Scattering of long-wavelength elastic waves from localized defects in solids

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It is shown that long-wavelength elastic scattering data from an arbitrary localized defect in a uniform isotropic medium has a maximum information content of 22 parameters which are characteristic of the defect. These parameters are shown to consist of the mass excess δM and the 21 independent components of a fourth-rank tensor D_{ijkl}, that depends on the elastic moduli variation δc_{ijkl} and static response properties of the defect region. This tensor and the contracted forms D_{ij} ($= D_{ijkk}/3$) and D ($= D_{kk}/3$) allow partial "inversion" of scattering data to determine properties of the defect. In particular, it is shown how to estimate the orientation and maximum stress intensity factor for defects in the form of planar cracks, to obtain lower bounds to maximum defect dimensions, and to represent defects in the form of inclusions or voids as approximately equivalent ellipsoids. The results are pertinent to the quantification of nondestructive examination of materials for defects in their interiors.

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I. INTRODUCTION

The scattering of elastic waves by defects obviously provides some information about the latter and therefore has been extensively used for nondestructive evaluation (NDE) of structural members. Ideally, measurements of the scattered waves would lead to a complete knowledge of the geometry, density distribution, and elastic properties of defects, be they cracks, voids, inclusions, regions of stress, etc. This would constitute a complete solution of the so-called inverse problem—the complete reconstruction of the defect from the scattering data.

At this time it has not been established whether such a complete inversion is even possible in principle. If so, it would certainly require measurements at all frequencies. The present paper has a much more limited scope. It restricts itself to the interpretation of scattering results with extremely long wavelengths in the sense of $kL \ll 1$, where k is the wave number and L is a characteristic dimension of the defect. We shall show first that even this limit yields surprisingly rich information, namely,[2] 22 independent parameters characteristic of the defect, one of which is the excess mass of the defect. We then show, by means of some representative examples, that the remaining 21 parameters yield some limited but still very useful information about the dimensions and elastic properties of the defect. We therefore believe that the 22 long-wavelength parameters, or at least some of them, can and should be incorporated in comprehensive NDE programs.

II. REVIEW OF SCATTERING THEORY

Gubernatis *et al.*[1] have derived formal expressions for the amplitudes of elastic waves scatted by a localized defect in a solid. Gubernatis *et al.*[2] have also presented some applications of these general expressions to the long-wavelength limit, including the scattering from a homogeneous ellipsoid and a circular (penny-shaped) crack. Here we shall start

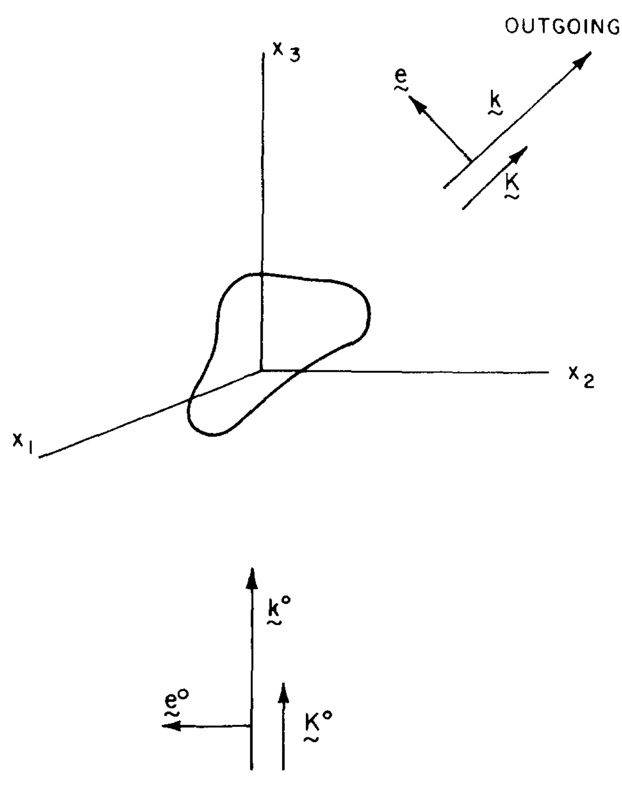

FIG. 1. The scattering geometry. $\mathbf{k}^0$ is the incident wave vector, $\mathbf{K}^0$ is the unit vector ($\mathbf{k}^0/k^0$), $\mathbf{e}^0$ is the incident polarization vector (shown for the case of a transverse wave), and **k**, **K**, and **e** have analogous meanings for the scattered wave.

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from their general expressions and then develop the theory of the long-wavelength limit in somewhat different directions.

We consider an experimental situation such as that shown in Fig. 1. We assume that the matrix is isotropic and specified by the density ρ and the Lamé stiffness constants λ and μ, in terms of which we can write the bulk modulus and Poisson's ratio as

$$K = \lambda + \tfrac{2}{3}\mu, \quad \nu = \tfrac{1}{2}\lambda/(\lambda + \mu). \tag{1}$$

Following Ref. 1, we write the incident wave in the form

$$\mathbf{u}^0(\mathbf{r}) = a\mathbf{e}^0 \exp(i\mathbf{k}^0\cdot\mathbf{r}), \tag{2}$$

where $\mathbf{u}^0(\mathbf{r})$ is the displacement vector, a its amplitude, $\mathbf{e}^0$ the unit polarization vector, $\mathbf{k}^0$ the wave vector of incidence, and the common time dependence $\exp(-i\omega t)$ is suppressed throughout. For a given ω, the magnitude of $\mathbf{k}^0$ will depend on whether the incident wave is longitudinal (L) or transverse (T):

$$k^0 \equiv |\mathbf{k}^0| = \begin{cases} \alpha = \omega/v_L, & L \text{ wave} \\ \beta = \omega/v_T, & T \text{ wave,} \end{cases} \tag{3}$$

where v_L and v_T are, respectively, the longitudinal and transverse sound velocities.

