abstract—This note examines two aspects of the theory which treats localization of deformation as a bifurcation from homogeneous deformation. The results are obtained for solids modelled as elastic–plastic and having smooth yield and plastic potential surfaces, but it is not required that inelastic strain increments be normal to the yield surface. First, it is demonstrated that discontinuous bifurcations, for which elastic unloading occurs outside the zone of incipient localization, first become possible at the point of continuous bifurcation, for which further plastic deformation is assumed to occur both inside and outside of the zone of localization. Second, we investigate an apparent paradox which arises in the rigid plastic limit of an elastic-plastic localization calculation if normality does not apply. This is resolved by consideration of the relative amounts of the bifurcation mode corresponding to elastic and to plastic deformation and it is demonstrated that even for very small amounts of elasticity, bifurcation modes which are inadmissible in the rigid plastic case become possible.

INTRODUCTION

An often-observed feature of the deformation of many solids is that the deformation becomes intensely concentrated in a narrow zone. Although such localization of deformation may often be considered to result from local inhomogeneities, stress concentrations or, more generally, from the onset of some physical mechanism which degrades abruptly the strength of the material, an alternative point of view is that this phenomenon may be explained as a bifurcation from a homogeneous (or smoothly varying) pattern of deformation. More specifically, this approach investigates whether the constitutive description of homogeneous deformation can admit a solution which is compatible with boundary conditions for further homogeneous deformation, but which corresponds to non-uniform deformation in a planar zone. The basic principles for this analysis were established by Hadamard[1] in his study of elastic stability and have been applied by Thomas[2] to rigid plastic solids, and by Hill[3] and Mandel[4] in studies of acceleration waves in elastic–plastic solids. In this latter case, localizations correspond to "stationary waves." Only more recently, however, have the consequences of this approach been examined for more specific forms of constitutive behavior. Beginning with the analysis by Rudnicki and Rice[5] of localization in pressure-sensitive, dilatant materials, several papers[6–8, 15, 16] have shown that predictions of localization are sensitive to details of the constitutive response, in particular to whether the material is idealized as having a smooth yield surface and as obeying plastic normality. Rice[8] has given a general review of localization of plastic deformation which includes examples of the results for a wide class of constitutive models and discussion of the inter-relation among conditions for localization, the general problem of uniqueness, and stability of acceleration waves.

This note examines two aspects of the theory of localization for rate-independent, elastic–plastic solids. We restrict attention to solids modelled as having smooth yield and plastic potential surfaces but we do not require that these coincide; that is, that inelastic strain increments be normal to the yield surface. Specifically, by these latter restrictions, we mean that the plastic strain rate is of fixed "direction" in an appropriate hyperspace, and that its magnitude is linear in stress rate provided that a condition for continued plastic deformation (vs elastic unloading) is met.

The first aspect of localization which we examine here is discontinuous bifurcations for which elastic unloading occurs outside the zone of localization while continued elastic-plastic deformation occurs within it. Although this is a mode of localization which seems to be observed frequently in experiments†, we will show that continuous bifurcations, for which further plastic loading occurs inside and outside of the zone of localization, provide the lower limit to the range of deformations for which discontinuous bifurcations can occur. This result is analogous to Shanley's[9] solution for plastic column buckling: The lowest bifurcation point corresponds to continued plastic flow throughout the column (except at an instantaneously non-deforming outermost fiber). This point is the lower limit of a range of discontinuous bifurcations, in which a finite portion of the column cross section unloads elastically. More generally, if the constitutive law is of a type which admits a rate potential (elastic–plastic solids having a smooth yield surface and satisfying normality), then it is possible to demonstrate[10, 11] that loss of uniqueness in the actual solid, which may be unloading elastically in some finite region, cannot precede loss of uniqueness in a "comparison solid"[10, 11], an identical body which responds to all deformations with the incremental moduli appropriate, in the actual solid, to plastic loading everywhere. However, the result which is derived here pertains even if normality does not apply.

The second aspect of localization which we examine here is motivated by an apparent paradox‡ which arises in the rigid plastic limit[8] of the elastic-plastic calculation of Rudnicki and Rice[5]. In this limit, localization appears to be possible at values of the plastic hardening modulus which are not admitted by the direct rigid-plastic analysis. The "paradox" is resolved by considering the magnitudes of the components of the bifurcation modes which correspond to elastic and to plastic deformation. We demonstrate that modes which appear to be possible in the rigid-plastic limit involve rigid components and hence, in fact, are non-admissible. This result reinforces the conclusion that the rigid plastic idealization is inadequate in many circumstances as a material model for localization calculations.

