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THE LOCALIZATION OF PLASTIC DEFORMATION

James R. Rice

Division of Engineering
Brown University
Providence, Rhode Island, U.S.A.

The localization of plastic deformation into a shear band is discussed as an instability of plastic flow and a precursor to rupture. Experimental observations are reviewed, a general theoretical framework is presented, and specific calculations of critical conditions are carried out for a variety of material models. The interplay between features of inelastic constitutive description, especially deviations from normality and vertex-like yielding, and the onset of localization is emphasized.

1. INTRODUCTION

It is remarkably common among ductile solids that when deformed sufficiently into the plastic range, an essentially smooth and continuously varying deformation pattern gives way to highly localized deformation in the form of a "shear band". Sometimes such bands, once formed, persist and the subsequent deformation proceeds in a markedly non-uniform manner. Often, however, such intense local deformation leads directly to ductile fracture, so that the onset of localization is synonymous with the inception of rupture.

While observed in ductile metals, polymers, and in rocks and granular aggregates under compressive stresses, there is little in the way of a comprehensive understanding of the phenomenon. Some basic theoretical principles follow from Hadamard's [1] studies of elastic stability, extended to the non-elastic context by Thomas [2], Hill [3], and Mandel [4]. But it is only recently that conditions for the onset of localization have been tied to realistic constitutive descriptions of inelastic response. Indeed, as will be realized from the work to be reviewed (section 3), these conditions depend critically on subtle features of these descriptions, specifically on vertex-like yielding effects and departures from plastic "normality", as well as on the tensorial nature of the pre-localization deformation field (e.g., plane strain vs. axially symmetric). Further, only some very elementary steps have been taken thus far toward as assessment of the role of deformation field non-uniformities or "imperfections" which, from experimental studies, seem often to be of great importance for the initiation and spreading of localized deformation zones.

1.1. Some examples of localization

In the plastic deformation of metals, Lüders band formation in materials with sharp yield points is perhaps the best known example of localization. Indeed, the name seems to be employed generically for the entire class of localization phenomena in [2,3,4] although, in many respects, the theoretical framework of those refences seems less suited to this case than any of the others to be cited. Nadai [5] has assembled an impressive array of photographs of Lüders yielding in steel specimens; occurrences often entail the crack-like propagation of bands in highly non-uniform conditions (and with pronounced sensitivity to imperfections) rather than a more-or-less spatially simultaneous bifurcation into a localized mode from a uniform or nearly uniform deformation field.

In a study of the ductile fracture of metal single crystals, Beevers and Honeycombe [6] observe that after some plastic deformation, the smoothly varying field of flow, often within a diffusely necked region of their specimens, gives way to a concentration of deformation in coarse slip bands, and localized flow within these leads, shortly, to ductile fracture (see their fig. 4b, also 5 and 7). Price and Kelly [7] studied this further in Al alloy single crystals; they pointed out that such localizations can occur under rising load (i.e., without a "non-hardening" state) and also observed an association of the onset of these localizations with the ease of "cross slip", an association which will be studied later for its possible connection with slight deviations from Schmid's law of critical resolved shear stress, and hence from plastic normality. On the other hand, Jackson and Basinski [8] study latent hardening for deformation of Cu crystals on slip systems intersecting an initial system on which the crystals were prestrained. The first increments of deformation on the new system correspond to near-zero initial rates of hardening, and the deformation is then initially concentrated in coarse slip bands (see their figs. 6,7). Price and Kelly [7] suggest also that there is no permanent weakening in coarse slip zones, at least in cases for which fracture does not immediately follow localization, for by unloading their specimens, machining away the steps where bands met the surface, and reloading into the plastic range, they found that localization occurred immediately upon plastic yielding, but not generally within a zone of previous localization. Hence the entire body of material in their specimens seemed to have been brought to a plastically unstable state.

