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THE MECHANICS OF QUASI-STATIC CRACK GROWTH

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Abstract

Results on the mechanics of quasi-static crack growth are reviewed. These include recent studies on the geometry and stability of crack paths in elastic-brittle solids, and on the thermodynamics of Griffith cracking, including environmental effects. The relation of crack growth criteria to non-elastic rheological models is considered and paradoxes with energy balance approaches, based on singular crack models, are discussed for visco-elastic, diffuso-elastic, and elastic-plastic materials. Also, recent approaches to prediction of stable crack growth in ductile, elastic-plastic solids are discussed.

1. INTRODUCTION

This is a review of studies, for the most part recent, on the mechanics of quasi-static crack growth. The following topics are considered:

(i) Elasticity analysis of slightly curved or kinked cracks and the condition for stability of a straight crack path under Mode I (tensile) loading,

(ii) Irreversible thermodynamics formulation of conditions for Griffith crack growth or healing in elastic-brittle solids, including consideration of reactive environments which adsorb on the fracture surfaces,

(iii) Relation of crack growth criteria to non-elastic rheological properties of the crack-containing body, and examination of paradoxes with energy balance approaches for sharp, structureless crack tip models in visco-elastic, diffuso-elastic and elastic-plastic solids, and

(iv) Formulation of criteria for ductile crack growth in elastic-plastic solids based on J integral methods and on asymptotic, incremental plasticity analysis of singular fields for growing cracks.

There is no attempt at setting a unified theme. Rather, the paper examines some topics which have been of interest to the writer in recent work.

For preliminaries, the following results from linear elastic fracture mechanics are useful. There is a characteristic inverse square root stress singularity at a crack tip so that if r and θ are local polar coordinates at the tip,

$$\sigma_{ij} \to \sum_{J=I}^{III} K_J f_{ij}^J(\theta)/\sqrt{2\pi r} \quad \text{as } r \to 0 , \qquad (1)$$

where the modes I, II and III refer respectively to tensile, in-plane shear, and anti-plane shear stress acting on the prolongation of the crack ($\theta = 0$); the functions $f_{ij}^J(\theta)$ are universal functions that are normalized so that if indices ij correspond to the primary stress component of

mode J on $\theta = 0$, then $f_{ij}^{J}(0) = 1$, and the K's are the elastic stress intensity factors. The elastic strain energy release rate G is defined, for example, under plane conditions and quasi-static growth of the crack in its own plane by distance δa, by

$$(\delta W)_{\text{displ.}} = - G\, \delta a \qquad (2)$$

to the first order in δa, where W is the strain energy per unit thickness and the variation is taken at fixed displacements of load points (for mixed load and displacement boundary conditions an equivalent definition of G is given by replacing W with the total potential energy, equal to W plus the potential energy of the prescribed loads, and taking the variation at fixed boundary values). G is a positive definite quadratic function of the stress intensity factors and, in the case of an isotropic material under plane strain conditions,

$$G = [(1-\nu)(K_I^2 + K_{II}^2) + K_{III}^2]/2\mu \qquad (3)$$

where ν is the Poisson ratio and μ the shear modulus.

2. GEOMETRY AND STABILITY OF CRACK GROWTH PATH IN ELASTIC-BRITTLE SOLIDS

Solutions for 2-dimensional elasticity problems for slightly curved or kinked cracks in unbounded bodies have been given by Banichuck [1] and Goldstein and Salganik [2,3], by developing a first-order perturbation solution in terms of the deviation of the crack from a straight cut. A recent re-examination of the problem by Cotterell and Rice [4] leads to a remarkably simple form for the stress intensity factors when the x and y axes are oriented parallel and perpendicular to the crack at its tip, as shown in fig. 1, and the loads are given as a distribution of tractions T_x and T_y. Then, where $\eta = \eta(r)$ measures the deviation of the crack from a straight line, it is found that [4]

$$\begin{Bmatrix} K_I \\ K_{II} \end{Bmatrix} = \sqrt{\frac{2}{\pi L}} \int_0^L \begin{Bmatrix} T_y(r) \\ T_x(r) \end{Bmatrix} \sqrt{\frac{L-r}{r}}\, dr \qquad (4)$$

$$+ \sqrt{\frac{L}{2\pi}} \int_0^L \begin{Bmatrix} T_x(r) \\ T_y(r) \end{Bmatrix} \left[\frac{\eta(L)}{L^2} - \frac{\eta(r)}{r^2} \right] \sqrt{\frac{r}{L-r}}\, dr$$

to first order in η. The first integral is a well-known result for a straight crack. The second represents the effect of non-straightness, and it is interesting to note that to first order, only loadings in the shear direction (i.e., T_x) contribute to the tensile mode and conversely.

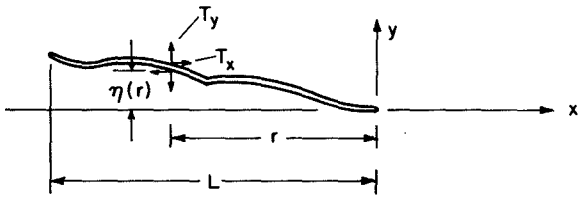


Fig. 1. Slightly curved or kinked two-dimensional crack. Coordinates are oriented so that x-axis is tangent to the crack at its tip; T_x and T_y are the surface tractions.

Cotterell and Rice [4] have used this result to derive the condition for stability of the straight path of crack growth for an initially straight crack loaded, nominally, in tension as in fig. 2a. It is supposed, however, that in addition to the primary stress intensity factor k_I at the tip, there is a small mode II intensity k_{II} which arises from some small imperfection of the loading device. The stress field on the x-axis directly ahead of the crack is indicated in fig. 2a, and it is to be noted that in addition to the inverse-square-root singular terms, there is a non-vanishing contribution to σ_{xx} at the tip, namely T, which represents a uniform tension acting parallel to the crack.

Fig. 2b shows the notation y(x) employed to describe a non-planar extension of the crack over distance a ahead of the tip and, when the forgoing perturbation solution is used, one finds that the mode II stress intensity factor at the tip is [4]

$$K_{II} = k_{II} + \frac{1}{2} y'(a) k_I - \sqrt{\frac{2}{\pi}}\, T \int_0^a \frac{y'(x)}{\sqrt{a-x}}\, dx \qquad (5)$$

where $y'(x) = dy/dx$ and, as appropriate for small extensions, the $0(\sqrt{x})$ terms in the original crack tip field displayed in fig. 2a have been neglected. Now, for continued growth of a crack along a smooth arc (of large curvature compared to the size of the fracture decohesion zone) in an isotropic, brittle solid, it may be argued [1-4] that the path selected is that for which $K_{II} = 0$ throughout the growth process.

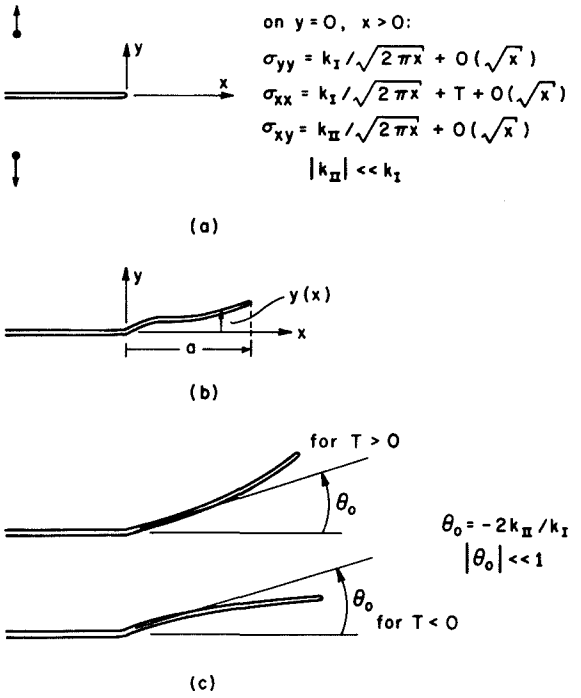


Fig. 2. Stability of the straight path of crack growth under mode I (tensile) loading. (a) Stress field near tip of crack loaded in tension; imperfection of loading system results in small mode II stress intensity. (b) Notation for describing extension of crack. (c) Crack path is unstable if $T > 0$, based on growth condition that $K_{II} = 0$ at advancing crack tip.

The condition $K_{II} = 0$ converts the above expresssion to an integral equation for the path, and the solution is [4]

$$y'(x) = \theta_0 \exp(8T^2 x/k_I^2)\ \mathrm{erfc}(-2\sqrt{2}\, T\sqrt{x}/k_I) \qquad (6)$$

where $\theta_o = -2k_{II}/k_I$ is the initial angle of crack growth and arises from the imperfection of the loading system. As illustrated in fig. 2c, the character of the solution is determined entirely by the sign of T. When $T > 0$ the crack tangent veers away from the original crack plane, with exponential growth at large x ($\mathrm{erfc} \to 2$ in that limit), and the straight crack path is unstable. When $T < 0$ the crack tangent gradually returns towards that for the initial crack, and the straight path is stable.

The derived stability criterion is in excellent agreement with experiment. For example, Radon et al. [5] observed crack paths in centrally precracked PMMA sheets loaded biaxially with tension σ perpendicular to the crack and $R\sigma$ parallel to it. In this case $T = (R-1)\sigma$ and, indeed, in all their tests with $R < 1$ (i.e., $T > 0$) the path veers outward from the initial crack plane, with the severity of the deviation increasing markedly with the excess of R over unity.

3. THERMODYNAMICS OF THE QUASI-STATIC GROWTH OF GRIFFITH CRACKS

In this section crack growth in highly brittle solids is considered. Further, the term "Griffith crack" is here used in the restricted sense of denoting a crack which separates an otherwise elastic material by direct decohesion of atomic bonds, leaving no trace of permanent deformation, e.g., dislocations, away from the crack plane. There is observational evidence that such conditions of crack growth exist in certain ceramic solids [6]. Also, there may be wider classes of materials in which conditions

immediately at the crack tip are of this type [7], i.e., no dislocation nucleation from the tip, despite the motion of existing dislocations in the concentrated crack tip stress field [8]. Here a thermodynamic formulation is given, following Rice [9], for the quasi-static growth of Griffith cracks, in a context that is wide enough to include thermally activated growth with lattice trapping [10-12] and environmentally influenced growth, e.g., in glasses and ceramics [13,14] in the presence of H_2O.