In the next section, we will briefly review the general theory of the approach to localization as a bifurcation from homogeneous deformation. Although, as mentioned earlier, the general principles have been enunciated by several authors[1–4], we will follow the presentation and notation of Rice[8].

OUTLINE OF GENERAL THEORY

Consider a homogeneous solid sustaining a uniform stress $\boldsymbol{\sigma}^\circ$ ($\boldsymbol{\sigma}$ is Cauchy stress). The response to a homogeneous velocity gradient field $(\partial \boldsymbol{v}/\partial \boldsymbol{x})^\circ$, where $\boldsymbol{v}$ is the velocity and $\boldsymbol{x}$ denotes the current position with components x_i in a fixed Cartesian coordinate system, is the homogeneous stress rate $\dot{\boldsymbol{\sigma}}^\circ$, which plainly satisfies the requisite quasi-static field equations. Conditions are sought for which the state of the solid allows the field equations to be satisfied for an alternate field

$$\partial \boldsymbol{v}/\partial \boldsymbol{x} = (\partial \boldsymbol{v}/\partial \boldsymbol{x})^\circ + \Delta(\partial \boldsymbol{v}/\partial \boldsymbol{x}) \tag{1}$$

in which $\Delta(\partial \boldsymbol{v}/\partial \boldsymbol{x})$ is a function only of distance across a planar band and vanishes outside the band. If the velocity is to be continuous at bifurcation, the compatibility condition[1–4]

$$\Delta(\partial \boldsymbol{v}/\partial \boldsymbol{x}) = \mathbf{g}\,\mathbf{n} \tag{2}$$

must be satisfied where $\mathbf{n}$ is the unit normal to the plane of the band, and the components of $\mathbf{g}$ are functions only of distance across the band $(\mathbf{n}\cdot\mathbf{x})$ and are zero outside.

Equilibrium must also be satisfied at the inception of bifurcation. This is expressed in rate form as

$$\frac{\partial}{\partial x_i}\left(\frac{\partial \sigma_{ij}}{\partial t}\right) = \frac{\partial}{\partial x_i}\left(\dot{\sigma}_{ij} - v_k \frac{\partial \sigma_{ij}}{\partial x_k}\right) = 0$$

†Consideration of this case was suggested in discussions on localization in rock by one of us (J.R.R.) with Dr. George Mandl and colleagues at the Shell Exploration and Production Laboratory, Rijswijk, Netherlands, August 1976.

‡This was pointed out in correspondence by post between J.R.R. and H. Lippmann (Lehrstuhl Fur Mechanik, Technische Universitat, Munich), 1976.

where the superposed dot denotes the material time rate. Because bifurcation is from the homogeneous stress state $\boldsymbol{\sigma} = \boldsymbol{\sigma}^\circ$, the rate equilibrium condition reduces at the instant considered to

$$\partial \dot{\sigma}_{ij} / \partial x_i = 0. \tag{3}$$

Furthermore, because of eqn (2), the stress rates also will be uniform outside the band and functions of only $\mathbf{n} \cdot \mathbf{x}$ inside. Consequently, eqn (3) requires that $\mathbf{n} \cdot \dot{\boldsymbol{\sigma}}$ have the same value inside and outside of the band or that

$$\mathbf{n} \cdot \Delta \dot{\boldsymbol{\sigma}} = 0 \quad \text{or} \quad n_i \Delta \dot{\sigma}_{ij} = 0 \tag{4}$$

where $\Delta \boldsymbol{\sigma} = \boldsymbol{\sigma} - \boldsymbol{\sigma}^\circ$. If instead, equilibrium is expressed in terms of the nominal or first Piola–Kirchoff stress tensor, the condition takes a form which, in view of (2), can be shown to be identical to that of eqn (4) [5].

Consider the class of materials whose constitutive behavior may be idealized as a piecewise-linear relation of the form

$$\overset{\nabla}{\boldsymbol{\sigma}} = \mathbf{L} : \mathbf{D} \quad \text{or} \quad \overset{\nabla}{\sigma}_{ij} = L_{ijkl} D_{kl}, \tag{5}$$

where $\mathbf{D} = \text{sym}\,(\partial \mathbf{v} / \partial \mathbf{x})$ and L_{ijkl} is symmetric with respect to the interchange of i and j and of k and l. The Jaumann co-rotational rate of stress (e.g. [12])

$$\overset{\nabla}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} + \boldsymbol{\sigma} \cdot \boldsymbol{\Omega} - \boldsymbol{\Omega} \cdot \boldsymbol{\sigma},$$

