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# AN INTEGRAL EQUATION FOR DYNAMIC ELASTIC RESPONSE OF AN ISOLATED 3-*D* CRACK

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By use of a steady state ($e^{-i\omega t}$) dynamic elastic representation theorem for fields created by relative motions $\Delta U_k$ on the faces of a crack, we reduce the problem of steady state response of an isolated three-dimensional planar crack, loaded by tractions on its surfaces, to an integral equation for $\Delta U_k$.

## 1. Integral representation

Let $U_{jk}(\boldsymbol{\xi}, \boldsymbol{x})\, e^{i\omega t}$ be the elastodynamic displacement $u_k$ at point $\boldsymbol{\xi}$ of a solid due to a concentrated force $e^{-i\omega t}$ acting in direction $j$ at point $\boldsymbol{x}$. Further, let $\Sigma_{jkl}(\boldsymbol{\xi}, \boldsymbol{x})\, e^{-i\omega t}$ be the stress $\sigma_{kl}$ generated at point $\boldsymbol{\xi}$ by the force. The fields satisfy

$$\frac{\partial \Sigma_{jkl}}{\partial \xi_l} + \rho\omega^2 U_{jk} + \delta_{jk}\delta_{\text{Dirac}}(\boldsymbol{\xi} - \boldsymbol{x}) = 0 \tag{1}$$

and, in the special version for an isotropic material,

$$\Sigma_{jkl} = \lambda \delta_{kl} \frac{\partial U_{ji}}{\partial \xi_i} + \mu\left(\frac{\partial U_{jk}}{\partial \xi_l} + \frac{\partial U_{jl}}{\partial \xi_k}\right). \tag{2}$$

Note that for an unbounded homogeneous body the fields $\boldsymbol{U}$ and $\boldsymbol{\Sigma}$ depend only on $\boldsymbol{\xi} - \boldsymbol{x}$.

Now consider an isolated planar crack in an unbounded body. Suppose the crack surfaces $A^+$ and $A^-$ (upper and lower) are subjected to tractions $t_j^+(\boldsymbol{\xi})\, e^{-i\omega t}$ and $t_j^-(\boldsymbol{\xi})\, e^{-i\omega t}$. These are assumed to satisfy $t_j^+ + t_j^- = 0$ (no net traction acts on the crack) and typically arise in the following way: Let $u_j^{\text{inc}}(\boldsymbol{\xi})\, e^{-i\omega t}$, $\sigma_{jk}^{\text{inc}}(\boldsymbol{\xi})\, e^{-i\omega t}$ represent an *incident* wave train in the cracked solid. Then by superposition the total field is the sum of the incident field, as it would exist in the uncracked solid, and the scattered field which is generated by applying tractions

$$t_j^+ = -n_k^+ \sigma_{jk}^{\text{inc}}, \qquad t_j^- = -n_k^- \sigma_{jk}^{\text{inc}} \tag{3}$$

to the crack surfaces. The $n$'s are outer normals, e.g., $\boldsymbol{n}^+$ is directed from the + to the − surface of the crack.

Such "scattered" fields, generated by tractions, may be given an integral representation based on a dynamic version of the Betti–Rayleigh reciprocal theorem (see, e.g., DeHoop [1] or, for the static case, Weaver [2]), or based directly on Green's function techniques (see Gubernatis et al. [3]). Thus if $u_j(\boldsymbol{x})\, e^{-i\omega t}$

is the displacement at $\boldsymbol{x}$ due to the traction loadings and if $\Delta u_i(\boldsymbol{\xi}) = u_i^+(\boldsymbol{\xi}) - u_i^-(\boldsymbol{\xi})$ is the displacement jump across the crack, then

$$u_i(\boldsymbol{x}) = -\int_A n_l^+ \Sigma_{jkl}(\boldsymbol{\xi}-\boldsymbol{x})\, \Delta u_k(\boldsymbol{\xi})\, \mathrm{d}A(\boldsymbol{\xi}). \tag{4}$$

An integral equation for $\Delta u_k(\boldsymbol{\xi})$ may now be established, following Weaver's work on the static problem, by demanding that the field $u_i(\boldsymbol{x})$ lead to stresses $\sigma_{jk}(\boldsymbol{x})$ that are consistent with the given tractions when $\boldsymbol{x}$ approaches a point of $A$.