Indeed, the Griffith criterion for crack growth is usually regarded as a thermodynamical criterion. Yet the typical presentation deduces the criterion as one of an equilibrium crack size (a condition of stationary free energy) and makes no reference to the second law of thermodynamics as a principle governing irreversible processes, although fracture, as typically encountered, is essentially irreversible. With reference to fig. 3, we consider for simplicity unit thickness of a cracked body in plane strain. Suppose that the body is in contact with surroundings at temperature T_o (represented by the heat reservoir), and further suppose that crack growth, or healing, proceeds slowly enough that inertial effects may be neglected and that the body is at an essentially uniform temperature, except perhaps for some microscopic-sized region adjoining the crack tip. (These considerations may seem unduly restrictive but it will be shown in the next section that serious paradoxes arise when one attempts to generalize the energy balance, or thermodynamic, approach for a mathematically sharp crack which grows in a solid with non-elastic rheology, or even in an elastic solid for which the time scale of deformation involves significant coupling with diffusive fields or with heat flow fields as in coupled thermoelasticity).

First we neglect chemical interactions with the surroundings. Under the conditions considered, the first and second principles of thermodynamics require that

$$\dot{P\Delta} + \dot{Q} = \dot{U}$$
$$\Lambda \equiv \dot{S} - \dot{Q}/T_o \geq 0 \qquad (7)$$

where U is internal energy, S the entropy, Λ the entropy production rate, Q the heat transferred, P the load, all per unit thickness, and Δ the work-conjugate displacement. Note that in the given circumstances, U and S are functions only of Δ, a (crack length), and T ($=T_o$). Eliminating heat flow in the usual way leads to

$$T_o\Lambda \equiv P\dot{\Delta} - \dot{\Phi} \geq 0 \text{ , where } \Phi = U - T_oS \qquad (8)$$

is the Helmholtz free energy. Adopting Griffith's [15] procedure for computing Φ, we write

$$\Phi = \Phi_{elastic} + \Phi_{surface} = W(\Delta,a) + 2\gamma a \qquad (9)$$

where $W(\Delta,a)$ is the strain energy (at temperature T_o) as calculated from continuum elasticity, and γ is the surface free energy, i.e., 2γ is the work of reversible, isothermal separation of atomic bonds over unit area. The expression for Φ is motivated [9] by recalling that Φ may be equated to the "reversible work" of attaining the current configuration of the body from some reference configuration.

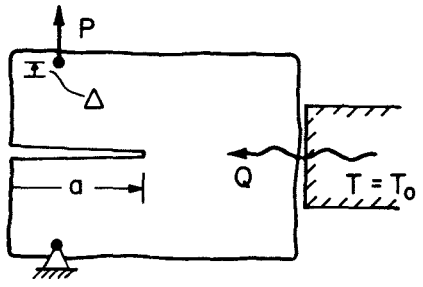


Fig. 3. Griffith crack in body at temperature T_o; for discussion of thermodynamic restrictions on crack growth.

When it is realized that $\partial W(\Delta,a)/\partial\Delta = P$, and the definition of Irwin's energy release rate, eq. (2) or

$$G = -\frac{\partial W(\Delta,a)}{\partial a} \quad , \tag{10}$$

is introduced, the foregoing expression for the entropy production rate becomes

$$T_o \Lambda \equiv (G - 2\gamma)\dot{a} \geq 0 . \tag{11}$$

This inequality is the proper expression of the consequence of the principles of thermodynamics for Griffith crack growth.

By contrast, the classical Griffith criterion [15], namely $G = 2\gamma$, corresponds to growth without entropy production, i.e., to fully reversible crack growth, and such may not be possible in real solids, even those in which cracks meet the definition of "Griffith cracks" adopted here.

Before proceeding to discussion of the inequality $(G - 2\gamma)\dot{a} \geq 0$, it is pertinent, especially to the discussion of the next section, to note that the Griffith criterion $(G = 2\gamma)$ is fully consistent with a more elaborate cohesive-zone fracture model, illustrated in fig. 4, in which surfaces are supposed to separate gradually at the crack tip. In this separation, the restraining stress σ is a function of opening displacement δ which, as shown, falls off to zero at a sufficiently great $\delta(=\delta_c)$. This consistency was first shown by Willis [16] through direct calculations based on linear elasticity, and in a "small strain" non-linear elastic context by Rice [17], later generalized by Eshelby [18] to include geometrical non-linearities also. Specifically, following Rice [17], the integral

$$J = \int_\Gamma (\phi n_x - \underset{\sim}{n}\cdot\underset{\sim}{\sigma}\cdot \partial\underset{\sim}{u}/\partial x)\, ds \tag{12}$$

(here ϕ is the strain energy density, $\underset{\sim}{\sigma}$ the stress, $\underset{\sim}{u}$ the displacement, and ds an element of arc length) is path-independent for all contours

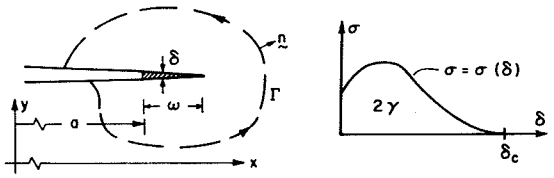


Fig. 4. Cohesive zone model for elastic-brittle fracture. Path Γ for evaluation of J integral may be shrunk to boundary of cohesive zone, showing equivalence of cohesive zone model and Griffith criterion when ω is small compared to overall geometric dimensions.

of the type Γ shown in homogeneous elastic materials. By taking the contour along the crack tip cohesive zone, Rice showed that the value of J when bonds at the crack tip are just pulled out of range of one another is

$$J = \int_0^{\delta_c} \sigma(\delta)\, d\delta \equiv 2\gamma . \tag{13}$$

But it is known also that J = G for a mathematically sharp crack [19], i.e., with no cohesive zone. When the cohesive zone is present,

$$J = G + O(\omega/a) \quad , \tag{14}$$

where ω is the size of the zone. Hence the crack growth condition for this cohesive zone model is that $G = 2\gamma$, in agreement with the Griffith criterion, in typical circumstances for which $\omega \ll a$.

However, when models for crack growth in idealized crystal lattices (with non-linear force-distance relations between atoms, falling to zero force at large distance) are considered [10-12], it is found that the value of G (say G^+) to grow a crack differs finitely from that (say G^-) for crack healing, and neither coincides with 2γ. This is the phenomena of "lattice trapping." Nevertheless, for cracks of macroscopic size the principles of thermodynamics must be respected and thus (11) requires that

$$G^- < 2\gamma < G^+ \quad , \tag{15}$$

i.e., that the reversible work of separation fall within the trapping range for G. The validity of this result was put into doubt by Esterling's [11] results for crack growth in crystal lattices: he concluded that in general the Griffith value of G(=2γ) did not fall within the trapping range. If such were so it would be a contradiction of thermodynamics, and close examination of Esterling's work [20] suggests that he arrived at this conclusion by comparing his lattice calculations of G^+ and G^- with the form of the Griffith criterion for an isotropic elastic continuum rather than for that which is the continuum limit of his lattice model. The effect is to increase all his reported values of the Griffith load by $\sqrt{2}$, bringing it into the trapping range except for a few cases which seem to be due to certain approximations in treatment of the non-linear force laws in his analysis, as remarked by Fuller and Thomson [12].

Evidently, the effect of lattice trapping is to give as a criterion of crack growth

$$G = G^+ = 2\bar{\gamma} \quad , \quad \text{where } 2\bar{\gamma} = 2\gamma + T_o\Lambda \geq 2\gamma . \tag{16}$$

That is, the effective fracture energy $2\bar{\gamma}$ includes 2γ plus the dissipation resulting from entropy production. Of course, in microscopic terms the entropy production corresponds to energy dissipation in phonon vibration waves, which are inevitably generated as the crack proceeds through the discrete atomic structure of a solid.

The concept of lattice trapping in the sense of a vanishing $\dot{a}$ for $G^- < G < G^+$ applies strictly at 0°K, leading to a growth rate versus G relation as in fig. 5a. However, at finite temperature, quasi-static crack growth (or healing) becomes possible by thermal activation processes [13,14], and the crack growth rate relation has the form

$$\dot{a} = \dot{a}(G, T_o) \tag{17}$$

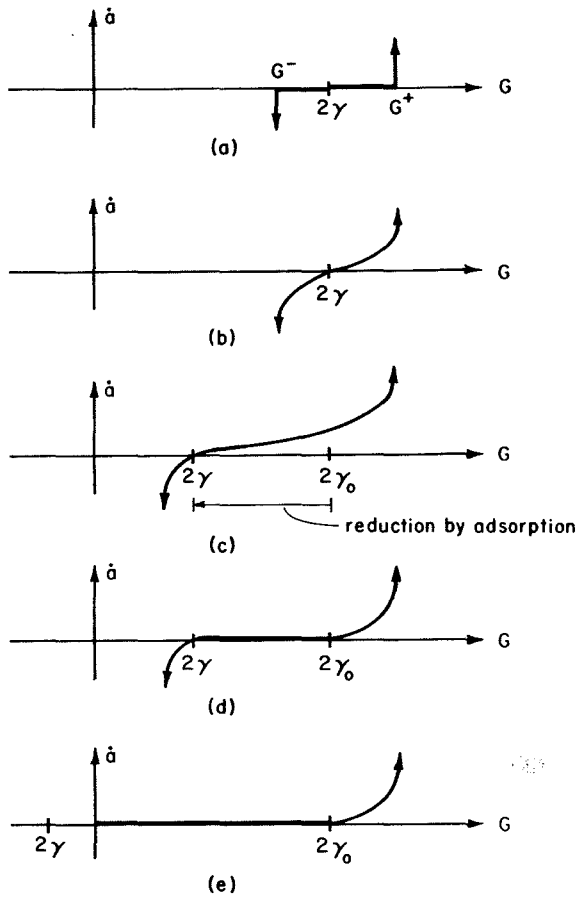


Fig. 5. Thermodynamically admissible kinetic relations, $\dot{a} = \dot{a}(G)$, for Griffith crack growth. (a) Lattice trapping; 2γ must lie in trapping range. (b) Thermally activated growth with lattice trapping. (c) Effect of reduction of γ by adsorption from environment. (d) Case of environment limited kinetically from access to separating crack tip bonds. (e) Case of kinetically-limited access but sufficiently strong adsorption so that 2γ is negative; crack healing is impossible.