which is the stress rate computed by an observer who is rigidly rotating with the material element, has been used rather than $\dot{\boldsymbol{\sigma}}$ because the latter is not invariant to rigid spins; $\boldsymbol{\Omega}$ is the anti-symmetric part of $\partial \mathbf{v} / \partial \mathbf{x}$. In the present analysis, we will assume a smooth yield surface idealization for materials modelled as elastic–plastic, and consequently, the relation (5) has two branches which correspond to continued loading and to elastic unloading. More generally, studies based on microstructural mechanisms [5, 13, 14] have indicated that a vertex will form on the yield surface and that a relation such as (5) will have an infinite number of branches corresponding to "directions" of $\mathbf{D}$. Rudnicki and Rice [5] have shown that predictions of localization, at least for certain deformation states, are sensitive to whether the yield surface is idealized as smooth or as having a vertex.

If the constitutive behavior can be expressed in the form of eqn (5), the homogeneous field outside the band satisfies

$$\overset{\nabla}{\boldsymbol{\sigma}}{}^\circ = \mathbf{L}^\circ : \mathbf{D}^\circ$$

and the corresponding relation inside the band is

$$\overset{\nabla}{\boldsymbol{\sigma}} = \mathbf{L} : \mathbf{D},$$

since we wish to consider the possibility that the two zones correspond to different branches of constitutive response. Recognizing that the compatibility condition, eqn (2), expressed in terms of $\mathbf{D}$, is

$$\Delta \mathbf{D} = \mathbf{D} - \mathbf{D}^\circ = (\mathbf{gn} + \mathbf{ng})/2, \tag{6}$$

and applying eqn (4) yield

$$(n_i L_{ijkl} n_l + A_{jk}) g_k = n_i (\mathbf{L}^\circ - \mathbf{L})_{ijkl} D^\circ_{kl}, \tag{7}$$

where

$$2A_{ij} = -n_i(n_k\sigma_{kj}) + (n_p\sigma_{pq}n_q)\delta_{ij} + (n_k\sigma_{ki})n_j - \sigma_{ij}$$

is the term which arises due to the difference between $\overset{\nabla}{\boldsymbol{\sigma}}$ and $\dot{\boldsymbol{\sigma}}$. In the simplest case, the constitutive response remains continuous at the inception of localization, $\mathbf{L} = \mathbf{L}^\circ$, and the right-hand side of eqn (7) vanishes. Thus, the condition for localization is that a solution other than $\mathbf{g} \equiv \mathbf{0}$ exists:

$$\det[n_i L_{ijkl} n_l + A_{jk}] = 0. \tag{8}$$

DISCONTINUOUS BIFURCATIONS

A common observation in experiments is that the constitutive response at localization is not continuous; that is, the material outside the localized zone apparently does not continue loading, but rather unloads elastically. Typically, localization may begin in one portion of the body where conditions are locally favorable, and unloading outside the non-uniform zone will cause localization to accelerate through the remainder of the specimen. In this case, the right-hand side of eqn (7) is not zero. However, unless the condition (8) for continuous bifurcation is satisfied, the matrix $(\mathbf{n} \cdot \mathbf{L} \cdot \mathbf{n} + \mathbf{A})$ is invertible and a tentative non-trivial solution for $\mathbf{g}$ is

$$\mathbf{g} = (\mathbf{n} \cdot \mathbf{L} \cdot \mathbf{n} + \mathbf{A})^{-1} \cdot [\mathbf{n} \cdot (\mathbf{L}^\circ - \mathbf{L}) : \mathbf{D}^\circ]. \tag{9}$$

In order to determine the circumstances for which this solution is, in fact possible (i.e. corresponds to continued plastic loading in the localized zone when there is elastic unloading outside this zone), consider the following particular form of the constitutive rate relation (5):

$$\overset{\nabla}{\boldsymbol{\sigma}} = \mathbf{E} : \left[\mathbf{D} - \frac{1}{h}\mathbf{P}(\mathbf{Q} : \overset{\nabla}{\boldsymbol{\sigma}})\right], \tag{10}$$

where the notation of [8] has been used. $\mathbf{E}$ is the tensor of incremental elastic moduli, e.g. it is assumed to have the form

$$E_{ijkl} = \Lambda\delta_{ij}\delta_{kl} + G(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

if the material exhibits elastically isotropic response; Λ and G are the Lamé constants. The second term in brackets is the "plastic" part of $\mathbf{D}$; the plastic hardening modulus is h; $\mathbf{P}$ is the tensor giving the "direction" of the plastic part of $\mathbf{D}$, i.e. the normal to the "plastic potential surface," and $\mathbf{Q}$ is the normal to the "yield surface" in stress space. Hence, for $\mathbf{Q} = \mathbf{P}$, the normality assumption of classical plasticity is satisfied. The term $\mathbf{Q}:\mathbf{E}$ gives the directions of the normal to the "yield surface" in strain space, in that deformation which tends to make $\mathbf{Q}:\mathbf{E}:\mathbf{D} > 0$ corresponds to continued elastic–plastic deformation, whereas that which tends to make $\mathbf{Q}:\mathbf{E}:\mathbf{D} < 0$ corresponds to elastic unloading. (Note that $\mathbf{E}:\mathbf{D}$ is the stress rate that the material would exhibit if it responded elastically to $\mathbf{D}$.)

