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# On the Estimation of a Crack Fracture Parameter by Long-Wavelength Scattering

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Attention is focussed herein on the possibility of estimating the fracture-mechanics parameter $k_I = (K_I)_{max}/\sigma$ associated with a flat crack of initially unknown dimensions and orientation by using long-wavelength NDE measurements. Here $K_I$ is the mode I stress-intensity factor associated with tension $\sigma$ normal to the plane of the crack, and "max" denotes the largest value along the crack perimeter. The estimates will be made on the basis of the long wavelength studies by Gubernatis, et al. [1][3], and certain properties of elliptic cracks that are nearly shape invariant.

Consider an incoming longitudinal wave of frequency $\omega$ given by the displacements

$$u_j{}^0 = Ug_j \, Re\,[e^{i\omega x_i q_i/V_L} e^{-i\omega t}] \tag{1}$$

where the direction of propagation is denoted by the unit vector $q_i$, the frequency is $\omega$, the speed of longitudinal waves is

$$V_L = \left[\frac{E(1-\nu)}{\rho(1-2\nu)(1+\nu)}\right]^{1/2},$$

and $U$ is the real wave amplitude. The long-wave approximation for the spherical far-field scattered wave due to a flat crack $A$ is

$$u_i{}^S = \frac{1}{r}\,\mathrm{Re}\,[A_i e^{i\omega r/V_L} + B_i e^{i\omega r/V_T}]e^{-i\omega t} \tag{2}$$

where

$$A_i = e_i e_j g_j$$

$$B_i = \left(\frac{V_L}{V_T}\right)^3 (g_i - A_i) = \left(\frac{V_L}{V_T}\right)^3 (\delta_{ij} - e_i e_j) g_j,$$

$$r^2 = x_i x_i; \quad e_i = x_i/r; \quad V_T = (G/\rho)^{1/2};$$

and, for an isotropic material,

$$g_j = -\frac{iE\omega}{8\pi(1+\nu)\rho V_L{}^3} \times \int_A \left[n_j e_k \Delta u_k + n_k e_k \Delta u_j + \frac{2\nu}{1-2\nu} e_j n_k \Delta u_k\right] dA \tag{3}$$

Here $\Delta u_j \equiv u_j{}^+ - u_j{}^-$ is the complex amplitude of the periodic displacement jump across the flat crack and $n_j$ is the unit normal to the "top" of the crack surface, pointing from the "−" to the "+" side.[4] In the long-wave approximation we estimate $\Delta u_j$ on the basis of the static displacement produced by the stresses associated with the incoming wave. These complex stresses are

$$\sigma_{ij} = \frac{iE\omega U}{V_L(1+\nu)}\left[q_i q_j + \left(\frac{\nu}{1-2\nu}\right)\delta_{ij}\right] \tag{4}$$

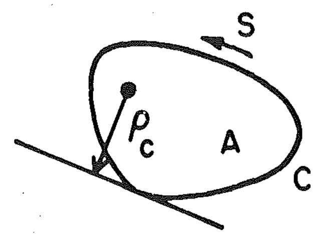


Fig. 1 Flat crack notation

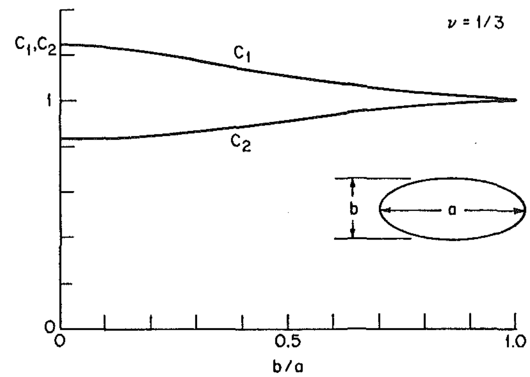


Fig. 2 Shear-mode correction factors: $C_i$ is the ratio of the left side of equation (9) to the right side

and they produce the stress

$$S_n = \sigma_{ij} n_i n_j = \frac{iE\omega U}{V_L(1+\nu)}\left[(q_i n_i)^2 + \frac{\nu}{1-2\nu}\right] \tag{5}$$

normal to the crack. The stress vector $T_j$ parallel to the plane of the crack is

$$T_j = \sigma_{ij} n_i - S_n n_j = \frac{iE\omega U}{V_L(1+\nu)}[q_j - n_j q_i n_i] q_r n_r \tag{6}$$

Now write $\Delta u_j$ as

$$\Delta u_j = n_j \Delta u^{(n)} + \Delta v_j \tag{7}$$

in terms of its normal and in-plane components. It has been shown (see, for example [2]) that (see Fig. 1)

$$\int_A \Delta u^{(n)} dA = \frac{2(1-\nu^2)}{3E} S_n \oint_c \rho_c k_I{}^2 ds \tag{8}$$

where $k_I = K_I/\sigma_n$ is the reduced mode I stress-intensity factor. We now introduce the simplifying approximation

$$\int_A \Delta v_i dA \simeq \left(\frac{2}{2-\nu}\right)\frac{T_i}{S_n}\int_A \Delta u^{(n)} dA \tag{9}$$

For elliptic cracks (see Appendix) this is exact for $\nu = 0$, and is pretty good for $\nu = 1/3$ as shown in Fig. 2. Substitution of eqs (5)–(9) into (3) gives

$$g_j = \frac{(1-2\nu)UP}{6\pi(2-\nu)}\left(\frac{\omega}{V_L}\right)^2 \times \left\{n_j\left(n_i q_i e_k q_k + \nu n_i e_i\left[\frac{2-\nu}{1-2\nu} - (n_k q_k)^2\right]\right) + q_j(n_i q_i n_k e_k) + \nu\frac{2-\nu}{1-2\nu} e_j\left[(n_i q_i)^2 + \frac{\nu}{1-2\nu}\right]\right\} \tag{10}$$

where

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[3] Numbers in brackets designate References at end of Note.