Fuller and Thomson [12] leave as an open question that of whether the G value corresponding to $\dot{a} = 0$ necessarily coincides with 2γ in general, although they note that it does for simple models that they analyze. However, if the thermodynamic requirement (11) is not be to violated it is seen that quite generally the G value for $\dot{a} = 0$ must equal 2γ, and this is illustrated by the schematic form of the $\dot{a}$ vs. G relation in fig. 5b.

Fig. 6a shows data of Wiederhorn, quoted by Wachtman [13], on crack growth in glass under high vacuum at several temperatures, and this may be an example of thermally activated growth against trapping barriers.

3.1 SURFACE CHEMICAL EFFECTS

The statement of thermodynamic restrictions on growth has been extended [9] to the case of a chemically reactive environment which adsorbs on the surfaces of the crack. The formulation begins by assuming that the adsorbing species is present in a fluid phase contained in a rigid chamber which surrounds the cracked body and heat source of fig. 3, and which is fitted with a piston to maintain a uniform pressure p. In the manner of Gibbs [21], an adsorbed mass $\Gamma = \Gamma(p,T)$ per unit area of crack surface is defined by first defining the volume of the uniform fluid phase as the difference between the volume of the container and that of a loaded elastic solid of crack length a. The mass of this uniform fluid phase is defined to be ρ times its volume, where $\rho = \rho(p,T)$ is the mass density of homogeneous fluid, and the excess fluid mass, not accountable for in this way, is written as Γ times the total area over which surface adsorption occurs, thus defining Γ.

In this case one writes

$$\Phi = \Phi_{elastic} + \Phi_{fluid} + \Phi_{surface} \qquad (18)$$

where the terms refer, respectively to the strain energy W of the loaded elastic solid, the strain energy (Helmholtz free energy) of the uniform fluid phase, and the surface excess of Helmholtz free energy, written as $2\hat{\phi}a$ where $\hat{\phi}$ is the surface free energy per unit area of the solid surface and adsorbate. By applying the principles of thermodynamics in this case, Rice [9] showed that the thermodynamic restriction on crack growth is that

$$T_o \Lambda \equiv (G - 2\gamma)\dot{a} \geq 0 \qquad (19)$$

where now the symbol γ has the meaning

$$\gamma = \hat{\phi} - \mu\Gamma \qquad (20)$$

where μ is the chemical potential (equal to the Gibbs free energy per unit mass of the uniform fluid phase, satisfying $d\mu = dp/\rho(p)$ at fixed T). Further, by recourse to the Gibbs adsorption equation [21], it was shown that

$$\gamma = \gamma_o - \int_0^p [\Gamma(p)/\rho(p)]\,dp \quad (T = \text{const.}) \qquad (21)$$

where γ_o, the value of γ for an indefinitely dilute fluid phase (i.e., a vacuum, p = 0), coincides with the term γ as used previously.

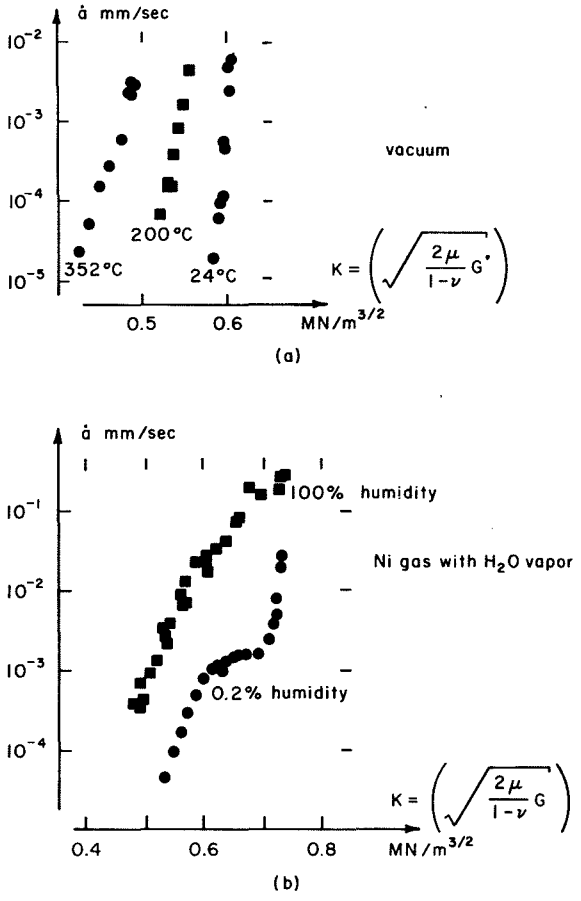


Fig. 6. Data of Wiederhorn on time-dependent crack growth in glass. (a) 61%-lead glass in vacuum. (b) Soda-lime-silica glass in environments of nitrogen gas with water vapor.

Thus, the effect of the adsorping species is to reduce the thermodynamic threshold for crack growth, and it is interesting that the amount of the reduction can be calculated from the apparently unrelated experiment of determining the adsorption isotherm (that is, the relation $\Gamma = \Gamma(p)$ at the temperature of interest). Fig. 5c shows schematically the reduction of the threshold for time dependent crack growth, and fig. 6b shows an example of environmentally influenced growth from data of Wiederhorn [13] on crack growth in glass in Ni gas containing different concentrations of H_2O (vapor), the chemically active agent. It is not definitively established that H_2O acts through the kind of surface adsorption process considered here, but this seems to be the most promising framework (versus, for example, one based on surface dissolution [22]) for explaining the effect.

The present framework also shows how environmental effects may contribute to the apparent irreversibility of crack growth even in solids which separate by "Griffith cracking." For example, fig. 5d shows schematically the effect of an environmental species which is inhibited by kinetic considerations from access to the bond-separation process at the crack tip. In this case, the threshold for growth is expected to be essentially unaltered from $2\gamma_o$, but the thermodynamic restriction rules out crack healing unless G is reduced below the adsorption-altered level 2γ. As a special case, which may in fact be rather typical of the effect of a strongly surface-reactive substance such as O_2, present in common environments, we note that 2γ may be negative. Indeed there is no reason, based on general thermodynamic principles, that adsorption could not be sufficiently strong to allow the integral in eq. (21) to exceed γ_o in value, making γ negative. This corresponds to the case in which the coherent solid is chemically unstable in its environment, but is preserved for long periods in a metastable state due to kinetic inhibitions against the environment gaining access to bonds between atoms of the solid. When 2γ is negative, as in fig. 5e, thermodynamics prohibits crack-healing, and cracking is an essentially irreversible process.

It should be noted, however, that if the crack surfaces are suddenly closed by removal of load, the thermodynamically favored desorption of the environmental species will be inhibited in the cases of figs. 5c and d, and in these cases as well as in that of fig. 5e it is possible that weak bonds (or an apparent, partial crack healing) can form on contact of the entrapped adsorbed layers. Further, for cases other than ideal Griffith cracks, with dislocation steps on the fractured surfaces or with the formation of small fracture debris particles, the resulting mechanical misfits may be the most important factors mitigating against crack healing.

It should be mentioned also that the effect of an adsorbed species along a material interface on the σ vs. δ relation (like that of fig. 4) for the interface has recently been analyzed [23]. Two limiting cases may be identified, namely "slow" separation at constant potential μ of the adsorbate, and "rapid" separation at constant adsorbate concentration Γ. In both limiting cases the effective reversible work, 2γ, of separation may be related to adsorption isotherms for the unstressed interface and for the two free surfaces created by separation.

4. PARADOXES IN ENERGY BALANCE APPROACHES TO CRACK GROWTH

The aim of this section is to emphasize an essentially negative result. Specifically, when perturbations due to lattice trapping were ignored, it was shown in the previous section (discussion in connection with fig. 4) that the cohesive zone fracture model leads to a result in agreement with the simpler Griffith energy balance for a mathematically sharp-tipped crack.

Specifically, one gets the "right" result in this case by equating the release rate, G, of mechanical energy, calculated from a continuum elasticity solution for a mathematically sharp-tipped crack (i.e., with no account of the actual finite-sized zone of gradual material separation, but instead with a point-singularity at the tip), to the energy 2γ absorbed in the separation process.

One is tempted to extend this energy balance procedure to cracks which grow in materials of non-elastic rheology, and there have been several attempts to do so. But it is important to note, as will be reviewed here, that every attempt to do so, e.g., in elastic-plastic [24], visco-elastic [25,26], and diffuso-elastic [27] (e.g., fluid infiltrated) materials has led to physically unacceptable results. There is, of course, no defect in the notion that energy must balance. Rather, the problem lies with the tacit assumption of the approach that the energy flow to the fracture zone can be calculated in an uncoupled manner, based on a continuum mechanical solution for growth of a macroscopically sharp crack that contains no reference to processes over the finite, if small, size scale of the separation zone.

4.1 ENERGY RELEASE RATE

We begin by writing an expression for the quasi-static rate G of energy release to the crack tip for crack growth in general non-elastic, single phase solids. The result is most simply obtained by generalizing the derivation by Rice [19] for crack growth in elastic solids, in a manner already adopted by Freund [28] for the elasto-dynamic case. Indeed, the principal result was obtained earlier by Cherepanov [29], although he did not emphasize certain restrictions on the expression and, while starting from a general thermodynamic approach rather than from the purely mechanical approach adopted here, seems to give an imprecise thermodynamical interpretation of the term which will be called w here.

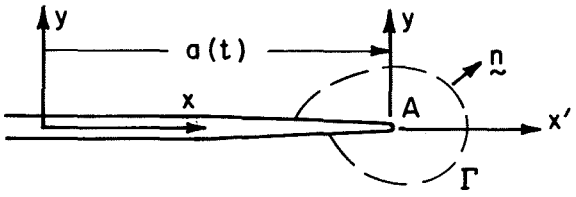


Fig. 7. Notation for discussing energy flux to a mathematically sharp, structureless crack tip; the x', y axes, region A, and contour Γ move through the material with the tip.

With reference to fig. 7, we consider a growing crack of length a(t). The x',y axis system moves with the tip, and the contour Γ, fixed relative to this system of axes, moves through the material. The region enclosed by Γ is A and this is, of course, a region of everchanging material points. Let

$$w = \int_{\underset{\sim}{0}}^{\underset{\sim}{\epsilon}} \underset{\sim}{\sigma} \, d\underset{\sim}{\epsilon} \equiv \int_0^t \sigma_{ij}(x,y,t)\, \partial \epsilon_{ij}(x,y,t)/\partial t \, dt \tag{22}$$

be the total density of accumulated stress working on the strain $\underset{\sim}{\epsilon}$ at a material point x,y. Then the time rate, $G\dot{a}$, of energy flow to the crack tip is the difference between the rate of traction working on the contour Γ and the rate of stress working on the fixed set of material points which coincide, instantaneously, with the time-dependent region denoted by A. Thus

$$G\dot{a} = \int_\Gamma \underset{\sim}{n}\cdot\underset{\sim}{\sigma}\cdot\dot{\underset{\sim}{u}} \, ds - \frac{d}{dt}\int_A w \, dA + \dot{a}\int_\Gamma w \, n_x \, ds \tag{23}$$

where the last integral arises because the region A moves relative to the material.