To carry this out, imagine that the crack is flat and that an $x_1$, $x_2$, $x_3$ coordinate system is affixed to the body so that the origin is on $A$, the $x_3$ axis is normal to $A$, and $x_3 = 0^+$ on the + side. Hence $n_l^+ = -\delta_{l3}$ and

$$u_i(\boldsymbol{x}) = \int_A \Sigma_{jk3}(\boldsymbol{\xi}-\boldsymbol{x})\, \Delta u_k(\boldsymbol{\xi})\, \mathrm{d}A(\boldsymbol{\xi}). \tag{5}$$

To compute displacement derivatives as required for stresses, we differentiate inside the integral on $\boldsymbol{x}$ and make use of

$$\partial\Sigma_{jk3}(\boldsymbol{\xi}-\boldsymbol{x})/\partial x_\alpha = -\partial\Sigma_{jk3}(\boldsymbol{\xi}-\boldsymbol{x})/\partial \xi_\alpha,$$

$$\partial\Sigma_{jk3}(\boldsymbol{\xi}-\boldsymbol{x})/\partial x_3 = -\partial\Sigma_{jk3}(\boldsymbol{\xi}-\boldsymbol{x})/\partial \xi_3 = \partial\Sigma_{jk\beta}(\boldsymbol{\xi}-\boldsymbol{x})/\partial \xi_\beta + \rho\omega^2 U_{jk}(\boldsymbol{\xi}-\boldsymbol{x})$$

where Greek indices have the range 1, 2, and where the last expression makes use of the equations of motion (1). There results

$$u_{i,\alpha}(\boldsymbol{x}) = \int_A \Sigma_{jk3}(\boldsymbol{\xi}-\boldsymbol{x})\, \Delta u_{k,\alpha}(\boldsymbol{\xi})\, \mathrm{d}A(\boldsymbol{\xi}), \tag{6}$$

$$u_{i,3}(\boldsymbol{x}) = -\int_A [\Sigma_{jk\beta}(\boldsymbol{\xi}-\boldsymbol{x})\, \Delta u_{k,\beta}(\boldsymbol{\xi}) - \rho\omega^2 U_{jk}(\boldsymbol{\xi}-\boldsymbol{x})\, \Delta u_k(\boldsymbol{\xi})]\, \mathrm{d}A(\boldsymbol{\xi}) \tag{7}$$

where the commas denote partial differentiation. Stresses are given by

$$\sigma_{jk} = \lambda\delta_{jk}u_{l,l} + \mu(u_{j,k} + u_{k,j}). \tag{8}$$

Those of interest for imposition of boundary conditions on the crack are

$$\sigma_{33}(\boldsymbol{x}) = -\int_A \{\lambda[\Sigma_{3k\alpha}(\boldsymbol{\xi}-\boldsymbol{x}) - \Sigma_{\alpha k3}(\boldsymbol{\xi}-\boldsymbol{x})]\, \Delta u_{k,\alpha}(\boldsymbol{\xi})$$
$$+ 2\mu\Sigma_{3k\alpha}(\boldsymbol{\xi}-\boldsymbol{x})\, \Delta u_{k,\alpha}(\boldsymbol{\xi}) - \rho\omega^2(\lambda+2\mu)U_{3k}(\boldsymbol{\xi}-\boldsymbol{x})\, \Delta u_k(\boldsymbol{\xi})\}\, \mathrm{d}A(\boldsymbol{\xi}). \tag{9}$$