The scattered waves in the far field at distance r from the defect, in the direction of the unit vector $\mathbf{K}$, are given by

$$\mathbf{u}(\mathbf{r}) = \mathbf{A}(\mathbf{K})\exp(i\alpha r)/r + \mathbf{B}(\mathbf{K})\exp(i\beta r)/r, \tag{4}$$

where $\mathbf{A}$ and $\mathbf{B}$ are, respectively, the vector amplitudes of the scattered L and T waves ($\mathbf{A}\|\mathbf{K};\mathbf{B}\perp\mathbf{K}$). These amplitudes have the form[1]

$$A_i(\mathbf{K}) = K_i K_j f_j(\boldsymbol{\alpha}),$$
$$B_i(\mathbf{K}) - (\delta_{ij} - K_i K_j) f_j(\boldsymbol{\beta}), \tag{5}$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are, respectively, the wave vectors of the scatted L and T waves:

$$\boldsymbol{\alpha} = \alpha\mathbf{K}, \quad \boldsymbol{\beta} = \beta\mathbf{K}, \tag{6}$$

and the vector quantity $f_i(\mathbf{k})$ (for $\mathbf{k}$ either $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$) is given by the expression

$$f_i(\mathbf{k}) = \frac{k^2}{4\pi\rho\omega^2}\Big(\omega^2 \int dV'\, \delta\rho(\mathbf{r}')u_i(\mathbf{r}')\exp(i\mathbf{k}\cdot\mathbf{r}')$$
$$+ ikK_j \int dV'\, \delta c_{ijkl}(\mathbf{r}')\epsilon_{kl}(\mathbf{r}')\exp(-i\mathbf{k}\cdot\mathbf{r}')\Big); \tag{7}$$

here $\delta\rho(\mathbf{r}')$ is the local density difference between the defect and matrix, $\delta c_{ijkl}(\mathbf{r}')$ are the differences (between defect and matrix) of the components of the stiffness tensor, $\epsilon_{kl}(\mathbf{r}')$ is the total local strain induced by the incident wave, and the integrations are carried out over a region containing the defect. Equation (7) is exact for arbitrary frequencies.

In the long-wavelength limit, one may set[2]

$$\exp(i\mathbf{k}\cdot\mathbf{r}')\to 1, \quad u_i(\mathbf{r}')\to u_i^0(0) = ae_i^0. \tag{8}$$

We also introduce the unperturbed normalized incident strain tensor near the origin, $\tilde{\epsilon}^{\,0}_{kl}$, as

$$\epsilon^{\,0}_{kl} = aik^0\tilde{\epsilon}^{\,0}_{kl}, \quad \tilde{\epsilon}^{\,0}_{kl} = \tfrac{1}{2}(e_k^0 K_l^0 + e_l^0 K_k^0) \tag{9}$$

with $\mathbf{K}^0 = \mathbf{k}^0/k^0$ and, similarly, the renormalized strain tensor in the presence of the defect, $\tilde{\epsilon}_{kl}$

$$\epsilon_{kl}(\mathbf{r}) = aik^0\tilde{\epsilon}_{kl}(\mathbf{r}). \tag{10}$$

The purpose is to extract the dependence on a and on k, as well as the factor i.

Substituting Eqs. (8)–(10) in Eq. (7) results in

$$f_i(\mathbf{k}) = \frac{a}{4\pi\rho}\Big(k^2\delta M e_i^0 - \frac{k^3k^0}{\omega^2}K_j\int dV'\delta c_{ijkl}(\mathbf{r}')\tilde{\epsilon}_{kl}(\mathbf{r}')\Big). \tag{11}$$

III. INFORMATION CONTENT OF LONG-WAVELENGTH SCATTERING DATA

A. The 22 independent parameters

We shall now show that for long-wavelength experiments the scattered displacement field of an arbitrary, localized defect can be completely characterized by 22 independent parameters and shall exhibit their definitions. These parameters constitute the maximum information about the defect which can be obtained from long-wavelength scattering experiments.

The strain tensor $\tilde{\epsilon}_{kl}(\mathbf{r})$ is linearly related to the normalized uniform strain tensor $\tilde{\epsilon}^0_{kj}$, which would be the strain at the positin of the defect if the defect were absent. Thus we can write

$$\tilde{\epsilon}_{kl}(\mathbf{r}) = \Gamma_{klmn}(\mathbf{r})\tilde{\epsilon}^{\,0}_{mn}, \tag{12}$$

where Γ_{klmn} is a fourth-rank tensor, symmetric under interchange of k and l and of m and n, which characterizes the static strain field induced within the defect when the solid is subjected to a remotely uniform strain state $\tilde{\epsilon}^{\,0}_{ij}$. Hence the second integral appearing in Eq. (11) can be written

$$\int dV\,\delta c_{ijkl}(\mathbf{r})\tilde{\epsilon}_{kl}(\mathbf{r}) = D_{ijmn}\tilde{\epsilon}^{\,0}_{mn}, \tag{13}$$

where the tensor D_{ijmn} is defined by

$$D_{ijmn} \equiv \int dV\,\delta c_{ijkl}(\mathbf{r})\Gamma_{klmn}(\mathbf{r}) \tag{14}$$

and is characteristic of the defect. In Appendix A it is shown that D_{ijmn} is also symmetric under simultaneous interchange of ij and mn. It can therefore be represented as a real symmetric 6×6 matrix with 21 independent components. The fact that there cannot be, in general, additional relationships between these 21 components follows from the special case of a small uniform δc_{ijkl}. In this case, to lowest-order $D_{klmn} \to V\delta c_{klmn}$, and for a general anisotropic defect, δc_{ijmn} is known to have 21 independent components. An effectively anisotropic δc_{ijmn} can also be obtained in a defect consisting of a composite of isotropic elements. We can now write

$$f_i(\mathbf{k}) = \frac{a}{4\pi\rho}\Big(k^2\,\delta M e_i^0 - \frac{k^0k^3}{\omega^2}K_j D_{ijmn}\tilde{\epsilon}^{\,0}_{mn}\Big). \tag{15}$$

In view of Eqs. (4) and (5), we therefore see that in the long-wavelength limit the factor $f_i(\mathbf{k})$ and hence all scattering results are characterized completely by the 21 independent components of D_{ijmn} and the mass excess δM, for all incident waves (ω, k^0, $\mathbf{e}^0$, $\tilde{\epsilon}^{\,0}_{mn}$) and all L and T scattered waves.

B. Sufficiency of longitudinal-wave experiments

We shall now show that *all* these 22 parameters can be determined by measurements of the scattering of L waves into L waves only. Measurements involving either incident or scattered T waves do not contain additional information. However, such measurements may be appropriate for practical reasons.

We substitute the Eq. (15) for $f_i(\mathbf{k})$ into Eq. (5) for the L-to-L scattering amplitude ($k^0 = k = \omega/v_L$):

$$\mathbf{A}(\mathbf{K}) = \frac{ak^2}{4\pi\rho}\mathbf{K}\Big(\delta M\,(e_j^0 K_j) - \frac{1}{v_L^2}\,K_i K_j D_{ijmn}\tilde{\epsilon}^{\,0}_{mn}\Big). \quad (16)$$

The parameters δM and D_{ijmn} can now be obtained by using waves incident from different directions and measuring the amplitude (positive or negative) of the scattered waves as a function of angle.