Localizations occur also within ductile metal polycrystals and structural alloys. Normally, when cleavage is precluded, such materials fail by the nucleation of holes from the brittle cracking or decohering of inclusions, with subsequent plastic growth to coalescence of the cavities thus created. Cox and Low [9] show in their fig. 16a a magnified section of a plastically deformed high-strength AISI-4340 steel near its point of rupture. Cavities have formed around some brittle inclusions and have enlarged with the plastic deformation. One might then assume that rupture will entail the large plastic growth of these holes until the remaining ligaments between them neck to zero thickness, for many such cases have been observed (e.g., McClintock [10]). But instead the hole growth process in the Cox and Low specimen has been terminated by the formation of a band of localized shearing extending between the large cavities. Within the band a number of very much smaller cavities have formed and grown toward coalescence. One cannot say whether these smaller rupture cavities first began to form, and this led to localization, or whether the plastic flow first localized and nucleated the small cavities. But certainly, there is no detectable evidence of the nucleation of the smaller cavities at points outside the shear band.

Indeed, such localizations seem to be crucial in setting the limits to achievable fracture ductility. For example, in sharply pre-cracked ductile solids, the onset of crack growth is expected to occur when the crack has been sufficiently opened at its tip so that the zone of large plastic strain extends over an adequate size, by comparison to the spacings of void nucleation sites, to grow a representative void to coalescence with the crack tip (Rice and Johnson [11]). In studies by Green and Knott [12] on steels and Hahn and Rosenfield [13] on aluminum alloys, it is shown that the predictions of such a model are often well followed experimentally, but that there are also cases in which the fracture ductility is strikingly less than predicted; these cases seem to involve the termination of the hole-joining process by strong localizations of the type revealed by Cox and Low.

At a more macroscopic level, Tanaka and Spretnak [14] subjected round bars of a high strength steel to large torsional strains and observed localizations which, they suggest, corresponded to the achievement of an "ideally plastic" state of stationary shear stress with ongoing deformation. Also, Berg [15] suggests the possibility of macroscopic localization in void-containing ductile materials, when the hardening of the solid matrix surrounding the voids, in an increment of deformation, is outweighed by the softening due to porosity increase through incremental void growth. The range of situations to which this model applies remains uncertain. Studies of ductile rupture-in-progress in the necks of Cu tensile specimens, nominally said to fail through void growth and coalescence, by Rogers [16] (his figs. 6,7,8,10) and Bluhm and Morrissey [17] (their fig. 42) reveal zones of highly localized deformation within which voids are indeed coalescing. But conditions are highly nonuniform across their specimens and it is difficult to determine whether a Berg-like localization, due to softening through porosity increase, has occurred or whether localization of plastic flow has caused the extensive void growth. Certainly, the final rupture surface is made up of a "void sheet", but this alone is not indicative of the process leading to it.

Further examples of localization instabilities are provided by the formation of narrow necked zones in ductile metal sheets deformed, at least prior to localization, in plane stress. Judged as 3-D problems, these are distinctly different from the other cases cited and involve "geometric" as opposed to "material" instability. However, to the extent that such ductile sheets are modelled as 2-D continua, the problem of localization may be treated through identical mathematical steps and is well considered within the same general framework.

Geological materials are rich also in examples of localization. For slope stability failures of overconsolidated clays and clay shales, it is a common observation that deformations concentrate in a narrow shear zone, perhaps only a few mm across, on which large downslope mass movements take place. Laboratory deformation of such clays reveal a corresponding concentration of deformation (e.g., Hvorslev [18], fig. 28), which seems to fit the concept of bifurcation into a localized mode. Field occurrences may, however, involve strongly non-uniform conditions with a crack-like mechanism for propagation of the shear band (Palmer and Rice [19]) after its initiation at a site of local strain concentration. Rowe [20] shows localizations within sand specimens (his figs. 15b,c), deformed in the "triaxial" apparatus under axially symmetric compression. The effect of end conditions is evident in that an arrangement intended to provide shear-free ends of the specimen leads to a significantly larger strain at localization than for an, effectively, fixed end-piece specimen. Also, it is evident that there is no necessary association of localization with the "ideally plastic" state, for the localizations occur when the material is well beyond that state and in the strain-softening range (significantly negative slope of the load-deformation diagram). Indeed the tensile fracture tests on metal specimens by Bluhm and Morrissey [17] were done in a stiff testing apparatus so that a strongly negative load-deformation slope could be accommodated without instability, and these suggest too that a strain-softening state (in terms of true stress) may well have prevailed, at the onset of macroscopic rupture through hole coalescence at the center of the necked region.