If we now <u>define</u> the integral J_Γ by

$$J_\Gamma = \int_\Gamma (w\, n_x - \underset{\sim}{n}\cdot\underset{\sim}{\sigma}\cdot\partial\underset{\sim}{u}/\partial x) \, ds \, , \tag{24}$$

and recall that

$$\dot{\underline{u}} \equiv \partial \underline{u}(x,y,t)/\partial t = \partial \underline{u}(x',y,t)/\partial t - \dot{a}\,\partial \underline{u}(x,y,t)/\partial x \ , \tag{25}$$

then the expression for G becomes

$$\dot{G}a = J_\Gamma \dot{a} + \left[\int_\Gamma \underline{n}\cdot\underline{\sigma}\cdot\partial \underline{u}(x',y,t)/\partial t \, ds - \frac{d}{dt}\int_A w(x',y,t)\,dx'dy \right] . \tag{26}$$

It is interesting that the bracketed quantity vanishes in certain cases, and for these cases

$$G = J_\Gamma = J \tag{27}$$

(i.e. independent of path Γ) .

Specifically, this occurs for

(i) elastic materials which are homogeneous, at least in the x direction; in that case a simple application of the divergence theorem (valid in the moving coordinates) shows that the two integrals of (26) cancel one another [19]; and

(ii) any material, elastic or non-elastic, but homogeneous in the x direction, in which crack growth takes place under conditions of steady state relative to the moving crack tip; i.e., $\underline{u}(x',y,t)$ and $w(x',y,t)$ are independent of t and both integrals in the brackets of (26) vanish.

This latter case is, of course, a rather idealized one, but it is to be expected (and, indeed, may be confirmed from known solutions) that for continuously growing cracks in elastic and non-elastic solids such "steady" conditions are approached asymptotically at the tip, in the sense that both integrals in brackets vanish as the region A is shrunk to zero size. In such cases the integral J_Γ will, generally, be path-dependent but one may write

$$G = \lim_{\Gamma \to 0} J_\Gamma \quad (=J_o \text{ , say}) . \tag{28}$$

Some care is, however, necessary in using this last result, and the nature of the crack tip singularity must be known in each case in at least enough detail to verify that the bracketed integrals of (26) do indeed vanish in the limit $\Gamma \to 0$. For example, the nature of strain singularities in elastic-ideally plastic materials is such that $G \neq J_o$ at the <u>onset</u> of growth, because the 1/r strain singularity associated with monotonic loading of a stationary crack changes to a weaker log (1/r) singularity for a continuously growing crack (see the next section). In this discussion "continuously growing" is taken to denote a process of crack growth in which the applied loads vary continuously with crack length; it excludes cases of load alteration at fixed crack length and hence excludes the first increment of crack growth subsequent to such load alterations.

Using the method of calculating G just outlined (or equivalent methods) for mathematically sharp-tipped cracks, one can now examine crack growth in materials of different non-elastic rheologies. What is found in each case is that the enforcement of an "energy balance" crack growth criterion in the form

$$G = 2\bar{\gamma} \ , \tag{29}$$

where $2\bar{\gamma}$ is some "non-continuum" work of fracture associated with the separation process, leads to results which are rather different from what one might expect on physical grounds. Moreover, the energy balance criterion, as implemented for a mathematically sharp crack, is shown to be in conflict with a cohesive zone model analogous to that of fig. 4. The intent is, of course, not to argue that the simple cohesive zone model is an adequate description of the fracture process in all cases. The important point is that it provides a mechanically self-consistent model which leads to predictions of crack growth without the necessity of introducing some postulate external to the model itself. As such, the failure of the energy balance approach to agree with it (in other than elastic materials) shows that the energy balance approach is inadequate as a general criterion of fracture, and suggests that

a proper continuum mechanical model of crack growth must include at least some details of coupling, over a finite size scale, with the microscale processes of separation.

4.2 LINEAR VISCOELASTIC SOLIDS

Although the first indications of defects in an energy balance approach were given for elastic-plastic solids [24], the case of crack growth in linear viscoelastic solids is better known and simpler to analyze in detail. In this case the paradoxical nature of the energy balance criterion was pointed out by Kostrov and Nikitin [25] and Barenblatt et al. [26], whereas formulations that recognized the importance of a finite-sized crack tip fracture process zone were further developed by Mueller and Knauss [30], Knauss [31] and Schapery [32].

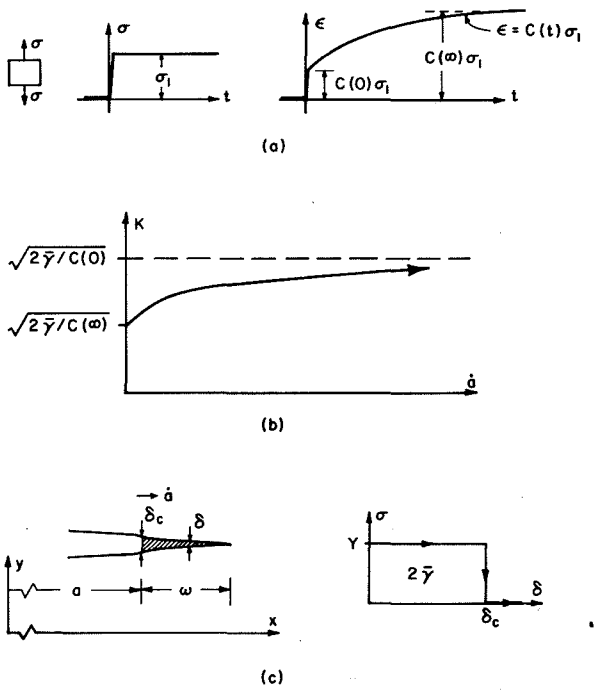


Fig. 8. Crack growth in linear viscoelastic solid. (a) Definition of creep compliance function C(t) for plane-strain tension test. (b) Expected form of relation between stress intensity factor and crack growth rate. (c) Cohesive zone fracture model.

For basic notation, an element of material subjected to plane strain tension is shown in fig. 8a, and the strain in response to a step in stress defines the creep compliance function C(t). The short and long time limits (if the latter exists) correspond to instantaneous and long time elastic response, and

$$C(0) = (1-\nu_o)/2\mu_o \quad , \quad C(\infty) = (1-\nu_\infty)/2\mu_\infty \qquad (30)$$

where ν_o, ν_∞ and μ_o, μ_∞ are the corresponding values of the Poisson ratio and shear modulus. In the simple loading cases which we will consider, i.e., plane-strain traction boundary value problems and isotropic materials, the in-plane stress field in a cracked body is the same as for a linear elastic solid of the same geometry. Indeed, the stress field has no dependence on constitutive parameters of the material, and is completely determined by the distribution of applied loadings and crack length.

If, for simplicity, we consider a material model for which the work of fracture, $2\bar{\gamma}$, is independent of the (quasi-static) crack speed, $\dot{a}$, then one expects the following results: for very slow growth speeds the material responds as an elastic solid with the long-time modulus $C(\infty)$ and, from (3,30), the stress intensity factor K required for growth is given by

$$G = C(\infty)\, K^2 = 2\bar{\gamma} \quad (\dot{a} \to 0^+) \quad . \qquad (31)$$

On the other hand, for rapid (but still quasi-static) growth the material responds with the instantaneous elastic properties and hence

$$G = C(0)\, K^2 = 2\bar{\gamma} \quad (\dot{a} \to \infty) \quad . \qquad (32)$$

The schematic form of an $\dot{a}$ versus K curve, consistent with these two limits, is shown in fig. 8b. Evidently, within the limits indicated time-dependent crack growth is to be expected, with $\dot{a}$ increasing with the level of the applied load (for a given crack size, K is proportional to the applied load).

However, this is not what the energy balance approach predicts. In particular, G can be evaluated from (28) as J_0 for a continuously growing crack. But the value of the J_Γ integral as $\Gamma \to 0$ is determined by the response of material elements immediately at the crack tip. Owing to the moving singularity of stress, these elements respond in the $\Gamma \to 0$ limit in a manner that depends only on the instantaneous elastic properties of the material. Thus, the energy balance approach as implemented for a mathematically sharp, structureless crack tip leads to the result [25,26]

$$G = C(0)\, K^2 \quad \text{for all } \dot{a} \,. \tag{33}$$

Since, for a given crack length, K is merely proportional to the applied load and not dependent on material properties, the energy balance criterion $G = 2\bar{\gamma}$ is seen to lead to the paradoxical prediction of a complete lack of crack speed dependence in the growth criterion; the expected form of the result, as indicated schematically in fig. 8b, is not obtained.

The situation is entirely different when we examine the solution based on a cohesive zone model. For simplicity, the cohesive strength is taken as constant at $\sigma = Y$ out to a critical separation δ_c, as in fig. 8c. Obviously, $Y\delta_c = 2\bar{\gamma}$, the work of fracture. Further, we shall deal only with the case for which the cohesive zone size ω is very small compared to crack length and other dimensions of the cracked body. In that case the external loadings can be described in the conventional "small scale yielding" sense [19] as the imposition of the surrounding inverse-square-root singular stress field, of intensity K, on an infinite body with semi-infinite crack. Then the cohesive zone size is related to K by [19, 31]

$$\omega = \pi K^2/8Y^2 \,, \tag{34}$$

and this condition removes the singularity at the end of the cohesive zone. To find the opening displacements δ in the cohesive zone, we observe that for an elastic material with properties correspoding to C(0) the opening within the cohesive zone is [19]

$$\delta = [C(0)\, K^2/Y]\; f\,[(x-a)/\omega] \tag{35}$$

(see fig. 8c for notation) where the function $f(\lambda)$ is [19], for $0 \le \lambda \le 1$,

$$f(\lambda) = (1-\lambda)^{1/2} - (\lambda/2)\log\left(\frac{[1+(1-\lambda)^{1/2}]}{[1-(1-\lambda)^{1/2}]}\right). \tag{36}$$

Note that $f(0) = 1$, $f(1) = 0$. Accordingly, by correspondence methods [31], the crack opening in a linear viscoelastic material can be given. In particular, one may solve for the displacement at the crack tip in the case of steady state crack growth (i.e., δ depends only on $x - \dot{a}t$, where $\dot{a}$ is constant). When this crack tip displacement is set equal to $\delta_c (=2\bar{\gamma}/Y)$ as a criterion for crack growth one obtains, by rearranging a result of Knauss [31], the equation

$$\left[\int_{\lambda=1}^{\lambda=0} C(\lambda\omega/\dot{a})\; df(\lambda)\right] K^2 = 2\bar{\gamma} \,, \tag{37}$$

where it is to be recalled that ω is related to K^2 by (34) and also that $\int df(\lambda) = 1$ over the interval considered, since $f(\lambda)$ increases from 0 to 1 as λ varies from the lower to upper limit on the integral.

