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Slightly curved or kinked cracks*

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ABSTRACT
A solution is presented for the elastic stress intensity factors at the tip of a slightly curved or kinked two-dimensional crack. The solution is accurate to first order in the deviation of the crack surface from a straight line and is carried out by perturbation procedures analogous to those of Banichuk [1] and Goldstein and Salganik [2, 3]. Comparison with exact solutions for circular arc cracks and straight cracks with kinks indicates that the first order solution is numerically accurate for considerable deviations from straightness. The solution is applied to fromulate an equation for the path of crack growth, on the assumption that the path is characterized by pure Mode I conditions (i.e., $K_{II} = 0$) at the advancing tip. This method confirms the dependence of the stability, under Mode I loading, of a straight crack path on the sign of the non-singular stress term, representing tensile stress T acting parallel to the crack, in the Irwin–Williams expansion of the crack tip field. The straight path is shown to be stable under Mode I loading for $T < 0$ and unstable for $T > 0$.

1. Introduction

When cracks grow in non-uniform stress fields, the path of the fracture is generally curved. The path taken by a crack in brittle homogeneous isotropic material can be assumed to be one for which the local stress field at the tip is of a Mode I type (that is, $K_{II} = 0$ at the tip), referred to as the "criterion of local symmetry" [1–3]. This result is consistent, in the sense that it follows by contradiction, with the various proposed mixed mode fracture criteria (maximum hoop) stress [4–8], maximum energy release rate [9–11], and stationary Sih energy density factor [12–14]). All of these criteria have in common the feature that, if $K_{II} \neq 0$ at the crack tip, the crack extends with an abrupt, non-zero, change in the tangent direction to the path. Hence, if a crack is observed to follow a path with a continuously turning tangent, these criteria all have the implication that $K_{II} = 0$ as the crack extends.

Much effort has been applied to crack initiation at an angle to a pre-existing crack [10, 11, 15–18] where, because of the abrupt change in tangent direction, it appears that locally the direction that releases the maximum energy or satisfies some other mixed mode criterion, need not necessarily coincide with $K_{II} = 0$ at the incipient kink. However, once the kink has been initiated, it extends so that $K_{II} = 0$. In practical terms the crack should come under the $K_{II} = 0$ criterion when the fracture process zone is no longer affected by the discontinuity.

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The curvature of naturally growing cracks is usually slight. In this case, the crack can be treated as a perturbed straight crack and Muskhelishvili's [19] method for straight cuts can be used as we show. Indeed, perturbation methods based on the Mushkelishvili approach have been used previously by Banichuk [1] and Goldstein and Salganik [2, 3] to solve the problems of slightly curved or kinked cracks that we treat here. Our approach to developing the solution is somewhat more direct and, in contrast to the rather lengthy expressions of [1–3], we have found a remarkably simple form for the first-order solution for the stress intensity factors (namely, eq. (27), to follow). Further, we compare the perturbation solution with exact results, especially for kinked cracks, and suggest a means of interpreting the first-order perturbation solution which is shown to lead to increased accuracy, by comparison to a strict first order solution, up to very considerable deviations from straightness in problems for which the loading is concentrated near the end of the crack. As in [1–3], the perturbation solution allows us to address the problem of the path taken by a fracture (i.e., the $K_{II} = 0$ path). We present a new application here by deriving the condition under which a straight crack path is stable under Mode I (tensile) loading.

2. Stress intensity factors for slightly curved or kinked cracks

The crack is arranged so that its tips lie on the x-axis at $\pm a$ (see Fig. 1) and the cracked two-dimensional body is assumed to be of infinite extent. It is assumed that the crack is opened by normal and shear tractions T_n, T_s at the surface, which are equal on top and bottom surfaces, and that at infinity the stresses are zero. This load configuration can be either the actual one, or that necessary to superimpose on the stress field for an uncracked body to remove the stresses from the boundary of the crack. For the moment it is assumed that T_n and T_s are everywhere bounded and differentiable (with respect to x). In the case of an extension of a pre-existing crack where T_n and T_s are the surface tractions necessary to remove the stress on the boundary of the extension, the singularity at the tip of the pre-existing crack is removed by replacing T_n and T_s by bounded functions that reduce continuously to zero over distances closer than a small distance ϵ to the pre-existing crack tip. Later it is shown that it is possible to let ϵ tend to zero, i.e., effectively, to remove the restriction of bounded and differentiable loadings T_n and T_s. Mushkelishvili's [19] analytic functions $\phi(z)$ and $\psi(z)$ express the stress field, for plane strain or generalized plane stress, as

$$\begin{aligned} \sigma_{xx} + \sigma_{yy} &= 2[\phi(z) + \overline{\phi(z)}] \\ \sigma_{yy} - \sigma_{xx} - 2\,\mathrm{i}\sigma_{xy} &= 2[z\overline{\phi'(z)} + \overline{\psi(x)}] \end{aligned} \tag{1}$$

where $z = x + \mathrm{i}y$, and the boundary condition on the crack is

$$\phi(z) + \overline{\phi(z)} + \mathrm{e}^{-2\mathrm{i}\theta}[z\overline{\phi'(z)} + \overline{\psi(z)}] = -(T_n + \mathrm{i}T_s) \tag{2}$$

where θ is the angle made by the crack with the x-axis. Introducing the analytic function

$$\Omega(z) = \bar{\phi}(z) + z\bar{\phi}'(z) + \bar{\psi}(z), \tag{3}$$

Eqn. (2) can be written as

$$\phi(z) + \overline{\phi(z)} + \mathrm{e}^{-2\mathrm{i}\theta}[(z - \bar{z})\overline{\phi'(z)} + \Omega(\bar{z}) - \overline{\phi(z)}] = -(T_n - \mathrm{i}T_s). \tag{4}$$

It is assumed that there are two functions $F(z)$, $W(z)$ which are analytic outside of a straight cut between the crack tips (Fig. 1) where they have boundary values

$F^{\pm}(t)$ and $W^{\pm}(t)$ on the upper and lower surfaces. These functions can be analytically continued to the corresponding upper and lower surfaces of the actual crack. When continued in this manner $F(z)$ and $W(z)$ correspond to $\phi(z)$ and $\Omega(z)$, the latter having their cuts along the actual crack. Using a perturbation expansion in the function λ, where $\lambda(x)$ describes the deviation of the actual crack from straightness (Fig. 1), the functions $F(z)$ and $W(z)$ can be written as

$$\begin{aligned} F(z) &= F_0(z) + F_1(z) + O(\lambda^2) \\ W(z) &= W_0(z) + W_1(z) + O(\lambda^2), \end{aligned} \tag{5}$$

where $F_0(z)$ and $W_0(z)$ are of zero order (that is, correspond to a straight crack, $\lambda = 0$, with the same normal and shear loadings T_n, T_s) and $F_1(z)$ and $W_1(z)$ are of first order λ. Thus on the boundary of the crack $\phi(z)$ at position $z = t + \mathrm{i}\lambda(t)$ is given by

$$\phi^{\pm}(z) = F_0^{\pm}(t) + \mathrm{i}\lambda\left(\frac{\mathrm{d}F_0(z)}{\mathrm{d}z}\right)_t^{\pm} + F_1^{\pm}(t) \tag{6}$$

if only first order terms are retained. Since

$$\left(\frac{\mathrm{d}F_0(z)}{\mathrm{d}z}\right)_t^{+} = \frac{\mathrm{d}F_0^{+}(t)}{\mathrm{d}t} = F_0^{+'}(t), \tag{7}$$