Finally, natural rock specimens tested under compressive principal stresses also show examples of localization, usually referred to as "faulting". Wawersik and Brace [21] study the post-"failure" behavior of specimens through a stiff testing apparatus, with provision for rapid unloading, and show (e.g., their plates 4a,b for a Frederick diabase) examples of localization. Here the inelastic deformation arises from frictional sliding on closed microcracks and progressive enlargement of the microcrack network through local fissuring; the final macroscopic fault links a large number of such microcracks, although their individual directions of growth do not coincide with that of the final fault. This case, like that of granular materials, is interesting because the Coulomb frictional nature of the yielding means that plastic normality will not apply (Mandel[4]) and this has interesting consequences for localization instabilities.

1.2. Mechanisms of localization

The work to follow explores a particular approach to explanation of the localization of deformation, viewing the process as an instability that can be predicted in terms of the pre-localization constitutive relations of the material. The material is modelled as rate-independent and critical conditions are sought at which its constitutive relations allow a bifurcation from homogeneous or smoothly varying deformation into a highly concentrated shear band mode, or, perhaps instead, at which the accelerated growth, with ongoing deformation, of some initially small non-uniformity of material properties can occur in such a manner that the same sort of shear band is the end result.

Of course, not all localization phenomena can be expected to fit this concept. An alternative hypothesis would be that some essentially new physical deformation mechanism comes into play, abruptly, and rapidly degrades the strength of the material. In such cases the pre-localization constitutive relations cannot be continued analytically at the critical point, and they provide no basis for prediction of localization. Indeed, to the extent that upper yield points arise from the sudden breaking free of dislocations from pinning obstacles, with only lightly impeded subsequent gliding, the Lüders band case must be considered as one which is dominated by onset of some new mechanism, and thus the bifurcation approach, explored here, does not apply to it. (It is, however, curious that use of a more sophisticated rate-dependent plastic flow theory would return this case to one for which localization could be understood in terms of constitutive relations, although the details of the analysis would be very different from what is to follow.)

Further, the approach to be explored here, being essentially a bifurcation approach, envisions a process of simultaneous or nearly simultaneous occurrence of concentrated deformation at all points of the (ultimate) zone of localization. But in contrast, there may be situations that are dominated by some strong local inhomogeneity, which concentrates deformation in its vicinity and causes the initiation of a localized zone which, subsequently, creates its own strain concentration and thereby traverses the material at nominal deformation conditions that are well removed from those for localization. This is, of course, the way in which a Griffith flaw can cause a crack to traverse a body at average stresses that are well below the strength level for an unflawed solid (although stresses at or near that level would be achieved locally at the propagating crack tip).

Such crack-like propagation processes do not invalidate the present approach to localization, but do require that it be generalized, in a way that is not yet fully clear, to encompass the highly non-uniform conditions prevailing near the tip of the crack-like zone. An elementary approach is to assume that once local deformations reach conditions for localization, the constitutive relations for continuum-like deformation are suspended in favor of a relation between tractions and relative displacements of the surfaces of the zone of localization (presumed thin). This approach is embodied in the Palmer and Rice [19] model for shear band propagation in overconsolidated clay soils: for very small initial flaws (or faults) the strength as so predicted approaches that for the unflawed body, whereas for large flaws the response tends to become independent of the strength for the unflawed body, and is expressible instead in terms of flaw size and parameters of the traction vs. separation-displacement relation (specifically, in terms of the net work of separation) for the localized material.