It is evident that the crack growth criterion (37) based on the cohesive zone model has the expected limiting behavior. As $\dot{a} \to 0^+$, the criterion reduces to (31); as $\dot{a} \to \infty$ it reduces to (32). Indeed, fig. 8b has been drawn to represent, approximately, the prediction of this criterion for a standard linear solid with $C(\infty) = 4\, C(0)$.

While the cohesive zone model leads to fundamentally different results from those of the energy balance criterion (again, as implemented for a sharp, structureless crack tip model), it is

possible to see how the prediction of (33) for the energy balance model arises. Indeed, if we consider the cohesive strength, Y, to increase without limit, then the zone size $\omega \to 0$ and, in this limit, the cohesive zone model leads to the same result as the energy balance model, namely $C(0)\, K^2 = 2\bar{\gamma}$ for all $\dot{a}$. But, from (37), it is appropriate to take this limit, corresponding to the sharp, structureless crack tip, only when the characteristic time, $\omega/\dot{a}$, associated with the decohesion process is very much smaller than any characteristic relaxation time of the material. Such a situation is obviously inappropriate for consideration of the long time strength of viscoelastic solids.

It would be difficult to argue that the simple cohesive zone model described here is an adequate representation of fracture in actual solids. The model is used here only to emphasize the inadequacy of the energy balance approach. In actual viscoelastic solids it may be inadequate to confine all material non-linearities to a single plane and to neglect time or rate dependence in the σ versus δ relation for the separation zone. Nevertheless, impressive correlations of crack growth data over a wide range of growth rates have been made by Meuller and Knauss [30], Knauss [31], and Schapery [32], based on the simple cohesive model described or, in the cae of Mueller and Knauss [30], on yet simpler ways of accounting for the size of the cohesive zone. For example, fig. 9 shows a plot by Schapery [32] of data from Mueller and Knauss on Solithane 50/50, where the points depict experimental data and the solid lines represent predictions of the cohesive zone model, based on writing C(t) as $C_0 + C_2\, t^{1/2}$ and on choosing values of Y and $\bar{\gamma}$ to best fit the data. A similar fit of the data without recourse to the approximation of C(t) is given by Knauss.

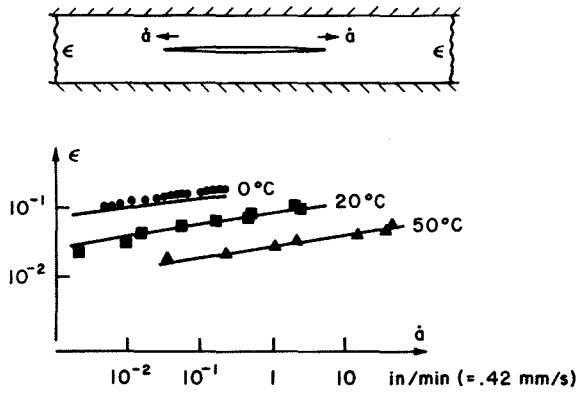


Fig. 9. Data of Mueller and Knauss on crack growth in Solithane 50/50; solid lines from analysis by Schapery based on cohesive zone model.

4.3 DIFFUSO-ELASTIC SOLIDS

This class of solids contains a mobile species which can diffuse under stress-induced changes in its chemical potential. The simplest case corresponds to the fluid-infiltrated solid of Biot [33], wherein state is characterized by the (total) stress tensor σ_{ij} and pore pressure p. Appropriate elastic constitutive laws relate the strain tensor ϵ_{ij} of the solid phase and the fluid mass content m, per unit volume of the porous material, to σ_{ij} and p, and a constitutive relation of the Darcy type relates the fluid diffusion rate to gradients in p; see Rice and Cleary [34] for a recent review. It is appropriate to note that, analogously to viscoelastic solids, there are two limiting cases in which response corresponds to that of a classical elastic solid. For slow deformations (by comparison to the time scale for diffusive transport) material elements which are connected by diffusion paths to a fluid source at fixed pressure can deform without associated changes in pore pressure. Such response is termed "drained," and in this limit an isotropic fluid-infiltrated material behaves as an elastic solid of shear modulus μ and Poisson ratio ν_d ("d" for drained).

In the other limit, that of very rapid deformation, there is no time for alteration of the fluid mass content m in material elements, and deformation causes associated alterations of p. Response in this limit is termed "undrained," and the material behaves as a classical elastic solid with shear modulus μ and Poisson ration ν_u ("u" for undrained). We note that $\nu_u \geq \nu_d$, which means that the volumetric stiffness is greater for undrained than for drained deformation, although the shear stiffness is the same in both cases. Values of ν_d and ν_u have been summarized from experiments on a variety of rocks [34] and from calculations based on flat, crack-like pore spaces, [35].

For plane strain deformations in the x,y plane the governing equations are [34]

$$\partial\sigma_{xx}/\partial x + \partial\sigma_{xy}/\partial y = 0 \quad , \quad \partial\sigma_{xy}/\partial x + \partial\sigma_{yy}/\partial y = 0$$

$$\nabla^2[\sigma_{xx} + \sigma_{yy} + 2\beta\,(\nu_u - \nu_d)p/(1-\nu_d)] = 0 \qquad (38)$$

$$[\nabla^2 - (1/c)\,\partial/\partial t]\,(\sigma_{xx} + \sigma_{yy} + 2\beta p) = 0 \ .$$

Here, c is the diffusivity; it is proportional to the permeability coefficient in Darcy's flow law and also depends on elastic constants of the solid and its fluid constituent [34]. The parameter β is most simply interpreted by observing that the alteration in fluid mass content m, from its value when $\underset{\sim}{\sigma} = p = 0$, is proportional to the quantity $(\sigma_{xx} + \sigma_{yy} + 2\beta p)$. Hence, for stress application under undrained plane strain conditions the induced pressure is

$$p = -(\sigma_{xx} + \sigma_{yy})/2\beta \ . \qquad (39)$$

Clearly, $0 \leq \beta \leq 1$, the upper limit being approached for the case of separately incompressible solid and fluid constituents (typical model of a water-saturated soil, in which case $\nu_u = 0.5$) and the lower limit for a highly compressible pore fluid (in which case $\nu_u \to \nu_d$).

Rice and Simmons [27] have solved the problem of a plane strain shear crack (mode II) which advances under steady state conditions, i.e., $\underset{\sim}{\sigma} = \underset{\sim}{\sigma}(x - \dot{a}t, y)$ where $\dot{a}$ is the constant crack speed, in such a material; the problem is discussed further by Simons [36,37]. The crack surfaces are loaded only over a distance L behind the crack tip by a uniform distribution of shearing traction τ. In a classical elastic solid this loading would cause a (mode II) stress intensity factor

$$K_{nom} = (8/\pi)^{1/2}\,\tau\,L^{1/2} \ , \qquad (40)$$

where the subscript "nom" indicates that this should be regarded only nominally as a stress intensity factor for a fluid-infiltrated material.

What Rice and Simons [27] find is that there is a $r^{-1/2}$ stress singularity at the tip, of identical form to that for a classical elastic solid and, further, the pore pressure p (or, better, alteration from ambient pore pressure) vanishes at the tip. That is, the $r^{-1/2}$ singularity corresponds to fully drained response. The stress intensity factor is found to be

$$K = K_{nom}\,h(aL/c) \qquad (41)$$

where the function h decreases monotonically with increasing crack speed and has the limits

$$h(0) = 1 \ , \qquad (42)$$

$$h(\infty) = (1-\nu_u)/(1-\nu_d) < 1 \ .$$

Now, the formalism for computing the energy release rate, eqs. (23-28), does not apply in this case because, in general, work of the pore pressure on fluid motion relative to the solid, and related energy alterations, has to be included. But owing to the fact that $p = 0(r^{1/2})$ at the tip and that the diffusive flow rate $q = 0(r^{-1/2})$, these terms make no contribution at the tip and G is given by J_o, which then coincides

with the ordinary expression (eq. 3) for the energy release rate in an elastic body having properties corresponding to drained response. Thus

$$G = (1-\nu_d)K^2/2\mu \qquad (43)$$

$$= [(1-\nu_d)K_{nom}^2/2\mu]\ h^2(\dot{a}L/c) \ ,$$

and the energy balance criterion $G = 2\bar{\gamma}$ leads to the result that

$$G_{nom} \equiv (1-\nu_d)K_{nom}^2/2\mu = 2\bar{\gamma}/h^2(\dot{a}L/c) \qquad (44)$$

in order to grow the crack at speed $\dot{a}$.

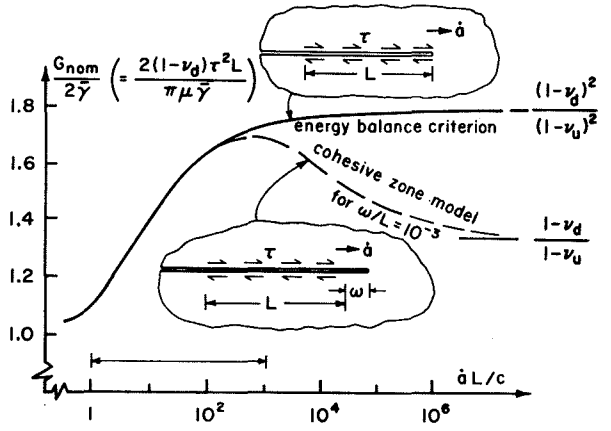


Fig. 10. Shear crack growth in a fluid-infiltrated elastic solid. Comparison of energy balance criterion with cohesive zone model. Curves are drawn for $(1-\nu_d)/(1-\nu_u) = 1.33$. Marked portion of speed axis corresponds to range of observed fault creep events; see text.