Eqn. (6) can be written as

$$\phi^{\pm}(z) = F_0^{\pm}(t) + \mathrm{i}\lambda F_0^{\pm'}(t) + F_1^{\pm}(t). \tag{8}$$

Using similar expressions and noting that $\theta = \lambda'(t)$ to first order, the boundary values on the straight cut, (4), become

$$\begin{aligned} &F_0^{\pm}(t) + W_0^{\mp}(t) + F_1^{\pm}(t) + W_1^{\mp}(t) + \mathrm{i}\lambda\,[F_0^{\pm}(t) + W_0^{\mp}(t)]' \\ &\quad + 2\,\mathrm{i}\{\lambda\,[\bar{F}_0^{\mp}(t) - W_0^{\mp}(t)]\}' = -(T_n - \mathrm{i}T_s), \end{aligned} \tag{9}$$

where again only first order terms have been retained. Thus separating zero and first order terms gives

$$F_0^{\pm}(t) + W_0^{\mp}(t) = -(T_n - \mathrm{i}T_s) \tag{10}$$

$$F_1^{\pm}(t) + W_1^{\mp}(t) = -\{\mathrm{i}\lambda\,[F_0^{\pm}(t) + W_0^{\mp}(t)]' + 2\,\mathrm{i}\{\lambda\bar{F}_0^{\mp}(t) - W_0^{\mp}(t)]\}'\} \tag{11}$$

The boundary values of $[F_0(z) + W_0(z)]$ and $[F_0(z) - W_0(z)]$ are given by

$$[F_0(t) + W_0(t)]^{+} + [F_0(t) + W_0(t)]^{-} = -2(T_n - \mathrm{i}T_s)$$

and (12)

$$[F_0(t) - W_0(t)]^{+} - [F_0(t) - W_0(t)]^{-} = 0.$$

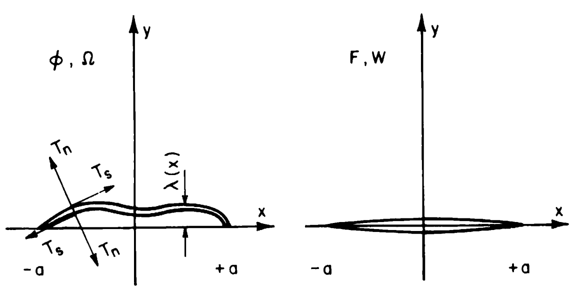

Figure 1. Crack configuration and cuts in F and W functions.

Therefore, $F_0(z) = W_0(z)$ and substitution of Eqns. (10) and (12) into (11) yields the boundary values of $[F_1(z) + W_1(z)]$,

$$[F_1(t) + W_1(t)]^+ + [F_1(t) + W_1(t)]^- = 2[\mathrm{i}\lambda(T_n - \mathrm{i}T_s)' - 2(\lambda T_s)']. \tag{13}$$

Thus using the formula of Mushkelishvili [19], the functions are given by

$$F_0(z) = W_0(z) = \frac{1}{2\pi(z^2 - a^2)^{1/2}} \int_{-a}^{a} (T_n - \mathrm{i}T_s) \frac{(a^2 - t^2)^{1/2}}{(z - t)} \, \mathrm{d}t \tag{14}$$

$$F_1(z) + W_1(z) = \frac{1}{\pi(z^2 - a^2)^{1/2}} \int_{-a}^{a} (\lambda T_s' + 2\lambda' T_s - \mathrm{i}\lambda T_n') \frac{(a^2 - t^2)^{1/2}}{(z - t)} \, \mathrm{d}t. \tag{15}$$

Letting ω be the slope of the crack tip at $x = a$ (given by $\omega = \lambda'(a)$ to a first order), the normal ($\sigma_{\omega\omega}$) and shear ($\sigma_{r\omega}$) stresses acting across the prolongation of the crack at a small distance r from the tip at $x = a$ are obtained by substitution of $F(z)$ and $W(z)$ into Eqn. (9), setting $\lambda = \omega r$, noting that F and W are single valued at the point considered, and identifying $-(T_n - \mathrm{i}T_s)$ with $\sigma_{\omega\omega} - \mathrm{i}\sigma_{r\omega}$. Thus

$$\begin{aligned}\sigma_{\omega\omega} - \mathrm{i}\sigma_{r\omega} = \{&2F_0(a + r)[1 - \mathrm{i}\omega] + 2\,\mathrm{i}\omega\bar{F}_0(a + r) + 2\,\mathrm{i}\omega r\bar{F}_0'(a + r) + F_1(a + r) \\ &+ W_1(a + r)\}.\end{aligned} \tag{16}$$

Substituting Eqns. (14) and (15) into (16) and letting r tend to zero yields

$$(\sigma_{\omega\omega} - \mathrm{i}\sigma_{r\omega}) = \frac{1}{\pi(2ar)^{1/2}} \int_{-a}^{+a} (q_{\mathrm{I}} - \mathrm{i}q_{\mathrm{II}}) \left(\frac{a + t}{a - t}\right)^{1/2} \mathrm{d}t \tag{17}$$

where

$$\begin{aligned} q_{\mathrm{I}} &= T_n - \tfrac{3}{2}\omega T_s + \lambda T_s' + 2\lambda' T_s \\ q_{\mathrm{II}} &= T_s + \lambda T_n' + \tfrac{1}{2}\omega T_n. \end{aligned} \tag{18}$$

Thus the stress intensity factors for a slightly curved or kinked crack are given by

$$K = K_{\mathrm{I}} - \mathrm{i}K_{\mathrm{II}} = \frac{1}{(\pi a)^{1/2}} \int_{-a}^{a} (q_{\mathrm{I}} - \mathrm{i}q_{\mathrm{II}}) \left(\frac{a + t}{a - t}\right)^{1/2} \mathrm{d}t \tag{19}$$

to first order in λ. This reduces to a well-known result in the case of a straight crack ($\lambda = \omega = 0$).