The right-hand side of eqn (10) may be written entirely in terms of $\mathbf{D}$, and comparing this form with eqn (5) makes it evident that

$$\mathbf{L} = \mathbf{E} - \frac{(\mathbf{E}:\mathbf{P})(\mathbf{Q}:\mathbf{E})}{h + \mathbf{Q}:\mathbf{E}:\mathbf{P}}. \tag{11a}$$

Because the material outside the band is assumed to unload elastically at the inception of localization

$$\mathbf{L}^\circ = \mathbf{E}. \tag{11b}$$

By substituting these into eqn (9) one obtains

$$\mathbf{g} = (\mathbf{n} \cdot \mathbf{L} \cdot \mathbf{n} + \mathbf{A})^{-1} \cdot \frac{[(\mathbf{n} \cdot \mathbf{E}:\mathbf{P})(\mathbf{Q}:\mathbf{E}:\mathbf{D}^{\circ})]}{h + \mathbf{Q}:\mathbf{E}:\mathbf{P}}. \tag{12}$$

The conditions for elastic unloading in the homogeneous field outside the zone of localization and, simultaneously, continued loading in the zone are

$$\mathbf{Q}:\mathbf{E}:\mathbf{D}^{\circ} < 0 \quad \text{and} \quad \mathbf{Q}:\mathbf{E}:\mathbf{D} > 0.$$

Substituting eqns (6, 12) into the second of the inequalities and using the first yield the following condition for a discontinuous bifurcation to be possible:

$$1 + \frac{(\mathbf{Q}:\mathbf{E} \cdot \mathbf{n}) \cdot (\mathbf{n} \cdot \mathbf{L} \cdot \mathbf{n} + \mathbf{A})^{-1} \cdot (\mathbf{n} \cdot \mathbf{E}:\mathbf{P})}{h + \mathbf{Q}:\mathbf{E}:\mathbf{P}} < 0. \tag{13}$$

Computing the inverse $(\mathbf{n} \cdot \mathbf{L} \cdot \mathbf{n} + \mathbf{A})^{-1}$ completes the calculation, and this computation is equivalent to solution of the system

$$(\mathbf{n} \cdot \mathbf{L} \cdot \mathbf{n} + \mathbf{A}) \cdot \mathbf{g} = \mathbf{f}. \tag{14}$$

For L as in eqn (11a), eqn (14) may be written

$$(\mathbf{n} \cdot \mathbf{E} \cdot \mathbf{n}) \cdot \mathbf{g} - \frac{1}{h + \mathbf{Q}:\mathbf{E}:\mathbf{P}} (\mathbf{n} \cdot \mathbf{E}:\mathbf{P})(\mathbf{Q}:\mathbf{E} \cdot \mathbf{n}) \cdot \mathbf{g} + \mathbf{A} \cdot \mathbf{g} = \mathbf{f}$$

or, in the concise notation of [8],

$$\mathbf{M} \cdot \left\{ (\mathbf{I} - \mathbf{B}) - \frac{1}{\lambda} \mathbf{ab} \right\} \cdot \mathbf{g} = \mathbf{f}, \tag{15}$$

where

$$\mathbf{a} = (\mathbf{n} \cdot \mathbf{E} \cdot \mathbf{n})^{-1} \cdot (\mathbf{n} \cdot \mathbf{E}:\mathbf{P}), \quad \mathbf{b} = \mathbf{Q}:\mathbf{E} \cdot \mathbf{n}, \quad \mathbf{M} = \mathbf{n} \cdot \mathbf{E} \cdot \mathbf{n}$$

$$\lambda = h + \mathbf{Q}:\mathbf{E}:\mathbf{P}, \quad \mathbf{B} = -(\mathbf{n} \cdot \mathbf{E} \cdot \mathbf{n})^{-1} \cdot \mathbf{A}, \quad (\mathbf{I})_{ij} = \delta_{ij}.$$