Let us take a given direction of incidence described by $\mathbf{K}^0$ ($=\mathbf{e}^0$). Clearly the term proportional to δM in Eq. (16) has an angle dependence

$$e_j^0 K_j = \cos\theta, \quad (17)$$

where θ is the angle between the direction of incidence and direction of scattering. This is, of course, a spherical harmonic with $l = 1$.

As for the second term of Eq. (16), let us write

$$D_{ijkl}\,\tilde{\epsilon}^{\,0}_{kl} = \tfrac{1}{3}\,\delta_{ij}\,D_{\rho\rho kl}\tilde{\epsilon}^{\,0}_{kl} + (D_{ijkl} - \tfrac{1}{3}\,\delta_{ij}\,D_{\rho\rho kl})\tilde{\epsilon}^{\,0}_{kl}. \quad (18)$$

Hence, the second term has the form

$$K_iK_jD_{ijkl}\,\tilde{\epsilon}^{\,0}_{kl} = \tfrac{1}{3}D_{\rho\rho kl}\,\tilde{\epsilon}^{\,0}_{kl} + K_iK_j(D_{ijkl} - \tfrac{1}{3}\,\delta_{ij}D_{\rho\rho kl})\tilde{\epsilon}^{\,0}_{kl}. \quad (19)$$

The first term on the right-hand side is independent of the angle of scattering, and thus corresponds to spherical harmonic $l = 0$; the second term corresponds to $l = 2$. Both of these are even under inversion of $\mathbf{K}$.

Thus by making a spherical harmonic analysis of the scattered amplitudes up to $l = 2$ and by picking out the $l = 1$ component, one can immediately determine the important parameter δM. Later we discuss the significance of the $l = 0$ component. To determine these harmonics, at most seven measurements are needed (rather than the $1 + 3 + 5 = 9$ which might be expected), since the $l = 1$ part is known from Eq. (17) to have $m = 0$ when the z axis is taken along $\mathbf{K}^0$. Indeed, in cases for which it is possible to make measurements for some direction $\mathbf{K}$ and its inverse $-\mathbf{K}$, the two measurements suffice to determine the $l = 1$ component. Once the $l = 1$ component is determined, measurements for vectors $\mathbf{K}$ in any three mutually perpendicular directions determine the $l = 0$ component.

More generally, once δM is determined the full set of remaining parameters, D_{ijmn}, can be determined by making measurements of $\mathbf{A}$ [Eq. (16)] for six independent directions of incidence, yielding by linear combination six independent components $\tilde{\epsilon}_{mn}$, and for six independent directions of observation yielding six independent values for $K_i K_j$. In fact, in view of the symmetries of D_{ijmn}, all but 21 of these 36 measurements are redundant.

For example, the component D_{2311} can be obtained by using a wave incident in the x_1 direction, for which

$$\tilde{\epsilon}^{\,0}_{mn} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (20)$$

and making measurements in the three directions

$$\mathbf{K} = (0,1,0), \quad \mathbf{K}' = (0,0,1), \quad \mathbf{K}'' = 2^{-1/2}(0,1,1). \quad (21)$$

From Eq. (16) we see that D_{2311} is proportional to the combination

$$A\,(\mathbf{K}'') - \tfrac{1}{2}[A\,(\mathbf{K}') + A\,(\mathbf{K})]. \quad (22)$$

All other components of D_{ijmn} can be similarly obtained.

IV. THE INVERSE PROBLEM

In the present section we examine what can be learned about the geometry and elastic properties of the defect from the measured values of δM and D_{ijmn}. Of course, δM itself is a very important physical characteristic. What additional insights can be obtained from D_{ijmn}?

A. Inseparability of defect volume and change of stiffness

First of all we note that, in general, we cannot from long-wavelength scattering data alone obtain independent information about the volume of the defect V and about the change of the stiffness constants δc_{ijkl}. This is mot directly seen from Eq. (14) in the limit in which the δc's are infinitesimal. In that case we may replace the tensor $\Gamma_{klmn}(\mathbf{r})$ by $\delta_{km}\delta_{lm}$, giving

$$D_{ijmn} = \int dV\,\delta c_{ijmn}(\mathbf{r}) + O\,(\delta c^2). \quad (23)$$

Thus an increase in the δc's can be compensated for by a corresponding decrease in V, and we can write, in this limit,

$$D_{ijmn} = V\langle\delta c_{ijmn}\rangle_{\mathrm{Av}} + O\,(\delta c^2). \quad (24)$$

This inseparability of V from the δc's persists also for finite values of the δc's. To obtain information about V and the δc's separately, indpendent information not available from long-wavelength scattering must be introduced. This could be independent knowledge of $\delta\rho$ (e.g., if the chemical composition of an inclusion is known) which, together with δM, determines the volume V, or an independent estimate of V by high-frequency acoustic shadow or diffraction effects. If the defect is known to be a void, then $V = -\delta M/\rho$. Finally, if $\delta M = 0$, the defect is in all likelihood a crack with $V \simeq 0$.

In spite of the fundamental inseparability of V and the δc's, we shall show that a knowledge of D_{ijmn} may permit setting a lower bound for the "diameter" d of the defect.

At the present time the only defects for which the tensor D_{ijmn} can be theoretically calculated are homogeneous ellipsoids; see Appendix B. To fully exploit the information contained in D_{ijmn} it would be useful to have available theoretical calculations for a broader class of physically interesting defects which could then be matched against the measured values of D_{ijmn}, subject to the inherent inseparability of defect volume and change of stiffness.

B. Contracted D tensors: D_{mn} and D

As we shall show in Sec. V, useful information concerning defects is contained in the following contractions of the tensor D_{ijmn}:

$$D_{mn} \equiv \tfrac{1}{3} D_{iimn} = \tfrac{1}{3} D_{mnii} = \frac{1}{3} \int dV' \delta c_{iikl} \Gamma_{klmn}, \qquad (25)$$

and

$$D \equiv \tfrac{1}{9} D_{iimm}. \qquad (26)$$

According to Eq. (19), they are experimentally obtainable by determining the $l = 0$ spherical-harmonic component of the outgoing longitudinal wave for six independent incident waves. For example, D_{11} is determined by the response to an L wave incident in the 1 direction, and D_{12} by a T wave incident in the same direction and polarized in the 2 direction.