$$\sigma_{\beta 3}(\boldsymbol{x}) = -\mu\int_A \{[\Sigma_{\beta k\alpha}(\boldsymbol{\xi}-\boldsymbol{x}) - \Sigma_{3k3}(\boldsymbol{\xi}-\boldsymbol{x})\delta_{\beta\alpha}]\, \Delta u_{k,\alpha}(\boldsymbol{\xi}) - \rho\omega^2 U_{\beta k}(\boldsymbol{\xi}-\boldsymbol{x})\, \Delta u_k(\boldsymbol{\xi})\}\, \mathrm{d}A(\boldsymbol{\xi}). \tag{10}$$

## 2. Formulae for Green's function

Before using these expressions to write integral equations for $\Delta u_k$ we examine the specific form of the Green's function formulae involved. From a variety of sources (Kupradze [4], Tan [5], Gubernatis et al. [3])

one has

$$U_{jk}(\boldsymbol{\xi}-\boldsymbol{x})=\frac{1}{4\pi\rho\omega^2}\left[\delta_{jk}\beta^2\frac{\mathrm{e}^{\mathrm{i}\beta R}}{R}-\frac{\partial}{\partial\xi_j}\frac{\partial}{\partial\xi_k}\left(\frac{\mathrm{e}^{\mathrm{i}\alpha R}-\mathrm{e}^{\mathrm{i}\beta R}}{R}\right)\right] \tag{11}$$

where $\alpha$, $\beta$, and $R\geqslant 0$ are defined by

$$\alpha^2=\rho\omega^2/(\lambda+2\mu),\qquad \beta^2=\rho\omega^2/\mu,\qquad R^2=(\xi_i-x_i)(\xi_i-x_i).$$

After evaluating the derivatives we write this expression as

$$\rho\omega^2U_{jk}(\boldsymbol{\xi}-\boldsymbol{x})=\delta_{jk}G_1(R)+(R_jR_k/R^2)G_2(R) \tag{12}$$

where $R_j=\xi_j-x_j$ and

$$G_1(R)=\frac{1}{4\pi}\left[\beta^2\frac{\mathrm{e}^{\mathrm{i}\beta R}}{R}+\beta^3f(\beta R)-\alpha^3f(\alpha R)\right], \tag{13}$$

$$G_2(R)=\frac{1}{4\pi}[\beta^3g(\beta R)-\alpha^3g(\alpha R)], \tag{14}$$

and where the new functions $f$ and $g$ are

$$f(z)=z^{-1}(\mathrm{d}/\mathrm{d}z)(z^{-1}\,\mathrm{e}^{\mathrm{i}z})=(-z^{-3}+\mathrm{i}z^{-2})\,\mathrm{e}^{\mathrm{i}z}, \tag{15}$$

$$g(z)=z\,\mathrm{d}f(z)/\mathrm{d}z=(3z^{-3}-3\mathrm{i}z^{-2}-z^{-1})\,\mathrm{e}^{\mathrm{i}z}. \tag{16}$$

For later purposes it is important to know explicitly the structure of $G_1$ and $G_2$ for $R$ near zero, and these results are

$$G_1(R)=(\beta^2+\alpha^2)/(8\pi R)+\mathrm{O}(1), \tag{17}$$

$$G_2(R)=(\beta^2-\alpha^2)/(8\pi R)+\mathrm{O}(1). \tag{18}$$

Further, one may calculate the $\Sigma$'s from (2) and there results

$$\Sigma_{jkl}(\boldsymbol{\xi}-\boldsymbol{x})=(R_j/R)\delta_{kl}F_1(R)+[(R_j\delta_{kl}+R_k\delta_{lj}+R_l\delta_{jk})/R]F_2(R)+(R_jR_kR_l/R^3)F_3(R) \tag{19}$$