It has been assumed that the elastic moduli themselves do not allow localization and, consequently, that the 3×3 matrix $(\mathbf{n} \cdot \mathbf{E} \cdot \mathbf{n})$ has an inverse. If the material exhibits elastically isotropic response,

$$(\mathbf{n} \cdot \mathbf{E} \cdot \mathbf{n})^{-1} = -\frac{\Lambda + G}{G(\Lambda + 2G)} \mathbf{nn} + \frac{1}{G} \mathbf{I}. \tag{16}$$

Furthermore, note that the matrix $\mathbf{B}$ comprises terms which arise from the difference between the co-rotational and material rates of stress. Because these terms have the magnitude of a typical stress component divided by an elastic modulus, they are generally small compared to unity. Consequently, the inverse of $(\mathbf{I} - \mathbf{B})$ generally exists and may be computed to the desired accuracy by

$$(\mathbf{I} - \mathbf{B})^{-1} = \mathbf{I} + \mathbf{B} + \mathbf{B} \cdot \mathbf{B} + \mathbf{B} \cdot \mathbf{B} \cdot \mathbf{B} + \cdots$$

To solve eqn (15), first note that the homogeneous equation with $f = 0$ has a non-trivial solution when [8]

$$\lambda = \lambda_c \equiv \mathbf{b} \cdot (\mathbf{I} - \mathbf{B})^{-1} \cdot \mathbf{a}, \quad \mathbf{g} \propto (\mathbf{I} - \mathbf{B})^{-1} \cdot \mathbf{a} \tag{17}$$

where λ_c is the value of λ at which continuous bifurcation is first possible. Furthermore, it is straightforward to verify that the left eigenvector, which satisfies

$$\hat{\mathbf{g}} \cdot (\mathbf{n} \cdot \mathbf{L} \cdot \mathbf{n} + \mathbf{A}) = 0,$$

is

$$\hat{\mathbf{g}} = \mathbf{b} \cdot (\mathbf{I} - \mathbf{B})^{-1} \cdot (\mathbf{n} \cdot \mathbf{E} \cdot \mathbf{n})^{-1}.$$

Forming the inner product of eqn (15) and $\hat{\mathbf{g}}$ yields

$$(\mathbf{b} \cdot \mathbf{g})(1 - \lambda_c/\lambda) = \hat{\mathbf{g}} \cdot \mathbf{f}.$$

By using this in eqn (15) and solving for

$$\mathbf{g} = \left\{\mathbf{I} + \frac{(\mathbf{I} - \mathbf{B})^{-1} \cdot \mathbf{ab}}{\lambda - \lambda_c}\right\} \cdot (\mathbf{I} - \mathbf{B})^{-1} \cdot \mathbf{M}^{-1} \cdot \mathbf{f}$$

one obtains the inverse

$$(\mathbf{n} \cdot \mathbf{L} \cdot \mathbf{n} + \mathbf{A})^{-1} = \left\{\mathbf{I} + \frac{(\mathbf{I} - \mathbf{B})^{-1} \cdot \mathbf{ab}}{\lambda - \lambda_c}\right\} \cdot (\mathbf{I} - \mathbf{B})^{-1} \cdot (\mathbf{n} \cdot \mathbf{E} \cdot \mathbf{n})^{-1}.$$

Substituting this expression into eqn (13) reduces the condition for discontinuous bifurcation to

$$\frac{\lambda}{\lambda - \lambda_c} < 0.$$

This condition is rewritten in terms of the original constitutive parameters as

$$\frac{h + \mathbf{Q}:\mathbf{E}:\mathbf{P}}{h - h_c} < 0$$

where h_c is the critical value of the hardening modulus which would allow the criterion (8) for a continuous bifurcation to be met, i.e. [8]

$$h_c = (\mathbf{Q}:\mathbf{E} \cdot \mathbf{n}) \cdot (\mathbf{I} - \mathbf{B})^{-1} \cdot (\mathbf{n} \cdot \mathbf{E} \cdot \mathbf{n})^{-1} \cdot (\mathbf{n} \cdot \mathbf{E}:\mathbf{P}) - \mathbf{Q}:\mathbf{E}:\mathbf{P}.$$

Now, it is reasonable to assume that $h + \mathbf{Q}:\mathbf{E}:\mathbf{P} > 0$ since $\mathbf{Q}:\mathbf{E}:\mathbf{P}$ is of the order of elastic moduli (shortly, we shall show that a set of constitutive parameters violating this inequality leads to unacceptable behavior). Thus, localization with elastic unloading outside the zone of non-uniform deformation is possible only when $h < h_c$. Because h decreases in value with ongoing plastic deformation, this inequality means that localization with elastic unloading outside the zone first becomes possible when the condition for continuous bifurcation ($h = h_c$) is met. Thus, the calculation for continuous bifurcation, in which the material inside and outside the localized zone is assumed to continue loading at the inception of localization, sets the lower limit to the range of discontinuous bifurcation. This result supports the interpretation which was suggested at the outset: localization first occurs as a continuous bifurcation at a point in the body where conditions are locally favorable; however, as the deformation proceeds an infinitesimal amount past the continuous bifurcation point, elastic unloading occurs and, at least for certain geometries of loading, this can cause the localized zone to concentrate further its deformation. As noted in the Introduction, the result is analogous to Shanley's [9] solution for plastic column buckling.