By a suitable rotation, D_{mn} can be brought into diagonal form:

$$D_{mn} = \begin{bmatrix} D_1 & 0 & 0 \\ 0 & D_2 & o \\ 0 & 0 & D_3 \end{bmatrix}. \qquad (27)$$

The axes of the rotated system can be defined as the *principal axes* of the defect, and the three numbers D_1, D_2, and D_3 can be defined as its *three-diagonal D values.* We suggest that these quantities are useful and simple partial characterizations of a defect.

One anticipates that in most practical cases the diagonal matrix elements D_m will be either positive (stiffer defect) or negative (softer defect). Now one expects from the exact solutions of Eshelby[3,4] for homogeneous isotropic ellipsoidal defects that any set of negative diagonal D values D_1, D_2, and D_3 can be reproduced by a unique vanishing stiff (or soft) ellipsoidal inclusion (such as a void) of principal axis lengths a, b, and c, while any set of positive diagonal D values can be reproduced by a unique infinitely stiff ellipsoidal inclusion. Thus every defect whose diagonal D values are all negative (positive) is associated with a unique ellipsoidal soft (hard) inclusion.

Similarly, if the defect is *a priori* known to be homogeneous and isotropic with Láme stiffness coefficients λ' and μ' (e.g., $\lambda' = \mu' = 0$ for voids and cracks), one may again seek the equivalent Eshelby ellipsoid with the given λ', μ'.

Such geometric representations of the D_{ij} tensor in terms of equivalent Eshelby ellipsoids should be useful for NDE purposes.

Finally, we may be satisfied with a characterization of the defect by the full trace D [Eq. (26)]. Any given $D < 0$ can be reproduced by a unique vanishingly stiff sphere, and any $D > 0$ by a unique infinitely stiff sphere. Again, if D *and* δK (the bulk-modulus excess) are known (the latter from some *a priori* source), and assuming that D and δK have the same sign, D can be associated with a unique sphere of the given δK (see Appendix B).

V. APPLICATIONS TO DEFECT CHARACTERIZATION

A. Detection of planar cracks

We examine the intepretation of scattering results due to cracks, assuming that the experiments have been carried out with enough superposed static load to keep the crack surfaces open. Thus the crack is to be modeled as a thin void (i.e., traction-free crack surfaces). A consistency check that the observed scattering does indeed result from a crack can be made by observing δM, which vanishes for a crack.

We assume that the crack is planar and that its orientation and shape are unknown. For planar cracks it is elementary to see that principal axes of the contracted tensor D_{ij}, based on the scattering observations, have the property that one axis (say, x_3) is normal to the crack plane while the other two (x_1 and x_2) lie in the plane of the crack. It remains to determine which of the three directions is normal to the crack.

To do so we first observe that a uniform stress σ^0_{11} or σ^0_{22} acting parallel to the crack does not tend to open it. Thus $D_1 (= D_{11})$ and $D_2 (= D_{22})$ satisfy the relations

$$\begin{aligned} &D_1 - \nu(D_2 + D_3) = 0, \\ &D_2 - \nu(D_1 + D_3) = 0, \end{aligned} \qquad (28)$$

which follow because the strains ϵ^0_{ij} associated with σ^0_{11} have the relation $\epsilon^0_{22} = \epsilon^0_{33} = -\nu\epsilon^0_{11}$. From these expressions it is seen that

$$D_1 = D_2 = \nu D_3/(1-\nu) = 3\nu D/(1+\nu), \qquad (29)$$

where $D = \frac{1}{3}(D_1 + D_2 + D_3)$ is the fully contracted D tensor. Thus another consistency check that the defect is indeed a planar crack is provided by the observation of two equal principal values of D_{ij}; the corresponding principal axes define the plane of the crack. Note that information about the in-plane shape of the crack is lost by use of the contracted tensor D_{ij}.

However, the available data suffices to estimate the radius a of an "equivalent" penny-shaped crack since

$$D = -\frac{16}{9} \frac{1-\nu^2}{1-2\nu} K a^3 \qquad (30)$$

in this case. (The result emerges as a limting case of that for the general ellipsoidal void; see Appendix B.)

In addition, by following procedures analagous to those of Budiansky and Rice,[5] it is possible to estimate closely the maximum normalized stress-intensity factor $k_{\rm I}$ around the crack border without knowing the detailed shape of the crack ($k_{\rm I} = K_{\rm I}/\sigma_n$, where $K_{\rm I}$ is the actual stress intensity factor under quasistatic normal-stress loading of amount σ_n). For a void,

$$D = -\tfrac{1}{9} C_{kkrs} \int dV' \, \Gamma_{rs\rho\rho}(\mathbf{x}') = -\tfrac{1}{3} K \int dV' \, \Gamma_{kk\rho\rho}(\mathbf{x}'), \qquad (31)$$

where $\Gamma_{kk\rho\rho}$ is the "dilation" of the void interior due to a quasistatic far-field strain in the form $\epsilon^0_{ii} = \delta_{ij}$. This strain corresponds to a far-field stress $\sigma_{ij}^0 = 3K\delta_{ij}$. Further, in the limit in which the void degenerates to a planar crack the integral of dilation over volume becomes an integral of opening displacement across the crack faces. Thus, if $\Delta U^{(n)}$ is the normal opening across the crack faces when the remote loading is a tensile stress S_n normal to the crack plane,

$$D = -\frac{K^2}{S_n}\left(\int_A dx_1\, dx_2\, \Delta U^{(n)}(x_1,x_2)\right). \tag{32}$$

But Budiansky and Rice define a parameter P [see Ref. 6, Eqs. (8) and (11)] by

$$P = \frac{9(1-2\nu)}{2(1-\nu^2)}K\frac{\int_A dx_1\, dx_2\, \Delta U^{(n)}}{S_n}$$

$$= -\frac{9(1-2\nu)}{2(1-\nu^2)}\frac{D}{K}, \tag{33}$$

and observe that P is a certain weighted integral of $k_{\rm I}$ around the crack border. Further, for elliptical cracks with aspect ratios betweeen 0.06 and 15, they show that one can write with an error not greater than approximately 10% that

$$(k_{\rm I})_{\max} \approx (8P/\pi^3)^{1/6}. \tag{34}$$

The same formula would, presumably, be applicable for any convex crack shape; it is exact for a penny-shaped crack. Note also that if only an estimate of the maximum $k_{\rm I}$ is required, it suffices to know only $D\ (=\frac{1}{3}D_{kk})$. The full tensor D_{ij} and the procedure of locating its principal axes is necessary only if, in addition, the orientation of the crack plane is to be determined.