3. The circular arc crack

Because an exact solution exists, the circular arc crack under a uniform stress field provides a simple check on the accuracy of the approximate solution. The solution given by Sih, Paris and Erdogan [20] for a circular arc contains an error in transcription of results from Mushkelishvili [19]. Expressed in different terms, the exact solution for a circular arc crack (see Fig. 2) is given by

$$\begin{aligned} K_{\mathrm{I}} = (\pi a)^{1/2} \Big\{ &\Big[\Big(\frac{\sigma_{yy} + \sigma_{xx}}{2}\Big) - \Big(\frac{\sigma_{yy} - \sigma_{xx}}{2}\Big) \sin^2(\alpha/2)\cos^2(\alpha/2) \Big] \frac{\cos(\alpha/2)}{[1 + \sin^2(\alpha/2)]} \\ &+ \Big(\frac{\sigma_{yy} - \sigma_{xx}}{2}\Big) \cos(3\alpha/2) - \sigma_{xy}[\sin(3\alpha/2) + \sin^3(\alpha/2)] \Big\} \end{aligned} \tag{20}$$

$$\begin{aligned} K_{\mathrm{II}} = (\pi a)^{1/2} \Big\{ &\Big[\Big(\frac{\sigma_{yy} + \sigma_{xx}}{2}\Big) - \Big(\frac{\sigma_{yy} - \sigma_{xx}}{2}\Big) \sin^2(\alpha/2)\cos^2(\alpha/2) \Big] \frac{\sin(\alpha/2)}{[1 + \sin^2(\alpha/2)]} \\ &+ \Big(\frac{\sigma_{yy} - \sigma_{xx}}{2}\Big) \sin(3\alpha/2) + \sigma_{xy}[\cos(3\alpha/2) + \cos(\alpha/2)\sin^2(\alpha/2)] \Big\}. \end{aligned} \tag{21}$$

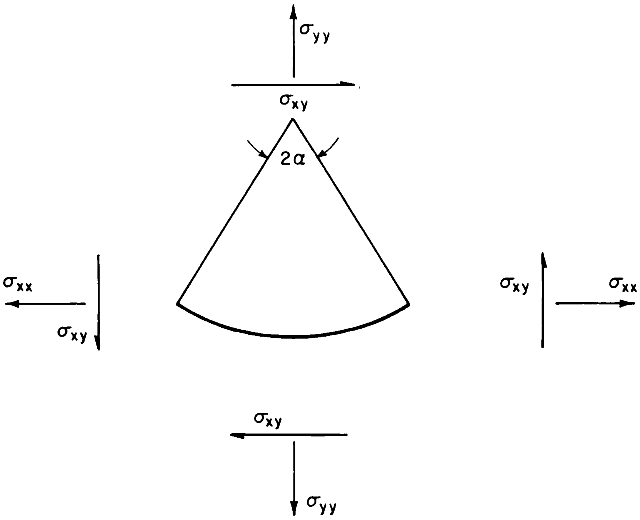

Figure 2. Configuration of circular arc crack.

When α tends to zero the solution to the first order,

$$K_{\mathrm{I}} = (\pi a)^{1/2}[\sigma_{yy} - \tfrac{3}{2}\alpha\sigma_{xy}] \tag{22}$$

$$K_{\mathrm{II}} = (\pi a)^{1/2}\left[\sigma_{xy} + \left(\sigma_{yy} - \frac{\sigma_{xx}}{2}\right)\alpha\right], \tag{23}$$

is identical to that obtained from the first order solution given by (19). The results for the exact and first order solutions are compared in Figs. 3 and 4 for biaxial and shear stress loading at infinity. In both cases the approximate solution is least accurate for the leading stress intensity factor, where the first order solution gives a value which is independent of α and is accurate to within 5% for $\alpha < 15°$. The secondary stress intensity factors are accurate to 5% for $\alpha < 40°$.

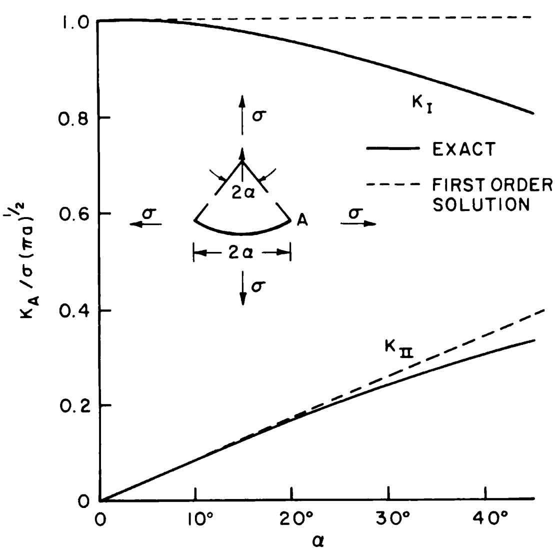

Figure 3. Stress intensity factors for a circular arc under pure biaxial stress.

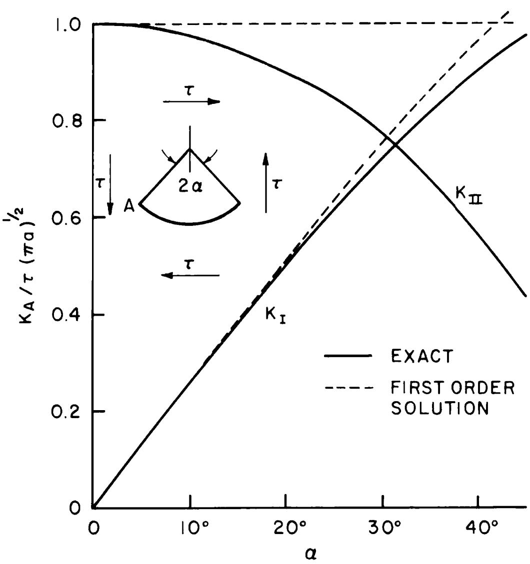

Figure 4. Stress intensity factors for a circular arc under pure shear.

4. An alternative form of the first order solution

If the crack surfaces are loaded only in the vicinity of the crack tip, the first order solution can be given in a form that gives improved accuracy. The problem for a curved crack is, in the limit as the portion of the crack surfaces that are loaded shrinks to the crack tip, identical to that for a semi-infinite straight crack. Consequently, in this alternative form of the solution, the crack tip is aligned with the x-axis, with the coordinate origin fixed at the tip, and the surface tractions are given in terms of the components T_y and T_x (see Fig. 5).