This expression is plotted as the solid curve in fig. 10; what is paradoxical about it is that in the limit $\dot{a} \to \infty$, the expected result is not recovered. Indeed, in this limit it is to be expected that the material response is everywhere undrained, so that the material responds as a classical elastic solid with constants ν_u and μ. The K for any such classical elastic solid is K_{nom}, and hence the expected result as $\dot{a} \to \infty$ is

$$G = (1-\nu_u)K_{nom}^2/2\mu = 2\bar{\gamma} \ , \quad (\dot{a} \to \infty) \ , \qquad (45)$$

or

$$G_{nom} \equiv (1-\nu_d)K_{nom}^2/2\mu \qquad (46)$$

$$= 2\bar{\gamma}\ (1-\nu_d)/(1-\nu_u) \ , \quad (\dot{a} \to \infty) \ .$$

But, by contrast, what the energy balance criterion (for a sharp-tipped, structureless crack model) actually predicts is, from (44), using (42) for $h(\infty)$,

$$G_{nom} = 2\bar{\gamma}(1-\nu_d)^2/(1-\nu_u)^2 \ , \quad (\dot{a} \to \infty) \ . \qquad (47)$$

Again, the paradox is resolved by appeal to a cohesive zone fracture model. In particular, the lower inset in fig. 10 shows a cohesive zone of size ω at the end of the crack, and Rice and Simons [27] solved the case in which the shear strength is constant within the cohesive zone (analogous to the tensile case of fig. 8c). Their result, for the case $\omega = 10^{-3}L$, is shown by the dashed line in fig. 10 and this does seem to have the correct asymptotic behavior (i.e., to agree with the prediction of eq. (46) as $\dot{a} \to \infty$).

Indeed, while the singularity at a sharp, structureless crack tip is always of a drained type, it is nevertheless true that at high velocities an effectively undrained field results outside the tip region. The distance to this effectively undrained zone, i.e., a kind of diffusion penetration distance, is of the order $c/\dot{a}$. When $c/\dot{a}$ is much larger than any size scale involved in the fracture process, the sharp crack model and energy balance criterion are quite reasonable. But as speed increases, $c/\dot{a}$ decreases and must finally become comparable to the fracture process zone. Indeed, at dimensionless speeds greater than 10^3 in fig. 10, $c/\dot{a}$ is smaller than ω, and in this range the energy balance and

cohesive zone model diverge widely. Ruina [38] gives an analogous discussion of size effects for tensile crack growth in fluid-infiltrated solids. Note that by contrast to the viscoelastic case, fig. 8b, there is not a monotonic increase, but rather a peak, in the (nominal) driving force versus velocity relation.

It is interesting to observe that the portion of the velocity axis between approximately 1 and 10^3 corresponds to the range of observed "creep" events [27,39,40] on the San Andreas fault in Central California, assuming a value of $c = 1\ m^2/sec$, which is thought to be representative of field conditions [27], and identifying L with the length of the slipping region (0.1 to 10 km) and $\dot{a}$ with the propagation speed (1 to 10 km/day). This prompts the suggestion [27] that the stabilization of a shear crack by pore fluid effects (i.e., the fact that the required driving force is an increasing function of speed, initially) may have something to do with making possible stable creep propagations of slip offsets along faults. On the other hand, the analysis also suggests that if the process is overdriven (e.g., loading in excess of the peak of the curve in fig. 10) then no quasi-static solution will exist and, presumably, an unstable seismic propagation of the slip offset occurs.

4.4 THERMOELASTICITY

It may be noted also that the linearized equations of a Biot fluid-infiltrated elastic solid are analogous, term by term, to the linearized equations of coupled thermoelasticity [41,34]. Hence it is to be expected that when coupling to the temperature field is considered, the correct results will not emerge from an energy balance approach in the adiabatic limit of rapid crack growth (i.e., analogous to the undrained limit, whereas isothermal response is analogous to drained).

The problem of a proper thermoelastic formulation is further complicated by the fact that the fracture process zone must be regarded, effectively, as a source of heat. But for the sharp-tipped, structureless crack model, this heat supply is in the form of a point source at the crack tip and hence leads to a temperature singularity there. It is not known if the laws of thermodynamics, when applied to a vanishingly small zone around this singularity, will lead to sensible results concerning crack growth. The preceding examples suggest that it will be necessary to include some account of the finite size of the fracture process zone.

4.4 ELASTIC-PLASTIC SOLIDS

The paradox associated with adaptation of the Griffith energy balance procedure to non-elastic materials was first noted in connection with elastic-ideally plastic solids. Indeed, as was shown for that case by Rice [24], the sharp-tipped, structureless crack model leads to the result $G = 0$ for crack growth, independently of the load level. Hence, there is no energy surplus, from the continuum solution for crack growth, which can be equated to the separation work, $2\bar{\gamma}$, and for this class of materials an energy balance criterion predicts that crack growth cannot occur.

A different way of seeing the result is provided by eq. [28], expressing G as J_o for a continuously growing crack (recall that this way of calculating G is not valid for the first increment of growth following load increase). It is easy to see that J_o, and hence G, will be zero whenever

$$r\, \underset{\sim}{\sigma}\, \partial \underset{\sim}{u}/\partial \underset{\sim}{x} \to 0 \quad \text{as} \quad r \to 0 \tag{48}$$

at the tip. This vanishing limit does indeed result according to available incremental plasticity solutions for growing cracks in non-hardening [19,42-44] and linearly strain hardening [45] materials. For example, the strain and displacement gradient fields in non-hardening materials are singular only as $\log(1/r)$ and the

stress is non-singular, so the condition of eq. (48) is met and G = 0.

This result has often been obscured in finite element or finite difference studies in which the crack is advanced grid point by grid point. Then, a finite energy release inevitably results in each growth step, owing to the finiteness of the grid spacing, and one can be led to quite erroneous results for G. On the other hand, sensible plotting of finite-element results as in studies by Kfouri and collaboraors [46,47] serves to support the theoretical result G = 0. For example, fig. 11 shows a quantity denoted by G^{Δ}, which is the energy released in a one-element growth step divided by the new crack area of that step, as a function of the step size (i.e., element size) $\Delta \ell$. The results are based on an incremental elastic-plastic analysis for a center-cracked bar in plane strain tension. The material is of the Mises type with linear strain hardening. All results shown are for a range of loading in which the yield zone is very small compared to specimen dimensions. In the figure G^{Δ} is normalized by the energy release rate for a similarly loaded elastic material, and the step size $\Delta \ell$ is normalized by a quantity which measures, approximately, the maximum radius of the plastic zone under small scale yielding conditions. It is seen that the numerical results do indeed seem consistent (dashed-line extrapolation) with the theoretically expected limit of $G^{\Delta} \to 0$ for a vanishing ratio of step size to plastic zone dimension. The results shown correspond to an average of G^{Δ} for the last three of four growth steps beyond the crack length at which the specimen was first loaded. The inset figure shows typical boundaries of plastically deformed material and of the currently active plastic zone during such a growth process.

Kfouri and Miller [46] suggest that a viable fracture criterion can be obtained by an energy balance approach, effectively by considering the step size $\Delta \ell$ as a property of the material [47] and equating the resulting G^{Δ} to some critical value. This may be thought of as corresponding, in some very approximate way, to the inclusion of a finite-sized zone of decohesion in the fracture model. A more extensive discussion of quasi-static crack growth in ductile elastic-plastic solids is given in the following section.

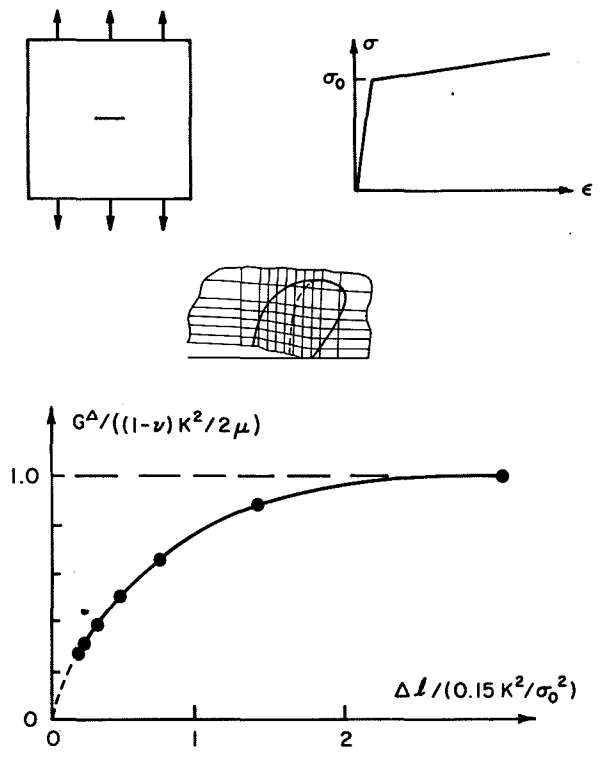


Fig. 11. Elastic-plastic finite-element results of Kfouri for energy release rate based on finite crack growth steps (equal to element size, $\Delta \ell$). Growth step is normalized by approximate measure of the maximum extent of plastic zone, and results seem consistent with theoretical result that G=0 for continuously growing crack.

5. ELASTIC-PLASTIC MODELS FOR STABLE CRACK GROWTH

For ductile structural metals in which crack growth occurs by a plastic "tearing" mechanism (e.g., involving microvoid nucleation and plastic growth, rather than brittle cleavage), it is

typical that the first increments of crack extension are not immediately unstable. Rather, stable crack growth occurs under increasing imposed force or, depending on the nature of the loading method, under increasing imposed displacement of the load point(s). Finally, a state is reached at which no further increment of the imposed loading quantity (force or displacement, as the case may be) is required in order to continue to meet the critical condition for growth at the crack tip, and at that state the process of quasi-static crack growth becomes unstable and gives rise to a running fracture. Here the concern is with elastic-plastic materials for which strain-rate effects are insignificant over the range of response considered and, accordingly, the stable growth phenomenon is analyzed within the context of rate-insensitive elastic-plastic constitutive models.