Clearly, the mere observation that a zone of localized deformation exists within a deformed solid is an inadequate basis for choosing which, if any, of the various mechanisms discussed is correct for its explanation.

2. GENERAL THEORY

We consider deformations which carry points $\underset{\sim}{X}$ of some reference state to positions $\underset{\sim}{x}$, where both $\underset{\sim}{X}$ and $\underset{\sim}{x}$ are coordinate sets (e.g., X_1, X_2, X_3) referred to a fixed Cartesian frame. The deformation gradient tensor is $\underset{\sim}{F} = \partial\underset{\sim}{x}/\partial\underset{\sim}{X}$ and stress is measured by the nominal stress tensor $\underset{\sim}{s}$, defined so that $\underset{\sim}{n}\cdot\underset{\sim}{s}$ is the force acting, per unit reference area, on an element of surface having normal vector $\underset{\sim}{n}$ in the reference state. It satisfies

$$s_{ij,i} + f_j = 0 \qquad (s_{ij,k} \equiv \partial s_{ij}/\partial X_k)$$

at equilibrium, where $\underset{\sim}{f}$ is the external force density based on the reference state. When needed, the true (or Cauchy) stress is denoted by $\underset{\sim}{\sigma}$; the measures are related by $\underset{\sim}{\sigma}\det(\underset{\sim}{F}) = \underset{\sim}{F}\cdot\underset{\sim}{s}$.

While a full analysis of rate effects on localization is well worth study, the theory is not yet well developed in that context and here we consider rate-independent, thermally decoupled (or else completely adiabatic) constitutive models, so that the stress rate $\dot{s}$ is a homogeneous function of degree one in the deformation gradient rate $\dot{F}$, the form of the functional relation being itself dependent on the prior history of deformation, and exhibiting proper variance for rigid spin.

Now, consider a homogeneous, homogeneously deformed solid subjected, quasi-statically, to increments of deformation which could give rise to the homogeneous and constitutively compatible quasi-static rate fields $\dot{F}^o$, $\dot{s}^o$. We wish to determine if a bifurcation within a localized band, of orientation n in the reference state, is possible, in such a manner that $\dot{F}$ varies with $n \cdot X$ (i.e., with position across the band) while the equations of continuing equilibrium, $\dot{s}_{ij,i} = 0$, (constant f) are met. There are two conditions to be met: First, if the velocity field is to be continuous, there is the kinematical condition that $\dot{F}$ must have the form

$$\dot{F} = \dot{F}^o + gn \,, \qquad \text{or} \qquad \Delta\dot{F} = gn \,, \tag{1}$$

where the vector g is some function of $X \equiv n \cdot X$, being non-zero within the band, and where the notation Δ denotes the difference between some function at a general point and within the homogeneous field outside the band. Given the kinematical condition, $\dot{s}$ can vary only with X, and thus the equations of continuing equilibrium take the form $n \cdot \partial\dot{s}/\partial X = 0$. This requires that

$$n \cdot \dot{s} = n \cdot \dot{s}^o \,, \qquad \text{or} \qquad n \cdot \Delta\dot{s} = 0 \,. \tag{2}$$

The last condition can be expressed also in terms of g. Choosing the reference state to coincide, instantaneously, with the current state, one may show [22] from the relation of g to s that $n \cdot \Delta\dot{s} = n \cdot \Delta\dot{\sigma}$ whenever (1) is satisfied, and hence the condition (2) is alternatively stated as $n \cdot \Delta\dot{\sigma} = 0$. This can, of course, be derived directly.