The coordinates used in the previous solution are given in terms of the new coordinates of Fig. 5, consistent to first order accuracy in λ, by

$$\begin{aligned} t &= a - r \\ \lambda &= \eta - \omega r. \end{aligned} \tag{24}$$

Using from now the prime to mean $\mathrm{d}/\mathrm{d}r(=-\mathrm{d}/\mathrm{d}t)$, (19) can, after integration by parts,

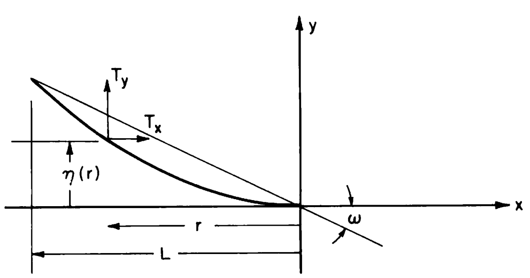

Figure 5. Crack configuration for an alternative form of the solution with x-axis tangent to crack at its tip.

be written as

$$\begin{Bmatrix} K_{\rm I} \\ K_{\rm II} \end{Bmatrix} = \left(\frac{2}{\pi L}\right)^{1/2} \int_0^L \left[\begin{Bmatrix} T_n - 2\eta' T_s - \eta T_s' \\ T_s - \eta T_n' \end{Bmatrix} \left(\frac{L-r}{r}\right)^{1/2} + \frac{\omega}{2} \begin{Bmatrix} T_s \\ T_n \end{Bmatrix} \left(\frac{r}{L-r}\right)^{1/2} \right] \mathrm{d}r. \tag{25}$$

These integrals are considerably simplified if the components of the tractions are taken in the directions of x, y axes. To first order

$$\begin{aligned} T_n &= T_y + \eta' T_x \\ T_s &= T_x - \eta' T_y. \end{aligned} \tag{26}$$

Substituting these components into Eqn. (25) and integration again by parts gives a symmetrical expression for the stress intensity factors,

$$\begin{Bmatrix} K_{\rm I} \\ K_{\rm II} \end{Bmatrix} = \left(\frac{2}{\pi L}\right)^{1/2} \int_0^L \left[\begin{Bmatrix} T_y \\ T_x \end{Bmatrix} \left(\frac{L-r}{r}\right)^{1/2} + \frac{1}{2} \begin{Bmatrix} T_x \\ T_y \end{Bmatrix} \left(\omega - \frac{\eta L}{r^2}\right) \left(\frac{r}{L-r}\right)^{1/2} \right] \mathrm{d}r \tag{27}$$

to first order. It should be noted that $\eta(0) = \eta'(0) = 0$, so there is no divergence at the lower limit. Also, $\omega = \eta(L)/L$. It is also seen that an integrable singularity (that is, weaker than $1/r$) can exist in T_y or T_x provided it is not at either of the crack tips. Thus T_y, T_x can be the tractions necessary to remove the stresses that exist on the prolongation of a pre-existing crack. Specifically, in terms of our earlier discussion, ϵ can be shrunk to zero and in that limit, the result of Eqn. (27) for $\{K_{\rm I}, K_{\rm II}\}$ approaches the result obtained by inserting directly into (27) the singular, actual traction distribution $\{T_x, T_y\}$. Such considerations, based essentially on the fact that the final result of (27) for $\{K_{\rm I}, K_{\rm II}\}$ contain only $\{T_x, T_y\}$ (and not $\{T_x', T_y'\}$) allow us to conclude that (27) is valid for all integrable traction distributions (i.e., not necessarily bounded or continuous) which are bounded at the crack ends.

5. Stress intensity factors for kinked cracks

When a crack is loaded asymmetrically, the new crack initiates at an angle to the old one. The calculation of stress intensity factors for kinked cracks is difficult and there have been many attempts at its solution [10, 11, 15–18, 21–24]. With most solutions, the analysis is such that the limit for an infinitesimally small kink cannot be obtained readily from the analysis for a finite kink. Recently, Lo [18] has presented a convincing solution that models the crack as a continuous distribution of dislocations, in a manner that can handle both the finite and the infinitesimal kink within the same formulation. Further, Lo compares results with those of previous investigators, and we use his work and favorable comparisons as a basis for choosing what we shall regard as accurate solutions of the kinked crack problem. For example, for the crack with a kink at one end, Lo is in agreement with the Refs. [11, 15, 16] for the infinitesimally small kink and [21–23] for finite kinks.

The stress intensity factors at the tip of a kinked crack can be calculated to the first order from the stresses that exist in the line of the pupative kink. For an infinitesimally small kink these stresses can be written in terms of the stress intensity factors of the main crack, $k_{\rm I}$ and $k_{\rm II}$, and the surface tractions (aligning the kink with the x-axis) are given by the polar coordinate hoop and shear stresses at a polar angle α with the prolongation of the main crack [25]

$$\begin{aligned} T_y &= C_{11} \frac{k_{\rm I}}{(2\pi s)^{1/2}} + C_{12} \frac{k_{\rm II}}{(2\pi s)^{1/2}} \\ T_x &= C_{21} \frac{k_{\rm I}}{(2\pi s)^{1/2}} + C_{22} \frac{k_{\rm II}}{(2\pi s)^{1/2}} \end{aligned} \tag{28}$$

where

$$C_{11} = \tfrac{1}{4}(3\cos\alpha/2 + \cos 3\alpha/2)$$
$$C_{12} = -\tfrac{3}{4}(\sin\alpha/2 + \sin 3\alpha/2)$$
$$C_{21} = \tfrac{1}{4}(\sin\alpha/2 + \sin 3\alpha/2)$$
$$C_{22} = \tfrac{1}{4}(\cos\alpha/2) + 3\cos 3\alpha/2). \tag{29}$$

Strictly for this order solution the coefficients C should be given only to the first order in α, but it will be seen that accuracies far greater than might be expected can be obtained by retaining the exact expressions for C. In the limit as the length ℓ of the kink tends to zero the stress intensity factors become to first order

$$\begin{Bmatrix} K_{\rm I} \\ K_{\rm II} \end{Bmatrix} = \left(\frac{2}{\pi}\right)^{1/2} \int_0^\ell \begin{Bmatrix} T_y \\ T_x \end{Bmatrix} \frac{\mathrm{d}s}{(\ell - s)^{1/2}}. \tag{30}$$

Substituting for T_y and T_x from Eqns. (29) gives the following expressions for the stress intensity factors at the tip of an infinitesimal branch

$$\begin{aligned} K_{\rm I} &= C_{11}k_{\rm I} + C_{12}k_{\rm II} \\ K_{\rm II} &= C_{21}k_{\rm I} + C_{22}k_{\rm II} \end{aligned} \tag{31}$$

which are accurate to the first order in α. These results are compared in Fig. 6 with those of Bilby, Cardew and Howard [16]. It is seen that the agreement is within 5% up to kink angles as large as 40°. The two functions C_{11} and C_{21} for a Mode I main crack are accurate to 5% for even much larger angles. Thus the previous suggestion [9, 24 and 26] that (31) is an approximation to the stress intensity factors has been shown to be rigorously correct to the first order, and a good approximation for even quite larger angles. Indeed, Eqns. (31) correspond to defining $K_{\rm I}$ and $K_{\rm II}$ from the coefficients of the $s^{-1/2}$ singularity of normal and shear stress acting along the line of the pupative kink before its introduction.