There has been extensive progress on characterizing the onset of crack growth, as reviewed recently by Rice [48], but at present there is no well agreed upon method of analyzing the subsequent stable crack growth. Several approaches are being explored in current research. For example, the use of an energy balance criterion in finite growth steps has been mentioned at the end of the previous section [46,47] and various approaches based on J-integral methods [49-51], critical near tip openings [44,51], and other, mostly-numerical-based, methods [52] are being explored. In this section progress in two of the more promising approaches are explored, in connection with the somewhat idealized case of plane strain crack growth (practical cases frequently involve significant 3-dimensional effects, e.g., formation of ductile shear lips adjoining, or even obscuring completely, the flat plane-strain-like fracture surface in cracked plates). These approaches are:

(i) Use of the J integral, based not on its interpretation as an energy release rate but rather on its interpretation within "deformation" plasticity theory as a parameter characterizing the strength of the crack tip deformation field [17,53,54]; such an approach is widely used for the onset of growth [55,48] and what is being considered here is its extension to at least small amounts of subsequent stable growth, and

(ii) Precise, incremental elastic-plastic analyses of fields near growing cracks, with a crack growth criterion being based on the intensity of some measure of the near tip field; approaches of this kind have been widely successful for the onset of growth, especially in relating macroscopic toughness parameters to microscale fracture mechanism [48]; for the growing crack, at present, the approach is reasonably well developed only within the ideally plastic (non-hardening) material model.

The approaches are summarized in turn.

J integral methods: The J integral is defined by [17,19]

$$J = \int_\Gamma (w n_x - \underline{n}\cdot\underline{\sigma}\cdot\partial\underline{u}/\partial x)\, ds \qquad (49)$$

and within an appropriate "deformation theory" version of the elastic-plastic Prandtl-Reuss equations (or other elastic-plastic constitutive equations) so that the stress working density w of (22) is a function only of strain $\underline{\epsilon}$, the integral is independent of path Γ. Alternately, within the same deformation theory approximation, J may be evaluated by "compliance" methods based on the difference between load-displacement curves for identically loaded bodies, with stationary cracks, that differ only with respect to crack size [48]. Since the contour Γ can be shrunk arbitrarily close to the crack tip, J can be interpreted as some integrated measure of the strength of the crack tip singular field, i.e., as a crack tip "characterizing parameter." Indeed, in the case of a monotonically loaded stationary crack in a power-law hardening material, i.e.,

$$\tau \propto \gamma^N \text{ in the plastic range,} \qquad (50)$$

where τ and γ are the Mises equivalent shear

stress and shear strain measures, Rice and Rosengren [53] and Hutchinson [54] observed that within a "small strain" elastic-plastic solution there were characteristic singularities of the form

$$\underset{\sim}{\epsilon} \to \frac{\sigma_o}{E}\left[\frac{EJ}{\sigma_o^2 r}\right]^{1/(1+N)} \underset{\sim}{F}(\theta;N) \quad \text{as } r \to 0$$

$$\underset{\sim}{\sigma} \to \sigma_o\left[\frac{EJ}{\sigma_o^2 r}\right]^{N/(1+N)} \underset{\sim}{G}(\theta;N) \quad \text{as } r \to 0 \qquad (51)$$

where σ_o is the tensile yield strength, E is Young's modulus, and $\underset{\sim}{F}$ and $\underset{\sim}{G}$ are certain universal functions. This field is referred to as the HRR singularity, and its strength is evidently determined by J. Hence, to the extent that the deformation plasticity model is appropriate, and that the HRR singularity actually does dominate the near tip field over a size scale inclusive of the fracture process zone (see McMeeking and Parks [56] for a fuller discussion of this requirement--it is not met in all fully plastic geometries), it is reasonable to phrase the condition for onset of crack growth as the attainment of a critical value of J.

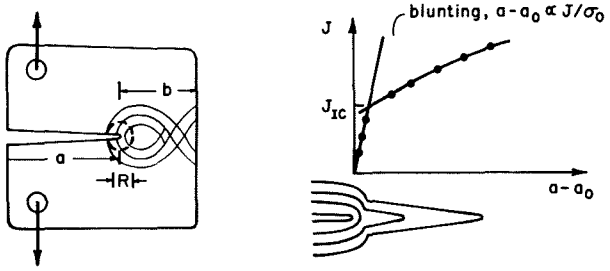


Fig. 12. Early stages of ductile crack growth and definition of J_{IC}. R denotes zone dominated by the J-characterized singularity at crack tip. See text for discussion of conditions under which the J versus $a-a_o$ relation may be considered as a "resistance curve," characteristic of the material.

Fig. 12 shows in schematic terms the typical analysis of experimental results to determine J_{IC} (the J value at onset of growth). The points on the diagram represent values of J versus change in crack length, and the sketch is made for the case in which there is a relatively sharp demarcation between growth associated with progressive plastic opening of the crack tip and that associated with the plastic tearing process. The extrapolated intersection with the blunting line defines J_{IC}, as shown.

However, in many materials the (nominal) J value continues to rise in such a steep manner with crack growth [49] that it is overly conservative to base a limiting strength prediction on J_{IC}. Accordingly, Hutchinson and Paris [50] have attempted to establish conditions under which the entire J versus $a-a_o$ curve, or at least some early portion of it corresponding to $a-a_o \ll b$ (the uncracked ligament size), might be considered as some universal characteristic of the material. In such cases stable growth may be predicted by equating J, viewed within the deformation plasticity model as some function of load and crack size, to the experimentally determined J versus $a-a_o$ curve. That is, if Q is some measure of the monotonically increasing load parameter, either force or load-point displacement as the case may be, and if $J_A(Q,a)$ is the "applied" J value, then within the formulation the crack growth criterion is

$$J_A(Q,a) = J_R(a-a_o) \quad , \qquad (52)$$

where the function $J_R(a-a_o)$ represents the experimental "resistance" curve as in fig. 12. This determines the relation Q versus a, and instability occurs when $dQ/da = 0$, i.e., when

$$\partial J_A(Q,a)/\partial a = d[J_R(a-a_o)]/d(a-a_o) \quad . \qquad (53)$$

This instability condition has a well known graphical interpretation: the critical point is reached at the value of Q and a for which the plot

of J_A versus a for fixed Q makes tangential contact with the resistance curve.

Of course, a limitation on any such approach, based on deformation plasticity, occurs because the process of crack growth necessarily involves elastic unloading and significantly non-proportional straining near the crack tip. Hutchinson and Paris [50] evaluate this restriction in the following way. With reference to fig. 12, let R be a measure of the size near the crack tip over which the J-characterized singularity (e.g., the HRR field) can be considered, within deformation plasticity, to dominate the overall deformation field. A basic requirement for validity of the J_{IC} concept is that R be large enough to fully envelop the crack tip fracture process zone. For ductile tearing mode fractures, extensive metallurgical studies suggest that this zone is typically of the same order of size as the crack tip opening displacement at the onset of growth [48], i.e., of the order of J_{IC}/σ_o. This consideration leads to the conclusion that the J_{IC} concept is valid for specimens which are sufficiently large that a typical dimension-like ligament size b meets the inequality [48]

$$b > f(\text{specimen type, } N)\, J_{IC}/\sigma_o \gg J_{IC}/\sigma_o \,, \tag{54}$$

and while the function f has not been determined experimentally, it is usually accepted on empirical grounds that for specimens of the type shown in fig. 12, $f \approx 25$ to 50.

With this background, Hutchinson and Paris consider the early stages of crack growth, in which J increases with a. The increments in J are considered to result in continued proportional straining of the HRR type, whereas the increments in a cause strongly non-proportional straining. The former will dominate over distances comparable in size to R if [50]

$$dJ/J \gg da/R \,. \tag{55}$$

When this condition is met it is argued that the stable growth process takes place over a size scale that is well embedded within the J-characterized zone, and hence that the concept of a specimen-independent resistance curve, J versus $a-a_o$, is valid. Since R scales approximately with overall size of the specimen when fully plastic conditions are attained, the condition for validity of the resistance curve concept may then be restated as

$$b \gg J/(dJ/da) \,. \tag{56}$$

The factor on the right in this inequality is approximately the amount of crack growth necessary for J to double in value over J_{IC}. For materials with a large resistance to crack extension—for example, exhibiting a doubling of J for growth of the order of 1 mm--it seems plausible that this size criterion could be met for at least the thicker of the typical range of sizes of specimens and structural parts (e.g., b greater than 25 mm or so). In such cases, which are not uncommon (see below), the concept of a J versus $a-a_o$ resistance curve, viewed as a fundamental material property, seems justified at least for limited, and not yet well quantified, amounts of growth. Of course, the approach does not seem to be viable for materials with significantly lower resistance to crack growth, i.e., higher values of J/(dJ/da), except when attention is limited to such unrealistically large sizes that response is essentially in the elastic fracture mechanics range.

Paris et al. [49] give an extensive tabulation of values of J_{IC} and dJ/da for structural alloys. From this data, some representative values of J/(dJ/da) at the onset of crack growth are: 0.6 mm for G.E. Ni-Mo-V rotor steel (ASTM-A469), 0.9 mm for Ni-Cr-Mo-V rotor steel (ASTM-A471), 1.6 mm for AISI-403 12 Cr stainless rotor steel, all at 150°C; 0.8 mm for AISI-4340 steel, 2.5 mm for HY-130 steel, 2.9 mm for 2024-T351 aluminum alloy, and 5.0 mm for 6061-T651 aluminum alloy, all at

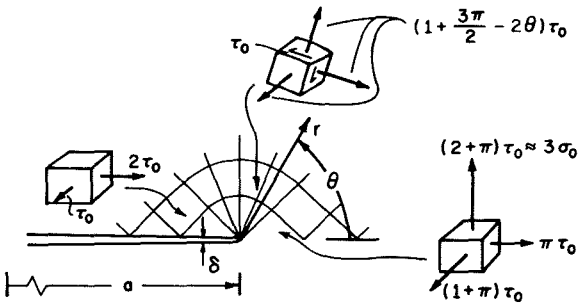
room temperature (the results for HY-130 and 2024-T351 were privately communicated by Dr. J. Landes of Westinghouse Research Laboratory). This range of values suggests that many, although certainly far from all, cases of plane strain ductile crack growth in specimens of substantial size (say, b > 25 mm) can be analyzed in the manner described, in terms of a J versus $a-a_o$ relation that is regarded as characteristic of the material. An experimentally-based understanding of restrictions on the approach has not yet been attained.