The present procedure for estimating $k_{\rm I}$ from scattering data differs from that of Ref. 6 in that Ref. 6 employs the additional approximation that the compliance of the crack in shear is proportional, by the factor $2/(2-\nu)$, to the compliance under normal loading. On the other hand, the procedure of Ref. 6 requires three observed parameters, whereas the six components of D_{ij} are required in the present approach.

We have not made use of the full tensor D_{ijkl} in this discussion. However, its components of the kind D_{1331}, D_{2332}, and D_{2331} contain information on the response of the crack to shear loadings, and hence on the orientation, e.g., of "long" and "short" diameters of the crack in the crack plane.

B. Lower bound for defect diameter

Consider a defect giving rise to certain value of D, positive or negative. Does this give us any information about the dimensions of the defect, in the absence of any knowledge of the δc's? We note that any given $D<0$ can be reproduced by a unique vanishing stiff sphere and any $D>0$ by a unique infinitely stiff sphere.

Consider first $D>0$. This value of D can be reproduced by an infinitely stiff sphere of diameter d^+. The relation between D and d^+ is obtained by using the results of Eq. (A30) with $K'=\infty$, and is given by

$$d^+ = \left(\frac{6\alpha}{\pi}\right)^{1/3}\left(\frac{D}{K}\right)^{1/3}, \tag{35}$$

where K is the bulk modulus of the matrix and α is given in terms of its Poisson's ratio ν as

$$\alpha \equiv (1+\nu)/3(1-\nu). \tag{36}$$

For a typical Poisson's ratio of $\nu=\frac{1}{4}$, $\alpha=\frac{5}{9}$, and $(6\alpha/\pi)^{1/3}=1.020$. Similarly, if $D<0$, it can be reproduced by a vanishing stiff sphere of diameter d^-. From Eq. (A30), with $K'=0$, we have

$$d^- = \left(\frac{6(1-\alpha)}{\pi}\right)^{1/3}\left(\frac{-D}{K}\right)^{1/3}, \quad D<0. \tag{37}$$

For $\nu=\frac{1}{4}$, $[6(1-\alpha)/\pi]^{1/3}=0.947$.

By comparison of the relation between D and d for a spherical defect of finite compressibility K' (as given in the Appendix) with the above expression, one may verify that

$$d > d^+, \quad D>0,$$
$$d > d^-, \quad D<0. \tag{38}$$

Next let us consider a defect of arbitrary shape and define its diameter d as the maximum distance between any two points in the defect. In Sec. IV we showed that for a thin penny-shaped void (crack)

$$d = \left[\frac{9}{2}\left(\frac{1-2\nu}{1-\nu^2}\right)\right]^{1/3}\left(\frac{-D}{K}\right)^{1/3}, \tag{39}$$

so that, by comparison with Eq. (37) for $0<\nu<\frac{1}{2}$,

$$d > d^-. \tag{40}$$

If the penny-shaped region has finite stiffness coefficients, one can show that for a given D its diameter exceeds the value in Eq. (38).

From these examples we propose the following conjecture. For a given positive (negative) D the defect of smallest diameter is an infinitely (vanishingly) stiff sphere. This would imply that for a given D, the diameter d of *any* defect satisfies the inequalities of Eq. (38), where $d^{\pm}$ are given in Eqs. (35) and (37).

C. Representation of scatterer by ellipsoidal inclusion or void

In Appendix B it is shown in Eq. (A24) that the fourth-rank tensor D_{ijkl}, for an arbitrarily anisotropic but homogeneous ellipsoidal inclusion, is

$$\mathbf{D} = V\delta\mathbf{C}{:}(\mathbf{I}+\mathbf{S}{:}\mathbf{C}^{-1}{:}\delta\mathbf{C})^{-1}, \tag{41}$$

where $\mathbf{S}$ is a certain tensor appearing in Eshelby's[3,4] solution to the "transformation problem" for an ellipsoidal region. The interpretation of $\mathbf{S}$ and of the dyadic notation employed is explained in Appendix B. It is seen that $\mathbf{D}$ depends nonlinearly on $\delta\mathbf{C}$, although the Born approximation $\mathbf{D}=V\delta\mathbf{C}$ is verified in the limit of vanishingly small $\delta\mathbf{C}$.

Evidently, the number of parameters necessary to describe the most general homogeneous ellipsoidal inclusion is 28: V, $\delta\rho$, the five geometric parameters in $\mathbf{S}$ (two ratios of principal axes, three for orientation of the ellipsoid relative to the coordinate system), and the 21 independent components of the most general $\delta\mathbf{C}$. Since the long-wavelength scattered field contains 22 parameters, a *unique* representation of an arbitrary scatterer by an inclusion of this class is not possible. On the other hand, if we assume either that (i) the material of the inclusion is known or that (ii) the inclusion is of unknown but isotropic material, the number of parameters reduces below the maximum of 22 observable, and the inverse problem is overdetermined. In this case appeal may be made to "least-squares"-fitting procedures to determine an "equivalent" ellipsoid.

For example, in case (i) $\delta\rho$ and $\delta\mathbf{C}$ are known, the latter only for a special choice of axes relative to the material if the inclusion is anisotropic. Thus the number of unknown parameters is 9: V, the five geometric parameters in $\mathbf{S}$, and three parameters to describe the rotation of principal axes of anisotropy relative to those of the ellipsoid. The last three do not enter if the ellipsoid is isotropic and the number reduces to six in that case. On the other hand, for case (ii) there are also nine unknown parameters: V, $\delta\rho$, δK, $\delta\mu$, and the five geometric parameters in $\mathbf{S}$.

As a specific case, assume that the purpose of the scattering investigation is to characterize geometrical inclusions of a known isotropic material. Hence $\delta\rho$, δK, and $\delta\mu$ (and thus $\delta\mathbf{C}$) are known and the problem is to determine the volume, shape, and orientation of the inclusion given that the inclusion is to be represented as an ellipsoid. In this case V is determined directly from δM as $V = \delta M/\delta\rho$. The other geometric information is contained in $\mathbf{S}$, which may be solved as,

$$\mathbf{S} = V\mathbf{D}^{-1}:\mathbf{C} - \delta\mathbf{C}^{-1}:\mathbf{C} \tag{42}$$

assuming that $\mathbf{D}^{-1}$ exists.