It is interesting to use the present results to examine whether, if a crack is propagating with a pure Mode I crack opening, the rate of release of energy is locally a maximum with respect to crack angle, as has been previously suggested [9]. Since the stress intensity factors given by (31) do not depend on the shape of the crack, the

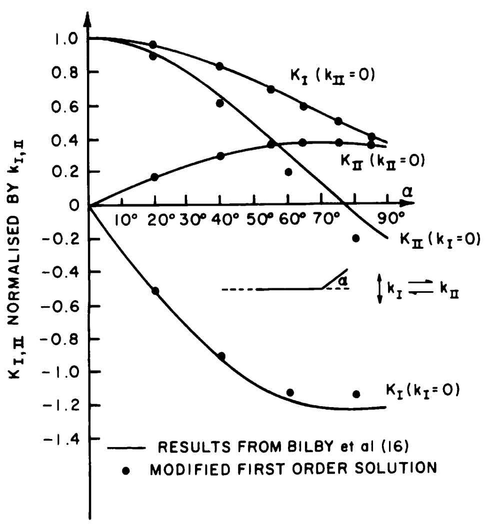

Figure 6. Stress intensity factors for infinitesimal kinks.

stress intensity factors for a crack that deviates by a small angle α from the Mode I path are given rigorously to a first order by

$$\begin{aligned} K_{\mathrm{I}} &= C_{11}k_{\mathrm{I}} = k_{\mathrm{I}} + O(\alpha^2) \\ K_{\mathrm{II}} &= C_{21}k_{\mathrm{I}} = -\frac{\alpha}{2}k_{\mathrm{I}} + O(\alpha^2). \end{aligned} \tag{32}$$

Once a kink is established, the crack opens so that the displacements near the tip are proportional to the square root of the distance from the tip and the rate of release of energy can be given in terms of the stress intensity factors by Irwin's expression

$$\mathcal{G} = \frac{(1-\nu^2)}{E}[K_{\mathrm{I}}^2 + K_{\mathrm{II}}^2). \tag{33}$$

Thus with the above results for K_{I} and K_{II},

$$\mathcal{G} = [1 + O(\alpha^2)]\,\frac{(1-\nu^2)}{E}\,k_{\mathrm{I}}^2, \tag{34}$$

proving that the energy release rate is stationary with respect to angle α when the crack propagates so that $K_{\mathrm{II}} = 0$. In fact, if Eqns. (31) with (29) are used as an approximation to the stress intensity factors,

$$K_{\mathrm{I}} = C_{11}k_{\mathrm{I}} = (1 - \tfrac{3}{8}\alpha^2)k_{\mathrm{I}} + O(\alpha^3) \tag{35}$$

(according to Fig. 6, this approximation seems to be accurate to second order in α), and

$$\mathcal{G} = \left(1 - \frac{\alpha^2}{2}\right)\frac{(1-\nu^2)}{E}k_I^2 + O(\alpha^3), \tag{36}$$

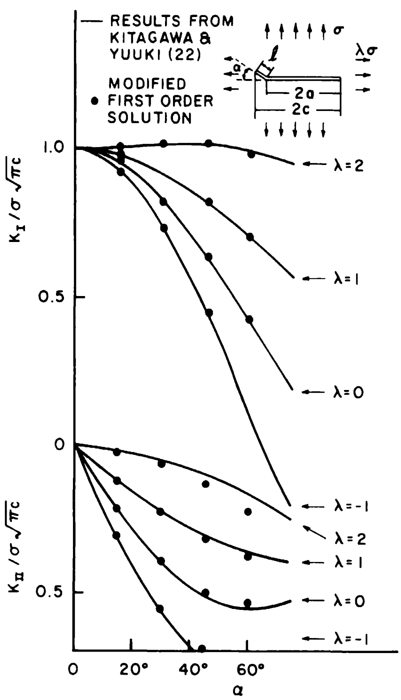

Figure 7. Stress intensity factors for kinked crack subjected to various biaxial stresses.

which shows that the rate of release is locally a maximum for the pure Mode I path, as previously proposed [9].

The problem of a main crack under a biaxial stress aligned with the line of the crack has beeh chosen to illustrate the accuracy of the first order solution for a crack with a finite kink. The stresses along the line of the pupative kink can be obtained from Westergaard's solution [27]. The stress intensity factors for a kink of length $\ell = 2a/10$ are compared in Fig. 7 with the results presented by Kitagawa and Yuuki [22] (those results agree with [21, 18]). The agreement is good except at large kink angles and high transverse stress. The results for uniaxial normal stress are compared with those of Kitagawa, Yuuki and Ohira [23] in Fig. 8. The results from Bilby, Cardew and Howard [16] for infinitesimally small kinks and an asymptotic value for the kink of infinite length have been added to Fig. 8. This latter value was obtained by noting that in the limit as ℓ tended to infinity, the stress intensity factor is that for a straight crack with a uniaxial stress at an angle $(\pi/2) - \alpha$ to the crack line. These results show that the modified first order solution is accurate to within 5% for all kink lengths and kink angles less than 30°. In the "modified" first order solutions for kinked cracks, we have calculated exactly the tractions acting along the prospective kink, from known solutions for the straight main crack, and used these exact tractions for T_x and T_y in Eqn. (27). To conform consistently with the first order solution, only first

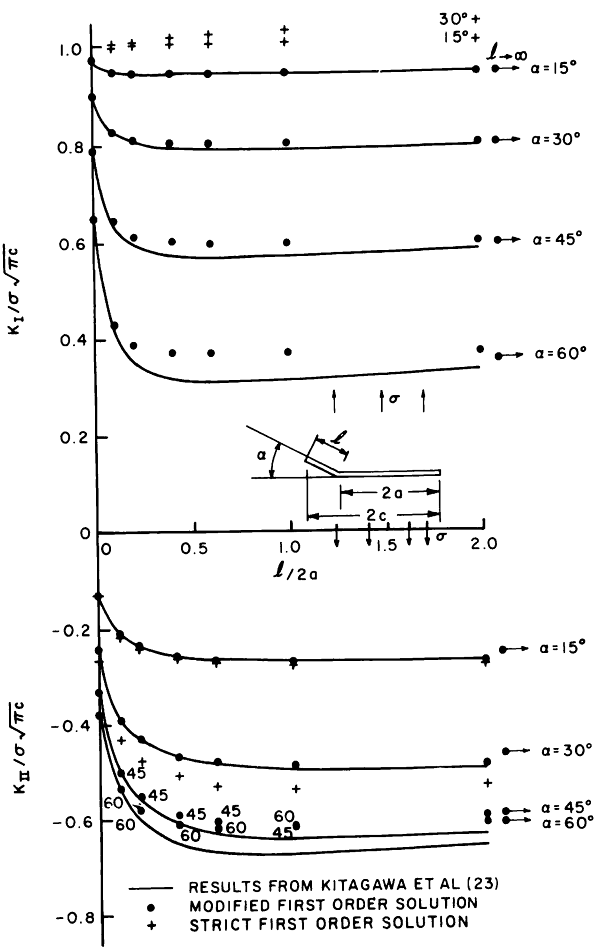

Figure 8. Stress intensity factors for kinked cracks subjected to uniaxial stress for various kink lengths.