where

$$F_1(R)=\frac{1}{4\pi\alpha^2\beta^2}\{(\beta^2-3\alpha^2)\beta^5Rf(\beta R)$$
$$+(\beta^2-2\alpha^2)[3\beta^3g(\beta R)/R-3\alpha^3g(\alpha R)/R+\beta^4h(\beta R)-\alpha^4h(\alpha R)]\}, \tag{20}$$

$$F_2(R)=\frac{1}{4\pi\beta^2}[\beta^5Rf(\beta R)+2\beta^3g(\beta R)/R-2\alpha^3g(\alpha R)/R], \tag{21}$$

$$F_3(R)=\frac{1}{2\pi\beta^2}[-2\beta^3g(\beta R)/R+2\alpha^3g(\alpha R)/R+\beta^4h(\beta R)-\alpha^4h(\alpha R)], \tag{22}$$

and where the new function $h$ is

$$h(z)=\mathrm{d}g(z)/\mathrm{d}z=(-9z^{-4}+9\mathrm{i}z^{-3}+4z^{-2}-\mathrm{i}z^{-1})\,\mathrm{e}^{\mathrm{i}z}. \tag{23}$$

The functions $F_1$, $F_2$, $F_3$ must also be known explicitly for $R$ near zero and these results are

$$F_1(R) = \alpha^2/(2\pi\beta^2R^2) + \mathrm{O}(1) = \mu/[2\pi(\lambda+2\mu)R^2] + \mathrm{O}(1), \tag{24}$$

$$F_2(R) = -\alpha^2/(4\pi\beta^2R^2) + \mathrm{O}(1) = -\mu/[4\pi(\lambda+2\mu)R^2] + \mathrm{O}(1), \tag{25}$$

$$F_3(R) = -3(\beta^2-\alpha^2)/(4\pi\beta^2R^2) + \mathrm{O}(1) = -3(\lambda+\mu)/[4\pi(\lambda+2\mu)R^2] + \mathrm{O}(1). \tag{26}$$

Of course the singular terms in the $F$'s are the same as for the static Green's function stresses.

## 3. Integral equations for crack opening

We obtain the integral equations for $\Delta u_j$ by substituting (19) for $\Sigma_{jkl}$ and (12) for $U_{jk}$ into (9, 10) and letting $x_3 \to 0^+$ or $0^-$. We remark that this involves expressions having discontinuous limits of the types

$$\lim_{x_3\to\pm 0} \int_A \left[\frac{x_3}{R^3}, \frac{x_3^3}{R^5}, \frac{x_3 R_\alpha R_\beta}{R^5}\right] \Delta u_{j,\gamma}(\boldsymbol{\xi})\,\mathrm{d}A(\boldsymbol{\xi}) = \pm 2\pi[1, \tfrac{1}{3}, \tfrac{1}{3}\delta_{\alpha\beta}]\Delta u_{j,\gamma}(x). \tag{27}$$

Other singular terms do not make similar contributions. Furthermore, we have verified by explicit calculation that when all of the terms similar to those on the right in (27) that arise on taking the limit $x_3 \to +0$ in (9, 10) are added up, their total contribution vanishes. (This is to be expected on general grounds since tractions are assumed to satisfy $t_j^+ + t_j^- = 0$, so that $\sigma_{j3}^+ = \sigma_{j3}^-$.)

The integral equations governing the crack opening then become

$$\sigma_{33}(\boldsymbol{x}) = \int_A \{[\lambda F_1(R) - 2\mu F_2(R)](R_\alpha/R)\,\Delta u_{3,\alpha}(\boldsymbol{\xi}) + (\lambda+2\mu)G_1(R)\,\Delta u_3(\boldsymbol{\xi})\}\,\mathrm{d}A(\boldsymbol{\xi}), \tag{28}$$