Obviously, unless the scattering actually arose from an ellipsoidal inclusion, the $\mathbf{S}$ so deduced will not be compatible with Eshelby's formulas (recall that for ellipsoids, $\mathbf{S}$ contains only five independent parameters: three for orientation and two for ratios of principal axes). One alternative, which we have not developed in detail, is to fit the $\mathbf{S}$ as deduced from observations in some least-squares sense to the five parameters, and in this way determine the most nearly equivalent ellipsoid.

Another approach, which seems simpler to implement, is to determine from observations the contracted tensor D_{ij} and to find its principal axes. If the scatterer was actually an ellipsoid, these directions would correspond to its principal axes (note that some information is lost in this approach for elliptical cracks since two principal values of D_{ij} coincide for planar cracks so the principal axes of the defect are indeterminate in this limit). Hence, if we define the equivalent ellipsoid as one whose principal axes coincide with those of D_{ij}, we can determine the axes ratios for the ellipsoid by using Eq. (42) to calculate specific components of $\mathbf{S}$. For example, by calculating S_{1111}, S_{2222}, and S_{3333} (i.e., components in principal directions) it would be possible to determine a, b, and c for the ellipsoid from Eq. (42). This would imply an independent estimate of V, which could be checked for consistency against that inferred from δM. A related approach would be to determine a, b, and c by using Eq. (42) to calculate S_{1212}, S_{2323}, and S_{3131}. Finally, consistency checks that the scatterer can be represented adequately as an ellipsoid could be made by evaluating the components of $\mathbf{S}$ relating shear to extension, extension to shear, and one kind of shear to another. If the scatterer does indeed correspond closely to an ellipsoid, all of these should vanish for axes aligned with principal directions.

VI. SUMMARIZING REMARKS

In this paper we have obtained the following results.

(1) For the purposes of all long-wave elastic scattering experiments from a localized defect in a given isotropic matrix, the defect is completely characterized by 22 parameters: its excess mass δM and the 21 independent elements of a tensor D_{ijmn}, symmetric under certain interchanges of indices and determined by distribution of the stiffness coefficients $c'_{ijkl}(\mathbf{r})$ of the defect region.

(2) All 22 parameters can be determined from measurements of the scattering of longitudinal incident waves into longitudinal outgoing waves.

(3) The 21 parameters of D_{ijmn} are, of course, insufficient to determine uniquely the stiffness field $c'_{ijkl}(\mathbf{r})$ of the defect region. In particular, without independent additional information, a knowledge of D_{ijmn} does not allow a separate determination of the volume of the defect and of the magnitude of the stiffness-coefficient differences.

(4) The contracted tensor $D_{mn}(\equiv\frac{1}{3}D_{iimn})$ represents a useful partial characterization of the defect. D_{mn} defines what may be called the principal axes and diagonal D values of the defect. It allows determination of the orientation of defects in the form of planar cracks and leads to an estimate of the maximum normalized stress intensity factor around the crack periphery. Also, it allows in a large class of cases the representation of inclusions as (partially) equivalent ellipsoids.

(5) The further-contracted tensor (scalar) D $(\equiv\frac{1}{3}D_{mm})$ gives the crudest average measure of the stiffness field of the defect region. It can be reproduced by a unique spherical defect of either infinite stiffness (for $D > 0$) or vanishing stiffness (for $D < 0$). We also make a plausible conjecture which gives a lower bound for the "diameter" of an arbitrary defect in terms of D.

(6) We give general results for D_{ijkl} for homogeneous ellipsoidal inclusions, and specific results for the contracted forms D_{ij} and D for planar cracks and spherical defects.

We believe that these results can be put to practical characterization of localized defects by scattering of long-wavelength elastic waves.

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APPENDIX A: SYMMETRIES OF THE TENSOR D_{ijmn}

In this Appendix we show that the tensor D_{ijmn} defined by Eqs. (14) and (12), in addition to the obvious symmetries

$$D_{ijmn} = D_{jimn}, \quad (A1)$$

$$D_{ijmn} = D_{ijnm}, \quad (A2)$$

also has the symmetry

$$D_{ijmn} = D_{mnij}. \quad (A3)$$

The symmetry Eq. (A1) follows at once from Eq. (14) and the symmetry of c_{ijkl} under the interchange of i and j. The symmetry Eq. (A2) follows from the fact that since $\tilde{\epsilon}^{\,0}_{mn}$ is symmetric in m and n, so is Γ_{klmn} [Eq. (12)] without loss of generality, and hence by Eq. (14), D_{ijmn} is also.

To demonstrate Eq. (A3) we begin by defining the tensor

$$K_{ij} = D_{ijkl}\,\epsilon^0_{kl} = \int dV\,\delta c_{ijkl}\,\epsilon_{kl}, \quad (A4)$$

where the second equality follows from Eq. (13). [In Eq. (A4) we omit for notational simplicity the tilde on both ϵ^0 and ϵ, which is legitimate since there is a common factor aik^0.]

We now denote the displacements corresponding to the strains ϵ^0_{kl} and $\epsilon_k(r)$ respectively by $u^0_k(\mathbf{r})$ and

$$u_k(\mathbf{r}) \equiv u^0_k(\mathbf{r}) + \delta u_k(\mathbf{r}). \quad (A5)$$

Then evidently

$$\epsilon_{kl} = \epsilon^0_{kl} + \tfrac{1}{2}(\delta u_{k,l} + \delta u_{l,k}), \quad (A6)$$

and since the system is in equilibrium in the absence of body forces, δu_k satisfies the equation

$$[(c_{ijkl} + \delta c_{ijkl})(\epsilon^0_{kl} + \delta u_{k,l})]_{,i} = 0, \quad (A7)$$

Eq. (A7) can also be rewritten as

$$[(c_{ijkl} + \delta c_{ijkl})\delta u_{k,l}]_{,i} + g_j = 0, \quad (A8)$$

where

$$g_j \equiv (\delta c_{ijkl}\,\epsilon^0_{kl})_{,i}. \quad (A9)$$

Thus δu_k may also be regarded as the displacement of the system with defect unstrained at infinity but subject to the localized distribution of body force g_j.