order terms in the deviation from straightness should be included in the expressions for T_x and T_y. These "strict" first order results are shown in Fig. 8 for a 15° and 30° kink, and can be calculated from (19) or (27). It is seen that $K_{\rm I}$ is over-estimated by 5% for $\ell > 2a/10$ for the 15° kink, and over-estimated by greater than 5% for all 30° kinks. However, $K_{\rm II}$ is given well within 5% accuracy for all 15° kinks. In the next section, it is necessary to use a strict first order approximation to the tractions on the crack.

6. Crack path prediction

The first order solution for slightly curved cracks can be used to predict the path of a fracture as in the work of Banichuk [1] and Goldstein and Salganik [2, 3]. Here we focus on the initial development of the crack path, in a manner which will enable us subsequently to establish the condition for stability of a straight crack path. It is assumed that the crack propagates so that at its tip $K_{\rm II} = 0$, which also implies that the energy release rate is locally a maximum and, as remarked earlier, is consistent with various proposed mixed-mode criteria. In order to determine the crack path it is necessary to write an expression for the stress intensity factor at a crack tip located at some arbitrary point $(\ell, \lambda(\ell))$. The initial form of the crack path from one tip is independent of the location or growth at the other, and thus the theory is developed for a semi-infinite crack which at the origin is tangential to the x-axis and has its tip at the point $(\ell, \lambda(\ell))$, (see Fig. 9). The appropriately transformed version of Eqn. (27) for the semi-infinite crack is obtained by letting $L \to \infty$, $\omega/L \to 0$, and by writing

$$\eta = \lambda(x) - \lambda(\ell) - \lambda'(\ell)(x - \ell) \tag{37}$$

where now $\lambda'(x)$ means $\mathrm{d}\lambda(x)/\mathrm{d}x$. Thus

$$\begin{Bmatrix} K_{\rm I} \\ K_{\rm II} \end{Bmatrix} = \left(\frac{2}{\pi}\right)^{1/2} \int_{-\infty}^{\ell} \left[\begin{Bmatrix} T_y - \lambda'(\ell) T_x \\ T_x + \lambda'(\ell) T_y \end{Bmatrix} \frac{1}{(\ell - x)^{1/2}} + \begin{Bmatrix} T_x \\ T_y \end{Bmatrix} \frac{[\lambda(\ell) - \lambda(x) - \lambda'(\ell)(\ell - x)]}{2(\ell - x)^{3/2}} \right] \mathrm{d}x, \tag{38}$$

where the terms with $\lambda'(\ell)$ in the first bracket of the integrand occur because the x-axis is rotated by $\lambda'(\ell)$ from the tangent to the (extended) crack tip, Fig. 9.

Near the tip of the original crack located at the origin in Fig. 9, the stress field on the x-axis can be written as [25]:

$$\begin{aligned} \sigma_{yy}(x, 0) &= k_{\rm I}/(2\pi x)^{1/2} + O(x^{1/2}) \\ \sigma_{xx}(x, 0) &= k_{\rm I}/(2\pi x)^{1/2} + T + O(x^{1/2}) \\ \sigma_{xy}(x, 0) &= k_{\rm II}/(2\pi x)^{1/2} + O(x^{1/2}) \end{aligned} \tag{39}$$

for $x > 0$, where $k_{\rm I}$, $k_{\rm II}$ are stress intensities at the original crack tip and T corresponds to a local stress acting parallel to the crack at its tip. For example, $T = -\sigma$ for a

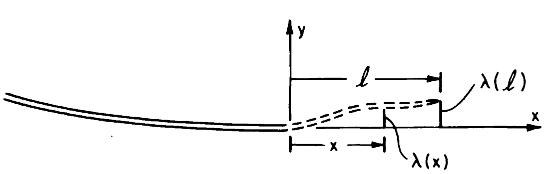

Figure 9. Small extension of a fracture from a pre-existing crack.

straight crack subjected to uniaxial normal stress σ. Sufficiently close to the original tip, the $O(x^{1/2})$ terms may be disregarded.

The tractions on a crack extension from the origin to a point $(\ell, \lambda(\ell))$ can be derived from this stress field and to first order in λ they are:

$$\begin{aligned} T_y &= \sigma_{yy}(x,0) + \lambda(x)\frac{\partial\sigma_{yy}(x,0)}{\partial y} - \lambda'(x)\sigma_{xy}(x,0) \\ &= \frac{1}{(2\pi x)^{1/2}}\left[k_{\mathrm{I}} + \frac{\lambda(x)}{2x}k_{\mathrm{II}} - \lambda'(x)k_{\mathrm{II}}\right], \\ T_x &= \sigma_{xy}(x,0) + \lambda(x)\frac{\partial\sigma_{xy}(x,0)}{\partial y} - \lambda'(x)\sigma_{xx}(x,0) \\ &= \frac{1}{(2\pi x)^{1/2}}\left[k_{\mathrm{II}} + \frac{\lambda(x)}{2x}k_{\mathrm{I}} - \lambda'(x)k_{\mathrm{I}}\right] - \lambda'(x)T \end{aligned} \tag{40}$$

where we have used the equilibrium equations, $\partial\sigma_{xx}/\partial x + \partial\sigma_{xy}/\partial y = \partial\sigma_{xy}/\partial x + \partial\sigma_{yy}/\partial y = 0$. Substitution of these tractions into (38) yields the following first order expression for K_{II} at the tip of the extending crack:

$$\begin{aligned} K_{\mathrm{II}} &= k_{\mathrm{II}} + \tfrac{1}{2}\lambda'(\ell)k_{\mathrm{I}} - \left(\frac{2}{\pi}\right)^{1/2} T\int_0^\ell \frac{\lambda'(x)}{(\ell-x)^{1/2}}\,\mathrm{d}x \\ &\quad + \frac{k_{\mathrm{I}}}{2\pi}\int_0^\ell \left[\frac{\lambda(\ell)-\lambda(x)}{\ell-x} + \frac{\lambda(x)}{x} - 2\lambda'(x)\right]\frac{\mathrm{d}x}{[x(\ell-x)]^{1/2}}. \end{aligned} \tag{41}$$

The second integral in this expression is identically zero for all functions $\lambda(x)$ and thus the first order expression for K_{II} at the tip of the crack is

$$K_{\mathrm{II}} = k_{\mathrm{II}} + \tfrac{1}{2}\lambda'(\ell)k_{\mathrm{I}} - \left(\frac{2}{\pi}\right)^{1/2} T\int_0^\ell \frac{\lambda'(x)}{(\ell-x)^{1/2}}\,\mathrm{d}x. \tag{42}$$

From the results of the previous section on kinked cracks, it is believed that this equation accurately gives K_{II} for crack paths with tangent angles lying within approximately 15° on either side of the prolongation of the crack.