To see the reasonableness of the requirement that $h + \mathbf{Q}:\mathbf{E}:\mathbf{P} > 0$, we observe that if the condition for continuing plastic flow, $\mathbf{Q}:\mathbf{E}:\mathbf{D} > 0$, is met, then $\mathbf{Q}:\overset{\nabla}{\boldsymbol{\sigma}}$ should be of the same sign as h (i.e. $\overset{\nabla}{\boldsymbol{\sigma}}$ directed outward relative to the yield surface when $h > 0$, inward when $h < 0$). But from $\overset{\nabla}{\boldsymbol{\sigma}} = \mathbf{L}:\mathbf{D}$ with $\mathbf{L}$ given by (11a), one readily finds that

$$\mathbf{Q}:\overset{\nabla}{\boldsymbol{\sigma}} = \frac{h}{h + \mathbf{Q}:\mathbf{E}:\mathbf{P}} \mathbf{Q}:\mathbf{E}:\mathbf{D},$$

so that such characteristics of the response will be exhibited only if $h + \mathbf{Q}:\mathbf{E}:\mathbf{P} > 0$.

ELASTIC AND PLASTIC PORTIONS OF THE LOCALIZATION MODES

The constitutive law for a material modelled as rigid-plastic may be written in the form

$$\mathbf{D} = h^{-1}\mathbf{P}(\mathbf{Q} : \overset{\nabla}{\boldsymbol{\sigma}}),$$

where h, $\mathbf{P}$ and $\mathbf{Q}$ have the same interpretations as in eqn (10). Rice[8] has investigated the conditions for localization for such a law (Hill[17] had previously discussed the case $\mathbf{P} = \mathbf{Q}$) and has demonstrated that the compatibility condition, eqn (2) or eqn (6), requires that $\mathbf{P}$ have the form

$$\mathbf{P} = \boldsymbol{\mu}\mathbf{n} + \mathbf{n}\boldsymbol{\mu}.$$

equivalently, this condition may be stated as the requirement that the intermediate principal value of $\mathbf{P}$ vanish:

$$P_{II} = 0, \tag{18}$$

that is, that an instantaneously non-deforming plane exists in the material. This condition is extremely restrictive. If it is satisfied and if normality is not satisfied ($\mathbf{P} \neq \mathbf{Q}$), localization is, in general, possible for any value of h; but if normality is satisfied localization can occur only for $h = 0$.

In their elastic-plastic calculation, Rudnicki and Rice[5] investigated localization for a constitutive law intended to model the behavior of brittle rock. This law has the form of eqn (10) in which

$$\mathbf{P} = \frac{\boldsymbol{\sigma}'}{2\tau} + \frac{\beta}{3}\mathbf{I}, \quad \mathbf{Q} = \frac{\boldsymbol{\sigma}'}{2\tau} + \frac{\mu}{3}\mathbf{I}, \tag{19}$$

where $\boldsymbol{\sigma}' = \boldsymbol{\sigma} - 1/3\,\mathbf{I}\,\mathrm{tr}\,\boldsymbol{\sigma}$ and $2\tau^2 = \boldsymbol{\sigma}' : \boldsymbol{\sigma}'$; μ is a friction coefficient which reflects the pressure dependence of inelasticity and β is a dilatancy factor which measures the ratio of inelastic volume strain to inelastic shear strain. Clearly, if $\beta = \mu$, the inelastic strain increment vector is normal to the yield surface.