Now consider two asymptotic strain fields ϵ^{0A}_{kl} and ϵ^{0B}_{kl}, which have associated with them the quantities g^A_j, δu^A_j, K^A_{kl} and g^B_j, δu^B_j, and K^B_{kl}, respectively. We recall that, by the Betti-Rayleigh reciprocal theorem, which follows from the existence of an elastic-strain energy function,

$$\int dV\,g^A_j\,\delta u^B_j = \int dV\,g^B_j\delta u^A_j. \quad (A10)$$

Observing that

$$\int dV\,g^A_j\,\delta u^B_j = \int dV(\delta c_{ijkl}\,\epsilon^{0A}_{kl})_{,i}\,\delta u^B_j$$
$$= -\int dV\,\delta c_{ijkl}\,\epsilon^{0A}_{kl}\,\delta u^B_{j,i}$$
$$= -\int dV\,\delta c_{ijkl}\,\epsilon^{0A}_{kl}(\epsilon^B_{ij} - \epsilon^{0B}_{ij}), \quad (A11)$$

interchanging A and B and substituting into Eq. (A10) gives

$$\int dV\,\delta c_{ijkl}\,\epsilon^{0A}_{kl}\epsilon^B_{ij} = \int dV\,\delta c_{ijkl}\,\epsilon^{0B}_{kl}\epsilon^A_{ij}. \quad (A12)$$

Here we have used the symmetry of δc to cancel the terms proportional to $\epsilon^{0A}_{ij}\epsilon^{0B}_{kl}$. Again, from the symmetry of δc_{ijkl} and from the definition of K_{ij} [Eq. (A4)], it follows that

$$\int dV\,\delta c_{ijkl}\,\epsilon^{0A}_{kl}\epsilon^{\,B}_{ij} = K^{\,B}_{kl}\,\epsilon^{0A}_{kl}, \quad (A13)$$

so that Eq. (A12) can be rewritten

$$K^{\,B}_{ij}\epsilon^{0A}_{ij} = K^{\,A}_{ij}\epsilon^{0B}_{ij}, \quad (A14)$$

or, by Eq. (A4),

$$D_{ijkl}\,\epsilon^{0B}_{kl}\epsilon^{0A}_{ij} = D_{ijkl}\,\epsilon^{0A}_{kl}\epsilon^{0B}_{ij}. \quad (A15)$$

Since this equality holds for arbitrary strains $\epsilon^{\,A}_{ij}$ and $\epsilon^{\,B}_{ij}$, the symmetry Eq. (A3) follows.

APPENDIX B: ELLIPSOIDAL INCLUSIONS, VOIDS, AND CRACKS

In the "transformation problem" discussed by Eshelby[3,4] an ellipsoidal region undergoes a change ϵ^T_{ij} in its "stress-free" strain with no change in its modulus tensor. That is,

$$\sigma_{ij} = c_{ijkl}(\epsilon_{kl} - \epsilon^T_{kl}) \quad (A16)$$

is the stress-strain relation within the transformed region, with the same c's (presumed isotropic) as outside the ellipsoid. Eshelby finds that the final "constrained" strain within the ellipsoid, due to its misfit with the surroundings, is spatially uniform and can be expressed in the form

$$\epsilon_{ij} = S_{ijkl}\,\epsilon^T_{kl}, \quad (A17)$$

where $S_{ijkl} = S_{ijlk} = S_{jikl}$ depends on the Poisson ratio ν of the material, and for coordinates aligned with principal directions of the ellipsoid, on the ratios of principal axes to one another.

The form of this solution allowed Eshelby to solve the problem of an ellipsoidal inclusion of arbitrary anisotropy in an isotropic solid under remotely uniform stress. Let σ^0_{ij} and ϵ^0_{ij} be the remote stress and strain field, and let σ_{ij} and ϵ_{ij} be the fields within the inclusion. Further, assume that there is no misfit of the inclusion when $\sigma^0_{ij} = 0$. A consequence of the transformation solution just discussed is that the inclusion undergoes a homogeneous deformation. Further, the mismatch between its strain and that of the remote field is related linearly to the corresponding mismatch in stress, in a manner independent of the properties of the inclusion material, viz.,

$$\epsilon_{ij} - \epsilon^0_{ij} = Q_{ijkl}(\sigma^0_{kl} - \sigma_{kl}), \quad (A18)$$

where $Q_{ijkl} = Q_{ijlk} = Q_{jikl}$. The Q's are related to the S's, and the relation is obtained by substituting the results of the transformation problem into the above expression. It is convenient to adopt a dyadic notation, such that the last equation is written

$$\boldsymbol{\epsilon} - \boldsymbol{\epsilon}^0 = \mathbf{Q}:(\boldsymbol{\sigma}^0 - \boldsymbol{\sigma}). \quad \text{(A19)}$$

Thus the result of the transformation problem is

$$\mathbf{S}:\boldsymbol{\epsilon}^T - \mathbf{0} = \mathbf{Q}:[\mathbf{0} - \mathbf{C}:(\mathbf{S}:\boldsymbol{\epsilon}^T - \boldsymbol{\epsilon}^T)],$$

and since this is valid for arbitrary symmetric tensors $\boldsymbol{\epsilon}^T$,

$$\mathbf{S} = \mathbf{Q}:\mathbf{C}:(\mathbf{I} - \mathbf{S}) \quad \text{or} \quad \mathbf{Q} = \mathbf{S}:(\mathbf{I} - \mathbf{S})^{-1}:\mathbf{C}^{-1}, \quad \text{(A20)}$$

where

$$I_{ijkl} = \tfrac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \quad \text{(A21)}$$

is the fourth-rank unit tensor and where the superscript -1 denotes "inverse" on the space of symmetric tensors. For example, if $\boldsymbol{\sigma} = \mathbf{C}:\boldsymbol{\epsilon}$, then $\boldsymbol{\epsilon} = \mathbf{C}^{-1}:\boldsymbol{\sigma}$.

The stress-strain relations for the inclusion and for the remote material are $\boldsymbol{\sigma} = (\mathbf{C} + \delta\mathbf{C}):\boldsymbol{\epsilon}$ and $\boldsymbol{\sigma}^0 = \mathbf{C}:\boldsymbol{\epsilon}^0$, respectively. By substituting in Eq. (A19), we obtain

$$\boldsymbol{\epsilon} - \boldsymbol{\epsilon}^0 = \mathbf{Q}:[\mathbf{C}:\boldsymbol{\epsilon}^0 - (\mathbf{C} + \delta\mathbf{C}):\boldsymbol{\epsilon}] \quad \text{(A22)}$$

for the relation between $\boldsymbol{\epsilon}$ and $\boldsymbol{\epsilon}^0$. The solution for $\boldsymbol{\epsilon}$ can be written in the form of Eq. (12), namely, $\boldsymbol{\epsilon} = \boldsymbol{\Gamma}:\boldsymbol{\epsilon}^0$, where $\boldsymbol{\Gamma}$ is the quasistatic response tensor, and we therefore find that

$$\begin{aligned}\boldsymbol{\Gamma} &= (\mathbf{I} + \mathbf{Q}:\mathbf{C} + \mathbf{Q}:\delta\mathbf{C})^{-1}:(\mathbf{I} + \mathbf{Q}:\mathbf{C}) \\ &= (\mathbf{I} + \mathbf{S}:\mathbf{C}^{-1}:\delta\mathbf{C})^{-1}. \quad \text{(A23)}\end{aligned}$$