Now, by imposing the criterion $K_{\mathrm{II}} = 0$ at the tip of the extending crack, (42) becomes an integral equation for the crack path, $y = \lambda(x)$, and can be rewritten as

$$\theta_0 = \lambda'(\ell) - \frac{\beta}{\pi^{1/2}}\int_0^\ell \frac{\lambda'(x)}{(\ell-x)^{1/2}}\,\mathrm{d}x \tag{43}$$

where

$$\theta_0 = -2k_{\mathrm{II}}/k_{\mathrm{I}}, \quad \beta = 2\sqrt{2}T/k_{\mathrm{I}}.$$

Clearly, θ_0 can be interpreted as the initial angle of crack growth (i.e., $\lambda'(0)$) and θ_0 is necessarily small for validity of the approach. Noticing that the last term of (43) is a convolution integral, the solution is readily obtained by Laplace transform techniques and is

$$\begin{aligned} \lambda'(x) &= \theta_0 \exp(\beta^2 x)\,\mathrm{erfc}(-\beta x^{1/2}), \quad \text{or} \\ \lambda(x) &= \frac{\theta_0}{\beta}\left[\exp(\beta^2 x)\,\mathrm{erfc}(-\beta x^{1/2}) - 1 - 2\beta\left(\frac{x}{\pi}\right)^{1/2}\right] \end{aligned} \tag{44}$$

where "erfc" denotes the complementary error function. For small $\beta^2 x$ this solution reduces to

$$\lambda(x) = \theta_0 x\left[1 + \frac{4}{3}\frac{T}{k_{\mathrm{I}}}\left(\frac{2x}{\pi}\right)^{1/2} + 4\frac{T^2 x}{k_{\mathrm{I}}^2} + \cdots\right], \tag{45}$$

whereas for large $\beta^2 x$ one has

$$\lambda(x) \to \frac{\theta_0 k_{\mathrm{I}}^2}{4T^2} \exp\left(\frac{8T^2 x}{k_{\mathrm{I}}^2}\right) \quad \text{for } \beta > 0 \quad \text{(i.e., for } T > 0\text{)},$$

and (46)

$$\lambda(x) \to \frac{\theta_0 k_{\mathrm{I}}}{|T|}\left(\frac{x}{2\pi}\right)^{1/2} \quad \text{for } \beta < 0 \quad \text{(i.e., for } T < 0\text{)}.$$

As illustrated in Fig. 10, the solution is such that when $T > 0$ the tangent angle of the crack path increases from θ_0 as the crack extends and, further, the rate at which the deviation from straightness occurs increases with increasing magnitude of T/k_{I}. But when $T < 0$, the tangent angle reduces from θ_0 and tends toward zero as the crack extends.

7. Stability of crack path under Mode I loading

Consider a straight crack subject to Mode I loading. Nominally, $k_{\mathrm{II}} = 0$, but we assume that due to inevitable imperfections in the loading system or in crack alignment, k_{II} will differ slightly from zero. Hence for Mode I loading we should regard θ_0 of (43), and Fig. 10, as some small random parameter characterizing the imperfection. Thus, while Eqns. (44–46) confirm the expected straight crack path for ideal Mode I loading ($\theta_0 = 0$), it is seen that in the actual case any initial imperfection will cause the crack path to deviate from straight when $T > 0$, whereas the initial imperfection has only a local effect on the path when $T < 0$. Thus for Mode I loading, the straight crack path is stable when $T < 0$ and unstable when $T > 0$. Further, in the unstable range the rate of deviation from straightness increases with T/k_{I}, (45, 46).

This result confirms previous, less precise attempts [6, 28] to show that the directional stability of cracks depends on the transverse stress T. Indeed, experimental evidence from double cantilever specimens [28] and biaxially stressed sheets [29] seems to be consistent with our criterion. For example, Fig. 11 has been redrawn from the results of Radon *et al.* [29] on centrally-cracked PMMA sheets loaded by a stress σ normal to the crack and $R\sigma$ parallel to it. For this case

$$T = (R-1)\sigma, \quad k_{\mathrm{I}} = \sigma(\pi L/2)^{1/2} \tag{47}$$

where L is crack length. The lines emanating from the crack in Fig. 11 are drawn to correspond with observed crack paths for different values of R. The theory predicts stability of the straight crack path when $R < 1$ (i.e., $T < 0$) and instability when $R > 1$ (i.e., $T > 0$). Further, since T/k_{I} is porportional to $R - 1$, the rate of deviation of the path from straightness should increase with the excess of R over unity. These features are seen to be consistent with the experimental results, although the theory loses applicability as the tangent angles increase beyond approximately 15° in Fig. 11.

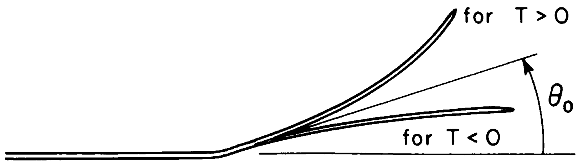

Figure 10. Path of crack growth and dependence on non-singular stress T acting parallel to the initial crack at its tip.

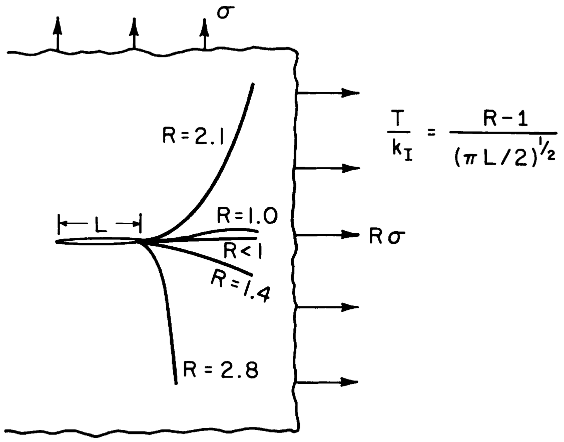

Figure 11. Crack paths observed by Radon *et al.* [29] in experiments on biaxially stressed PMMA sheets.