Rudnicki and Rice[5] expressed their results in terms of the following critical value of the plastic hardening modulus at which the localization condition eqn (8) is first met,

$$\frac{h_{cr}}{G} = \frac{1+\nu}{9(1-\nu)}(\beta - \mu)^2 - \frac{1+\nu}{2}\left(2P_{II} + \frac{\mu - \beta}{3}\right)^2 + 0(\tau/G), \tag{20}$$

where ν is Poisson's ratio, and G is the shear modulus for elastic unloading. Neglect of the terms denoted by $0(\tau/G)$ is equivalent to the approximation $\overset{\nabla}{\boldsymbol{\sigma}} \simeq \dot{\boldsymbol{\sigma}}$ in the constitutive law or to neglecting the contribution from $\mathbf{A}$ in eqn (8). In order to take the rigid plastic limit of eqn (20), let $G \to \infty$. Obviously, eqn (20) predicts that $h_{cr} \to +\infty, -\infty$, or $h_{cr} = 0$ if the right-hand side is positive, negative, or equal to zero, respectively. If, for convenience, ν is taken to be one-half, the result of the limit may be expressed as $h_{cr} \to +\infty$ for P_{II} in the range

$$-(\mu - \beta) < 2P_{II} < (\mu - \beta)/3 \quad (\mu > \beta > 0), \tag{21}$$

$h_{cr} \to -\infty$ for P_{II} outside this range, and $h_{cr} = 0$ at the transition points.

In other words, eqn (20) predicts that localization is possible for any value of h when P_{II} satisfies eqn (21), is possible only for $h = 0$ if $P_{II} = -(\mu - \beta)$, $(\mu - \beta)/3$, and is impossible otherwise. This conclusion disagrees with the direct rigid-plastic analysis except for the case in which normality is satisfied; if $\mu = \beta$ both approaches require that $h = 0$ for localization. The reason for this apparent contradiction is that, in taking the rigid plastic limit, it is necessary to consider not only the value of the critical hardening modulus but also the limiting form of the bifurcation mode or eigenmode. In particular, the components of $\Delta\mathbf{D}$ corresponding to elastic and plastic deformation will be determined and examined in the rigid-plastic limit. The constitutive rate law eqn (10) will be used for the calculation.

Equation (10) can be rewritten as

$$\mathbf{D} = \mathbf{E}^{-1} : \overset{\nabla}{\boldsymbol{\sigma}} + \mathbf{D}^P$$

where

$$\mathbf{D}^P = h^{-1}\mathbf{P}(\mathbf{Q} : \overset{\nabla}{\boldsymbol{\sigma}}) = \mathbf{P}\frac{\mathbf{Q}:\mathbf{E}:\mathbf{D}}{h + \mathbf{Q}:\mathbf{E}:\mathbf{P}}$$

is the plastic part of the rate of deformation. If the Δ-operator, whose meaning is defined by eqn (1), is applied to $\mathbf{D}^P$, the result is

$$\Delta\mathbf{D}^P = \lambda^{-1}\mathbf{P}(\mathbf{Q}:\mathbf{E}\cdot\mathbf{n})\cdot\mathbf{g} \tag{22}$$

where, again, $\lambda = h + \mathbf{Q}:\mathbf{E}:\mathbf{P}$. At localization, g is given by the second of eqn (17) where the constant of proportionality may be taken as arbitrary but positive. To simplify expressions we take it as unity. Substituting the expression for g into eqn (22) yields

$$\Delta\mathbf{D}^P = \mathbf{P} \tag{23}$$

for $\lambda = \lambda_c$ at localization.

Thus,

$$\Delta\mathbf{D}^e = \Delta\mathbf{D} - \Delta\mathbf{D}^P = \frac{1}{2}(\mathbf{ng} + \mathbf{gn}) - \mathbf{P}. \tag{24}$$

Assuming elastic isotropy and using eqn (16) in the second of eqns (17) yield

$$\mathbf{g} = (\mathbf{I} - \mathbf{B})^{-1}\cdot\left\{\mathbf{n}\frac{\Lambda}{(\Lambda + 2G)}[\operatorname{tr}\mathbf{P} - (\mathbf{n}\cdot\mathbf{P}\cdot\mathbf{n})] - \mathbf{n}(\mathbf{n}\cdot\mathbf{P}\cdot\mathbf{n}) + 2\mathbf{n}\cdot\mathbf{P}\right\}. \tag{25}$$

We can set $\mathbf{B} = 0$ for our present considerations since, as noted previously, the components of $\mathbf{B}$ are $0(\tau/G)$ and go to zero in the rigid plastic limit. Therefore, substituting in eqn (24) and rearranging yields

$$\Delta\mathbf{D}^e = \frac{\Lambda}{\Lambda + 2G}\mathbf{nn}(\operatorname{tr}\mathbf{P} - \mathbf{n}\cdot\mathbf{P}\cdot\mathbf{n}) + [(\mathbf{n}\cdot\mathbf{P})\mathbf{n} + \mathbf{n}(\mathbf{P}\cdot\mathbf{n}) - \mathbf{nn}(\mathbf{n}\cdot\mathbf{P}\cdot\mathbf{n}) - \mathbf{P}]$$