5.1 ELASTIC-PLASTIC INCREMENTAL ANALYSIS OF GROWING CRACKS

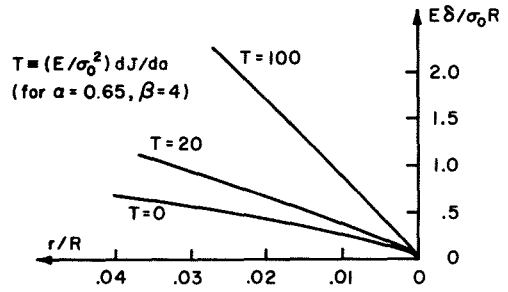
The second approach to be described is, as remarked, limited presently to the ideally plastic material model. The incremental Prandtl-Reuss equations are analyzed to establish the nature of the near tip singular field for a growing crack. For well contained plastic yielding near stationary cracks, subjected to monotonic load, it is now well established [48,57] that within "small strain" theory the limiting stress state as $r \to 0$ at the crack tip is the same field as that described by the Prandtl slip-line construction shown in fig. 13a. Indeed, the arguments of Rice [17] and Rice and Tracey [57] which lead to the Prandtl field at the tip seem, as remarked by Rice and Sorensen [44], to be equally valid for growing as for stationary cracks, and recent incremental elastic-plastic finite element solutions for crack growth under small scale yielding conditions by Sorensen [58] and, with a much finer mesh, by Sham [59] seem to give near tip stress states that are consistent with the Prandtl field.

Accordingly, the nature of the strain singularity can be established [19,42-44] by applying the Prandtl-Reuss equations to the stress state of fig. 13a, assumed to move through the material with the crack tip so that for small r, $\underset{\sim}{\sigma} = \underset{\sim}{\sigma}(\theta)$, where the origin of the polar coordinate system moves with the tip. Thus the strain rate $\dot{\underset{\sim}{\epsilon}}$ satisfies

$$\dot{\epsilon}_{ij} = \frac{1+\nu}{E}\dot{s}_{ij} + \frac{1-2\nu}{E}\dot{\sigma}\,\delta_{ij} + \dot{\Lambda}\, s_{ij} \qquad (57)$$



(a)



(b)

Fig. 13. (a) The Prandtl slip line field, shown, is assumed to provide the limiting stress state, $\underset{\sim}{\sigma} = \underset{\sim}{\sigma}(\theta)$ as $r \to 0$, at the tip of a stationary or growing plane-strain crack in an ideally plastic material with well-contained yielding; the Prandtl-Reuss incremental constitutive equations are integrated for this field in order to obtain the form of the near tip strains and crack opening displacement δ. (b) Crack opening δ versus distance from tip for continuously growing crack under small scale yielding conditions. R is undetermined by the asymptotic analysis but is assumed, on the basis of numerical solutions, to scale with the size of the crack tip plastic zone.

where σ is the mean normal stress, $\underset{\sim}{s}$ is the deviatoric stress tensor, and $\dot{\Lambda} \geq 0$ for plastic response ($\dot{\Lambda} = 0$ for elastic); the (Mises) yield condition is $3s_{ij}s_{ij} = 2\sigma_o^2$. Also, the stress rates are computed, for small r, by writing

$$\dot{\sigma}_{ij} = [d\sigma_{ij}/d\theta]\dot{\theta} = [d\sigma_{ij}/d\theta]\ (\dot{a}/r)\ \sin\theta \quad (58)$$

where $d\sigma_{ij}/d\theta$ denotes derivatives of Cartesian stress components of the Prandtl field and $\dot{a}$ is the crack growth rate.

The resulting expressions for the polar coordinate components of material velocities at small r are [44], within the centered fan zone of fig. 13a,

$$\begin{aligned} v_r &= (\beta\sigma_o/2E)\ \dot{a}\ \sin\theta\ \ln(\bar{R}/r) + df(\theta)/d\theta \\ v_\theta &= (\beta\sigma_o/2E)\ \dot{a}\ (\cos\theta - 1/\sqrt{2})\,[\ln(\bar{R}/r) \\ &\quad - 3\nu/(2-\nu)] - f(\theta) + g(r)\ , \end{aligned} \quad (59)$$

where $\beta = 4(2-\nu)/\sqrt{3}$ (= 3.93 for $\nu = 0.3$) and where the length $\bar{R}$ and (bounded) functions $f(\theta)$ and $g(r)$ are undetermined by the present analysis. Note that the terms which remain when $\dot{a} = 0$ correspond to the well known result for a stationary crack [17]. The nature of the singularity in plastic strain rates within the centered fan is such that only the $r\theta$ (shear) component is singular, and it has the form

$$(\dot{\underset{\sim}{\varepsilon}}^p)_{r\theta} = (\beta\sigma_o/4\sqrt{2}\ E)\,(\dot{a}/r)\ \ln(\bar{R}/r) + [f''(\theta)+f(\theta)]/2r \quad (60)$$

as $r \to 0$, for appropriate $\bar{R}$ and $f(\theta)$. Finally, the rate of change of opening displacement, $\dot{\delta}$, between material points of the upper and lower crack surfaces at small distances r behind the tip is [44]

$$\dot{\delta} = \dot{A} + \beta\,(\sigma_o/E)\ \dot{a}\ \ln(\bar{R}/r)\ , \quad r \to 0\ , \quad (61)$$

where $\dot{A}$, again undetermined by the asymptotic analysis, is independent of r.

Since the material model is rate independent, $\dot{A}$, as well f and g above, must be homogeneous of degree one in $\dot{a}$ and in the rate of some parameter measuring the rate of loading. For cases of contained yielding one may take this latter parameter as the far-field value of J and then, assuming linearity of $\dot{A}$ in $\dot{J}$ and $\dot{a}$,

$$\dot{\delta} = \alpha\,\dot{J}/\sigma_o + \beta(\sigma_o/E)\ \dot{a}\ \ln(R/r)\ , \quad r \to 0\ , \quad (62)$$

where α is undetermined by the analysis thus far and where R replaces $\bar{R}$ and accounts for the $\dot{a}$ dependence of $\dot{A}$. Integrations of (62) in two cases are of interest. First, for monotonic loading of a stationary crack the term with $\dot{a}$ vanishes and one obtains a discrete opening displacement at the tip. Indeed, in the "small scale yielding" limit it can be argued on dimensional grounds that α is constant and in this case

$$(\delta)_{r=0} = \alpha\,J/\sigma_o \approx 0.65\ J/\sigma_o \quad (63)$$

where the numerical value of α comes from recent numerical solutions [44,61]. Full details of the tip opening in this case can be resolved only on the basis of a finite strain analysis, as by Rice and Johnson [60] and McMeeking [61].

The other case is that of a continuously growing crack, i.e., J varies continuously with a, and in that case eq. (62) can be integrated [44] to give, for very small r,

$$\delta = \alpha\,(dJ/da)\,r/\sigma_o + \beta(\sigma_o/E)\,r[1 + \ln(R/r)]\ . \quad (64)$$

Note that for the continuously growing crack there is no discrete opening at the tip. The crack surface profile is shown in dimensionless form in fig. 13b for different values of the parameter

$$T = (E/\sigma_o^2)\ dJ/da\ , \quad (65)$$

assuming $\beta = 4$ and $\alpha = 0.65$. The value of α used is the same as for the stationary crack; Rice and Sorensen [44] observe that numerical results for crack opening at fixed a due to load increase, following various previous histories of J versus $a-a_o$, seem to suggest an approximately constant value of α. The parameter R used to scale lengths in fig. 13b is expected, on dimensional grounds, to scale with the size of the plastic zone, at least for well-contained yielding. Indeed, Rice and Sorensen [44] suggest tentatively, from their finite element results for crack growth, that

$$R \approx 0.16\ E\ J/\sigma_o^2 \qquad (66)$$

(which is also equal approximately to the maximum radius of the plastic zone) for small scale yielding. The factor 0.16 is likely to be revised with more accurate numerical solutions, intended to resolve the undetermined parameters and functions in eqs. (59-62). Note that the crack profiles shown in fig. 13b are not consistent with the notion of a crack tip opening angle. Instead, eq. (64) involves a vertical tangent at the tip. But for highly ductile materials, i.e., large values of T, the logarithmic term in eq. (64) is relatively unimportant except for extremely small values of r/R, and an effective crack tip opening angle can be defined (e.g., the case T = 100 in fig. 13b).

The strain fields are also different for the stationary versus the growing crack. In the first case, for small scale yielding [19]

$$\epsilon_{ij}^p \to \frac{J}{\sigma_o r} F_{ij}(\theta) \quad \text{as } r \to 0 \qquad (67)$$

within the fan region, whereas for the growing crack

$$\epsilon_{ij}^p \to \frac{1}{\sigma_o}\frac{dJ}{da} G_{ij}(\theta) + \frac{\sigma_o}{E} H_{ij}(\theta)\ \ell n\left[\frac{\tilde{R}(\theta)}{r}\right]. \qquad (68)$$

The functions $\underset{\sim}{F}$, $\underset{\sim}{G}$ and $\tilde{R}$ are not determined by the asymptotic analysis.

An approximate crack growth criterion can be based on the field very near the crack tip. One might expect that during the growth process a geometrically similar deformation pattern prevails at the crack tip as the crack grows. The equations for δ do not admit a solution of this type, since R in (62,64) seems to vary with the size of J. However, following Rice and Sorensen [44] an approximate condition of geometric similarity may be enforced by requiring that a certain fixed displacement, say δ_c, be achieved at a fixed material distance $\Delta\ell$, behind the tip, where $\Delta\ell$ might be regarded as a measure of the fracture process zone size [44]. Then eq. (64) leads to

$$\frac{\alpha}{\sigma_o}\frac{dJ}{da} = \frac{\delta_c}{\Delta\ell} - \beta\frac{\sigma_o}{E}\left[1 + \ell n\left[\frac{R}{\Delta\ell}\right]\right]. \qquad (69)$$

which may be regarded as a differential equation governing crack growth when the dependence of R on J (and, possibly, the previous growth history) is specified. Rice and Sorensen integrate this equation for the choice of R in eq. (66), intended to correspond to small scale yielding. This defines a "resistance curve" for the material, and instability conditions are phrased as in eqs. (52,53). The resistance curve, however, is not expected to be universal within this approach and would, for example, be different for fully plastic specimens. For fuller details and discussion in terms of observed crack growth parameters for structural steels, the reader is preferred to the paper by Rice and Sorensen [44]. Predicted resistance to growth increases strongly with the opening angle parameter, $\delta_c/\Delta\ell$, and decreases with material strength level, as measured by σ_o/E.

It may be mentioned in closing that the solution described for the growing crack may also be of interest for processes of stress corrosion cracking of ductile materials.

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