8. Conclusions

A first order solution for the stress intensity factors at the tip of a slightly curved or kinked crack opened by arbitrary tractions on its surface has been presented. From examination of the results for circular arcs and kinked cracks it is believed that this solution is accurate to within 5% for arbitrarily shaped cracks with local tangent angles differing from the average angle by up to approximately 15°. If the crack is only loaded near its tip greater accuracy can be obtained by orienting the crack so that its tip is aligned to the x-axis and using the actual tractions on the surface rather than first order approximation.

It has been shown that the various criteria of mixed mode crack propagation have the common feature that $K_{II} = 0$ at an extending crack tip, except at a finite kink. A method of determing the crack path under the condition $K_{II} = 0$ has been presented that is capable of predicting the path accurately within a wedge of approximately 15° on either side of the line of the pre-existing crack. For small crack growth under nominally Mode I loading, the straight crack path has been shown to be stable when $T < 0$ and unstable when $T > 0$, where T represents the uniform stress acting parallel to the crack at its tip in the second term of the Williams expansion [25].

REFERENCES

[1] N.V. Banichuk, "Determination of the Form of a Curvilinear Crack by Small Parameter Technique" Izv. An SSR, MTT 7, 2 (1970) 130–7 (in Russian).
[2] R.V. Goldstein and R.L. Salganik, "Plane Problem of Curvilinear Cracks in an Elastic Solid", Izv. An SSR, MIT 7, 3 (1970) 69–82 (in Russian).
[3] R.V. Goldstein and R.L. Salganik, *International Journal of Fracture* **10** (1974) 507–23.
[4] F. Erdogan and G.C. Sih, *Journal of Basic Engineering* 85 (1963) 519–27.
[5] J.G. Williams and P.D. Ewing, *International Journal of Fracture Mechanics* 8 (1972) 441–6.
[6] I. Finnie and A. Saich, *International Journal of Fracture* 9 (1973) RCR 484–6.
[7] P.D. Ewing and J.G. Williams, *International Journal of Fracture* 10 (1974) RCR 135.
[8] P.D. Ewing, J.L. Swedlow and J.G. Williams, *International Journal of Fracture* 12 (1976) 85–93.
[9] B. Cotterell, *International Journal of Fracture Mechanics* 1 (1965) 96–103.
[10] M.C. Hussain, S.L. Pu and J. Underwood, "Strain Energy Release Rate for a Crack Under Combined Mode I and Mode II", in *Fracture Analysis*, Proceedings of the 1973 National Symposium on Fracture Mechanics, Part II, ASTM Special Technical Publication 560 (1973) 2–28.

[11] K. Palaniswamy and W.G. Knauss, "On the Problem of Crack Extension in Brittle Solids Under General Loading", in *Mechanics Today*, S. Nemat-Nasser, Editor, Vol. 4, Pergamon Press (1978) 87–148.
[12] G.C. Sih, *Engineering Fracture Mechanics* 5 (1973) 365–77.
[13] G.C. Sih, "A Special Theory of Crack Propagation in Mechanics of Fracture, Methods, Analysis and Solutions of Crack Problems", Noordhoff (1973).
[14] G.C. Sih, *International Journal of Fracture* 10 (1974) 305–21.
[15] B.A. Bilby and G.E. Cardew, *International Journal of Fracture* 11 (1975) RCR 708–12.
[16] B.A. Bilby, G.E. Cardew and I.C. Howard, "Stress Intensity Factors at the Tip of Kinked and Forked Cracks", in *Fracture* 1977, Vol. 3, University of Waterloo Press (1977) 197.
[17] V.V. Dudukelanko and N.B. Romalis, "Direction of Crack Growth Under Plane Stress State Conditions", Izv. An SSR, MTT 8 (1973) 129–36.
[18] K.K. Lo, *Journal of Applied Mechanics* 45 (1978) 797–802.
[19] N.I. Muskhelishvili, "Some Basic Problems on the Mathematical Theory of Elasticity", Noordhoff (1952).
[20] G.C. Sih, P.C. Paris and F. Erdogan, *Journal of Applied Mechanics* 29 (1962) 306.
[21] S.N. Chatterjee, *International Journal of Solids and Structures* 11 (1975) 521–38.
[22] H. Kitagawa and R. Yuuki, "Anlaysis of Branched Cracks Under Biaxial Stresses", in *Fracture* 1977, Vol. 3, University of Waterloo Press (1977) 201.
[23] H. Kitagawa, R. Yuuki and T. Ohira, *Engineering Fracture Mechanics* 7 (1975) 515–29.
[24] V. Vitek, *International Journal of Fracture* 13 (1977) 481–501.
[25] M.L. Williams, *Journal of Applied Mechanics* 24 (1957) 109–114.
[26] B.R. Lawn and T.R. Wilshaw, "Fracture of Brittle Solids", Cambridge University Press (1975).
[27] H.M. Westergaard, *Journal of Applied Mechanics* 66 (1939) 49.
[28] B. Cotterell, *International Journal of Fracture* 2 (1966) 526–33.
[29] J.C. Radon, P.S. Lever and L.E. Culver, "Fracture Toughness of Pmma Under Biaxial Stress", in *Fracture* 1977, Vol. 3, University of Waterloo Press (1977) 1113–8.

RÉSUMÉ

On présente une solution pour les facteurs d'intensité de contrainte élastique à l'extrémité d'une fissure bi-dimensionnelle légèrement incurvée ou croquée.

La solution est exacte au premier degré pour la déviation à la surface de la fissure à partir d'une ligne droite, et est mise en oeuvre à l'aide de procédures de perturbations analogues à celles de Banichuck, et de Goldstein et Salganik.

Une comparaison avec les solutions exactes dans le cas de fissures en arc de cercle et de fissures droites avec un croquage indique que la solution du premier ordre est numériquement exacte pour des déviations déjà importantes par rapport à une fissure purement droite.

La solution est appliquée à la formulation d'une équation décrivant le parcours de la croissance d'une fissure, en supposant que ce parcours est caractérisé des conditions de Mode I pures (càd. $K_h = 0$) à l'extrémité de la fissure.

Cette méthode confirme la dépendance de la stabilité sous une mise en charge de Mode I d'un parcours d'une fissure droite par rapport aux signes du terme de contrainte non singulière représentant la contrainte de traction agissant parallèlement à la fissure dans un champ de Irwin–Williams décrivant l'expansion à l'extrémité de celle-ci. On montre que le chemin en ligne droite est stable sous une mise en charge de Mode I lorsque T est inférieur à O et instable lorsque T est supérieur à O.