Of course, $\boldsymbol{\Gamma} \approx \mathbf{I}$ for small $\delta\mathbf{C}$, as expected. The tensor $\mathbf{D}$ of the scattered field, defined by Eq. (14), is therefore given by

$$\begin{aligned}\mathbf{D} &= \int dV\, \delta\mathbf{C}(\mathbf{r}'):\boldsymbol{\Gamma}(\mathbf{r}') = V\delta\mathbf{C}:\boldsymbol{\Gamma} \\ &= V\delta\mathbf{C}:(\mathbf{I} + \mathbf{Q}:\mathbf{C} + \mathbf{Q}:\delta\mathbf{C})^{-1}:(\mathbf{I} + \mathbf{Q}:\mathbf{C}) \\ &= V\delta\mathbf{C}:(\mathbf{I} + \mathbf{S}:\mathbf{C}^{-1}:\delta\mathbf{C})^{-1}. \quad \text{(A24)}\end{aligned}$$

Here V is the volume of the ellipsoidal inclusion.

For the case of a void, $\delta\mathbf{C} = -\mathbf{C}$, and

$$\mathbf{D} = -V(\mathbf{C} + \mathbf{C}:\mathbf{Q}:\mathbf{C}) = -V\mathbf{C}:(\mathbf{I} - \mathbf{S})^{-1}. \quad \text{(A25)}$$

Note that for a narrow void in the form of a crack, $V\mathbf{Q}$ has a finite limit as the least principal axis is shrunk to zero (so that $V \to 0$).

We note that $\mathbf{S}$ is not symmetric under interchange of its first and last two indices. However, since $\mathbf{D}$ has this symmetry it can be shown that

$$S_{ijpq}C^{-1}_{pqkl} = S_{klpq}C^{-1}_{pqij} \quad \text{and} \quad Q_{ijkl} = Q_{klij}. \quad \text{(A26)}$$

Letting the 1, 2, and 3 axes coincide with the principal axes a, b, and c of the ellipsoid, Eshelby[3] remarks that coefficients S_{ijkl} coupling shear to extension (e.g., S_{1211}), or extension to shear (S_{1112}), or independent shears to one another (S_{1223}), are zero. The remaining coefficients are given in terms of elliptic integrals (Ref. 3, pp. 384–386, Eqs. 3.7–3.17).

The fully contracted scalar $D = \frac{1}{9}D_{iijj}$ is needed for some applications. When the inclusion is isotropic, this corresponds to

$$D = \tfrac{1}{9}V\delta C_{iikl}\Gamma_{kljj} = \tfrac{1}{3}V\delta K\Gamma_{kkjj}, \quad \text{(A27)}$$

where Γ_{kkjj} is the dilation in the inclusion due to the remote quasistatic strain field $\epsilon^0_{ij} = \delta_{ij}$ (with corresponding stresses $\sigma^0_{ij} = 3K\delta_{ij}$). The calculation of this dilation is straightforward for a spherical inclusion, and one obtains

$$\Gamma_{kkjj} = 3/(1 + \alpha\delta K/K), \quad \text{(A28)}$$

where $\alpha = (1 + \nu)/3(1 - \nu)$. Thus for a spherical inclusion

$$D = V\delta K/(1 + \alpha\delta K/K) \quad \text{(A29)}$$

or

$$D = \pi d^3(K' - K)K/[6(1 - \alpha)K + 6\alpha K'], \quad \text{(A30)}$$

where d is the diameter of the sphere and $K'\,(= K + \delta K)$ is the bulk modulus of the inclusion. This leads to the expressions given in the text for d^+ and d^- in terms of D when we set $K' = \infty$ and 0, respectively.

For a void $\delta K = -K$. Further, $V\Gamma_{kkjj}$ is the net volume change of the void, and in the special case for which the void degenerates into a planar crack, it is the integral over the crack area of the opening displacement $\Delta U^{(n)}$ between the crack surfaces under a normal stress of $3K$. For a penny-shaped crack of radius a the opening under this stress is[6]

$$\Delta U^{(n)} = \frac{8}{\pi}\,\frac{(1 - \nu^2)}{(1 - 2\nu)}\,(a^2 - r^2)^{1/2} \quad \text{(A31)}$$

at distance r from the crack center. Thus, for a penny-shaped crack,

$$D = -\tfrac{1}{3}K\int_0^a 2\pi r\Delta U^{(n)}\,dr d = -\frac{16}{9}Ka^3\frac{1 - \nu^2}{1 - 2\nu}. \quad \text{(A32)}$$

When the crack is elliptical with semiaxes a and b (with $a > b$) the corresponding area integral of $\Delta U^{(n)}$ is given in Refs. 5 and 6. There results

$$D = -\frac{8\pi}{9E(k)}Kab^2\frac{1 - \nu^2}{1 - 2\nu}, \quad \text{(A33)}$$

where $E(k)$ is the elliptic integral of second kind with modulus $k^2 = 1 - b^2/a^2$. As shown in the text, for planar cracks the contracted tensor D_{ij} has principal axes in the plane of and normal to the crack surface, and its principal values can be expressed in terms of D. Thus the last two formulas imply the full tensor D_{ij} for penny-shaped and elliptical cracks.

[1] J.E. Gubernatis, E. Domany, and J.A. Krumhansl, J. Appl. Phys. **48**, 2804 (1977).

[2] J.E. Gubernatis, J.A. Krunhansl, and R.M. Thomson, preceding paper, J. Appl. Phys. **50**, 3338 (1979).

[3] J.D. Eshelby, Proc. R. Soc. London A **241**, 376–396 (1957).

[4] J.D. Eshelby, in *Progress of Solid Mechanics*, edited by I.N. Sneddon and R. Hill (North-Holland, Amsterdam, 1961), Vol. 2, pp. 89–140.

[5] B. Budiansky and J.R. Rice, Trans. ASME, J. Appl. Mech. **45**, 453 (1978).

[6] B. Budiansky and R.J. O'Connell, Int. J. Solids Struct. **12**, 453 (1976).