$$\sigma_{\beta 3}(\boldsymbol{x}) = \mu \int_A \{-[F_1(R)+F_2(R)](R_\beta/R)\,\Delta u_{\alpha,\alpha}(\boldsymbol{\xi}) - F_2(R)(R_\alpha/R)\,\Delta u_{\beta,\alpha}(\boldsymbol{\xi})$$
$$- F_3(R)(R_\alpha R_\beta R_\gamma/R^3)\,\Delta u_{\gamma,\alpha}(\boldsymbol{\xi}) + G_1(R)\,\Delta u_\beta(\boldsymbol{\xi}) + G_2(R)(R_\beta R_\alpha/R^2)\,\Delta u_\alpha(\boldsymbol{\xi})\}\,\mathrm{d}A(\boldsymbol{\xi}) \tag{29}$$

where the integrals are understood to represent Cauchy principal values. The left sides of each equation are known on the crack surfaces. Note that analogously to Weaver's results in the static case, [2], the determination of the normal displacement $\Delta u_3$ decouples from that of the shear displacements $\Delta u_1$, $\Delta u_2$.

The integral equations for the corresponding static problem ($\omega \to 0$) are obtained by letting $\alpha \to 0$, $\beta \to 0$ in Eqs. (13, 14) and (20, 21, 22). There results

$$\sigma_{33}(\boldsymbol{x}) = \frac{\mu(\lambda+\mu)}{2\pi(\lambda+2\mu)} \int_A \frac{R_\alpha\,\Delta u_{3,\alpha}(\boldsymbol{\xi})}{R^3}\,\mathrm{d}A(\boldsymbol{\xi}), \tag{30}$$

$$\sigma_{\beta 3}(\boldsymbol{x}) = \frac{\mu(\lambda+\mu)}{4\pi(\lambda+2\mu)} \int_A \left\{\frac{\mu}{\lambda+\mu}\left[\frac{R_\alpha\,\Delta u_{\beta,\alpha}(\boldsymbol{\xi}) - R_\beta\,\Delta u_{\alpha,\alpha}(\boldsymbol{\xi})}{R^3}\right] + \frac{3R_\alpha R_\beta R_\gamma\,\Delta u_{\gamma,\alpha}(\boldsymbol{\xi})}{R^5}\right\}\mathrm{d}A(\boldsymbol{\xi}). \tag{31}$$

When it is realized that

$$\mu(\lambda+\mu)/(\lambda+2\mu) = E/[4(1-\nu^2)], \qquad \mu/(\lambda+\mu) = 1-2\nu,$$

these equations are seen to be identical to those given by Weaver, with the following exception: Where we have $R_\beta\,\Delta u_{\alpha,\alpha}$ in (31), Weaver has instead $R_\alpha\,\Delta u_{\alpha,\beta}$. But one may note the identity

$$\frac{R_\beta\,\Delta u_{\alpha,\alpha}(\boldsymbol{\xi})}{R^3}+\frac{\partial}{\partial\xi_\beta}\left(\frac{R_\alpha\,\Delta u_\alpha(\boldsymbol{\xi})}{R^3}\right)=\frac{R_\alpha\,\Delta u_{\alpha,\beta}(\boldsymbol{\xi})}{R^3}+\frac{\partial}{\partial\xi_\alpha}\left(\frac{R_\beta\,\Delta u_\alpha(\boldsymbol{\xi})}{R^3}\right)$$

and then use the divergence theorem, together with the fact that $\Delta u_\alpha$ vanishes on the crack edge, to show that our equation (31) is equivalent to Weaver's. We note, finally, that Tan [6] has deduced the analogous integral equation for $2D$ dynamic crack problems.

The singular integral equations (28), (29) for the crack openings $\Delta u_3$, $\Delta u_\gamma$ must, of course, be solved numerically, in general. Reasonable care must naturally be taken in an appropriate numerical procedure to account for the square-root behavior of the displacement jumps along crack edges, and due care must also be taken in the integration over the singularities of the kernels at $\boldsymbol{x}=\boldsymbol{\xi}$.

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