[4] The result in equation (A-4) of [1] on which equation (3) is based contains an extra integral, involving gradients of $\Delta u_k$, which can be shown to vanish as a consequence of the fact that the crack faces are free of traction. Note also that our $n_i$ is the negative of the $n_i{}^+$ of [1]. Finally, to terms of order $\omega^2$, it is permissible to drop the exponential function in the integrand shown in [1].

Manuscript received by ASME Applied Mechanics Division, December, 1977; final revision, February, 1978.

$$P = \oint_c \rho_c k_I{}^2 ds \qquad (11)$$

Consider now that the approximate location of the crack is established (presumably by short-wavelength probing), so that $e_i$ and $r$ are known for each choice of location of a scattered field measurement. It would appear then that for a given incoming signal, with $U$ and $q_i$ also known, it should be possible to determine $n_i$ (the crack normal) and $P$ from measurements of three independent scalar quantities in the far-field scattered wave equation (2). The final step in this primitive inversion process is to note that the maximum value of $k_I$ along the crack perimeter may be estimated from $P$ according to the formula

$$(k_I)_{max} \simeq \left(\frac{8P}{\pi^3}\right)^{1/6} \qquad (12)$$

This result is quite good for most elliptic cracks (see the Appendix). It is exact for a circle, and is fairly accurate down to a very elongated ellipse with axis ratio 0.06, with errors of only about 10 percent (see Fig. 3).

We note, finally, that the effect on $P$ of errors in the estimate of the crack location, and hence in $r$ and $q_i$, appear to be of the same order as those inherent in the long-wavelength approximation.

## Acknowledgment

This research was supported by the Defense Advanced Research Projects Agency of the Department of Defense under Contract No. MDA903-76C-0250 with The University of Michigan.

## References

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# APPENDIX

For an elliptic crack in mode I

$$\int_A \Delta u^{(n)} dA = \frac{8\pi ab^2 S_n}{3E(k)}\left(\frac{1-\nu^2}{E}\right) \qquad (13)$$

where $a$ and $b$ are the semiaxes $(a > b)$, $k^2 = 1 - b^2/a^2$, and $E(k)$ is the elliptic integral of the second kind (reference [2]). Under shear stress $T_i$ parallel to the crack ($T_i$ in the direction of the major axis) we have

$$\int \Delta v_1 dA = \frac{8\pi ab^2}{3}\frac{(1-\nu^2)}{E} R(k,\nu)T_1$$
$$\int \Delta v_2 dA = \frac{8\pi ab^2}{3}\frac{1-\nu^2}{E} Q(k,\nu)T_2 \qquad (14)$$

where [2]

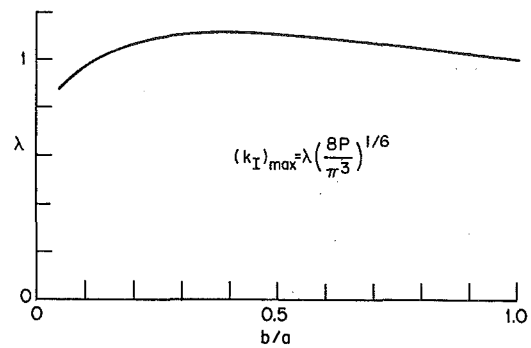


**Fig. 3 $(k_I)_{max}$ = correction factor**

$$R(k,\nu) = k^2\{(k^2-\nu)E(k) + \nu(1-k^2)K(k)\}^{-1}$$
$$Q(k,\nu) = k^2\{[k^2+\nu(1-k^2)]E(k) - \nu(1-k^2)K(k)\}^{-1} \qquad (15)$$

and $K(k)$ is the elliptic integral of the first kind. Consequently, the hypothesis (9) would require that

$$C_1 = \left(\frac{2-\nu}{2}\right)R(k,\nu)E(k)$$

and

$$C_2 = \left(\frac{2-\nu}{2}\right)Q(k,\nu)E(k) \qquad (16)$$

be roughly independent of $b/a$. Their actual variation, for $\nu = 1/3$, is shown in Fig. 2. From (8) and (13), we have, for ellipses

$$P = \frac{4\pi ab^2}{E(k)} \qquad (17)$$

Also [2]

$$(k_I)_{max} = \frac{\sqrt{\pi b}}{E(k)} \qquad (a > b) \qquad (18)$$

It follows that

$$(k_I)_{max} = \lambda\left(\frac{8P}{\pi^3}\right)^{1/6} \qquad (19)$$

where

$$\lambda = \left[\frac{\pi}{2E(k)}\right]^{5/6}\left(\frac{b}{a}\right)^{1/6} \qquad (20)$$

The estimate (12) would require $\lambda$ to be close to unity, and as Fig. 3 shows, it is for $1 \geqq b/a \geqq 0.05$.

# Errata

Erratum on "On the Estimation of a Crack Fracture Parameter by Long-Wavelength Scattering," by Budiansky, B., and Rice, J. R., and published in the June, 1978, issue of ASME JOURNAL OF APPLIED MECHANICS, Vol. 45, pp. 453–454.

In equation (1), $g_j$ should be $q_j$, so that the corrected equation (1) should read

$$u_j{}^0 = Uq_j \operatorname{Re}\left[e^{i\omega x_i q_i/V_L} e^{-i\omega t}\right]$$

  