where the neglected terms are order σ/G. The result is most readily interpreted relative to a system of axes x_1, x_2, x_3 with x_3 perpendicular to the prospective plane of localization. Letting Greek indices α, β have the range $1, 2$, the result is

$$\Delta D^e_{33} = \frac{\Lambda}{\Lambda + 2G}P_{\alpha\alpha}$$

$$\Delta D^e_{3\alpha} = \Delta D^e_{\alpha 3} = 0$$

$$\Delta D^e_{\alpha\beta} = -P_{\alpha\beta}.$$

Thus, all localizations for which some components of $P_{\alpha\beta}$ differ from zero necessarily involve elastic as well as plastic contributions to the bifurcation mode. That is, the predicted localizations involve elastic distortions (and thus become rigid, and hence non-admissible in the rigid-plastic limit) unless the kinematical condition of eqn (18) is met.

Consequently, taking the rigid plastic limit rules out modes which involve elastic strain rate non-uniformities that can be small fractions of the total strain rate non-uniformity (because, for them, the non-vanishing components of $P_{\alpha\beta}$ can still be very small compared to unity). Of course, in such cases, the idealization of an actual elastic–plastic material by a rigid-plastic model is inappropriate, since for even small amounts of elasticity, bifurcation modes which are inadmissible for the rigid-plastic case become possible.

CONCLUSIONS

Both of the results which have been obtained here are significant primarily because they do not require the assumption of normality of inelastic strain increments to the yield surface. Only if normality is not satisfied does the "paradox" arise in the rigid plastic limit of the localization calculation of [5]. If we had assumed normality, our result for discontinuous bifurcations would be a special case of Hill's [10, 11] that loss of uniqueness in the actual solid cannot precede that for the "comparison solid" defined to follow always the loading branch of the constitutive relation. However, as is well known, if normality does not apply, the failure of a sufficient condition for uniqueness in the form of Hill's criterion need not coincide with the existence of a non-uniform solution. The importance of deviations from normality in localization calculations has already been demonstrated [5, 7, 8, 16] and the apparent existence of materials for which normality is not satisfied is reason for consideration of these effects. Frictional materials are foremost among these (e.g. [4, 5, 18]), but hole growth with stress-dependent nucleation in ductile metals [16] and cross slip in single crystals [7, 8] also provide examples.

Acknowledgement—The work was supported at Brown University by the NSF Geophysics and USGS Earthquake Hazards Reduction Programs.

REFERENCES

1. J. Hadamard, *Lecons sur la Propagation des Ondes et les Equations de L'Hydrodynamique*, Paris (1903).
2. T. Y. Thomas, *Plastic Flow and Fracture in Solids.* Academic Press, New York (1961).
3. R. Hill, *J. Mech. Phys. Solids* **10**, 1 (1962).
4. J. Mandel, in *Rheology and Soil Mechanics* (Edited by J. Kravtchenko and P. M. Sirieys), p. 58. Springer Verlag, Berlin (1966).
5. J. W. Rudnicki and J. R. Rice, *J. Mech. Phys. Solids* **23**, 371 (1975).
6. S. Stören and J. R. Rice, *J. Mech. Phys. Solids* **3**, 421 (1975).
7. R. J. Asaro and J. R. Rice, *J. Mech. Phys. Solids* **25**, 309 (1977).
8. J. R. Rice, in *Proc. 14th Int. Cong. Theoretical and Appl. Mech. Delft* (Edited by W. T. Koiter), Vol. 1, p. 207. North-Holland, Amsterdam (1976).
9. F. R. Shanley, *J. Aero. Sci.* **14**, 261 (1947).
10. R. Hill, *J. Mech. Phys. Solids* **6**, 236 (1958).
11. R. Hill, in *Problems of Continuum Mechanics*, SIAM (1960).
12. W. Prager, *Introduction to Mechanics of Continua*, Dover (1961).
13. R. Hill, *J. Mech. Phys. Solids* **15**, 79 (1967).
14. J. W. Hutchinson, *Proc. R. Soc. Lond.* **A319**, 247 (1970).
15. R. Hill and J. W. Hutchinson, *J. Mech. Phys. Solids* **23**, 239 (1975).
16. A. Needleman and J. R. Rice, in *Mechanics of Sheet Metal Forming* (Edited by D. P. Koistinen and N-M. Wang), p. 237. Plenum Press, New York (1978).
17. R. Hill, *J. Mech. Phys. Solids* **10**, 89 (1962).
18. V. N. Nikolaevskii and J. R. Rice, in *High-Pressure Science and Technology* (Edited by K. D. Timmerhaus and M. S. Barber), Vol. 2, p. 455. Plenum Press, New York (1979).