2.1. Piecewise-linear constitutive rate laws

To proceed further, we assume that the relation of $\dot{s}$ to $\dot{F}$ is not only homogeneous of degree one but also piecewise-linear. This is so in the sense that "F-space" can be divided into a family of cones emanating from the current state, at each instant, in a manner such that

$$\dot{s}_{ij} = L_{ijk\ell} \dot{F}_{k\ell} \tag{3}$$

for some fixed set of moduli L whenever $\dot{F}$ points within a given cone; different L are assigned to the different cones and these are such that $\dot{s}$ is continuous across cone boundaries.

For certain states, the immediate neighborhood of F-space is accessible elastically. Then there is a single constitutive cone, i.e. the same L applies for all directions of $\dot{F}$, and if an elastic potential is to be admitted, there applies the symmetry

$$L_{ijk\ell} = L_{\ell kji} \tag{4}$$

For the simplest model of elastic-plastic response, there are two cones separated by a plane through the current state in F-space. This corresponds to a locally "smooth" yield-locus; $\dot{F}$ directions into one cone induce elastic response, to which (4) applies, whereas directions into the other induces an inelastic response governed by moduli L that may or may not meet (4), depending on the physical origin of the inelasticity. Specifically, if the constitutive relations satisfy plastic "normality" in conjugate deformation variables, then (4) applies (see Hill [23]). Precise conditions that are sufficient for such normality (or its generalization at a "vertex") are given in terms of structural rearrangement mechanisms by Rice [24] and of macroscopic work inequalities of the Drucker/Il'yushin type by Hill and Rice [25]. In short, such normality is to be expected, at least approximately, for polycrystals deforming by Schmid-like slip under critical resolved shear stresses in each grain but not [4] for systems with Coulomb (i.e., normal stress dependent) frictional resistance to slip.

More generally, the neighborhood of the current state in F-space is divided into many cones, one of which may (but need not) correspond to elastic response, and (4) applies or not in each cone according to whether the normality postulate is appropriate to the material at hand. One can admit the limiting case of an unbounded number of cones subtending infinitesimal angles in F-space. Indeed, realistic microstructure-based models seem universally to predict this [26,27,22]. The case corresponds to thoroughly non-linear behavior, but even then relations of the kind (3) may be considered to apply for groups of $\dot{F}$'s differing infinitesimally in direction from one another. When one of the cones is elastic (and of non-planar boundary, as expected universally) the yield surface is said to have a vertex at the current deformation state.

Now, returning to the bifurcation calculation, assume that the homogeneous fields $\dot{F}^o, \dot{s}^o$ outside the band are related by moduli L^o, whereas within the band the corresponding cone has moduli L. Then by (1)

$$\dot{s}^o_{ij} = L^o_{ijk\ell}\dot{F}^o_{k\ell} \,, \qquad \dot{s}_{ij} = L_{ijk\ell}(\dot{F}^o_{k\ell} + g_k n_\ell) \,,$$

and by (2) the kinematical non-uniformity g must satisfy

$$(n_i L_{ijk\ell} n_\ell) g_k = n_i (L^o - L)_{ijk\ell} \dot{F}^o_{k\ell} \,. \tag{5}$$

This has always the trivial solution $g = 0$; the onset of localization occurs at the first point in the deformation history for which a non-trivial solution exists.

In the simplest case, the band and the region outside it correspond to the same constitutive cone, $L = L^o$. Then (5) requires

$$(nLn)_{jk}\, g_k = 0 \;, \quad \text{or} \quad \det(nLn) = 0 \tag{6}$$

for the onset of localization. Here the concise notation nLn is introduced for the matrix having the jk component given by the term in parentheses at the left of (5). Later, in applications, it will be preferable to write constitutive rate laws in the form

$$\dot{\sigma}_{ij} = \bar{L}_{ijk\ell}\partial\dot{x}_k/\partial x_\ell \equiv C_{ijk\ell}D_{k\ell} - \sigma_{ik}\Omega_{kj} + \Omega_{ik}\sigma_{kj} \;, \tag{7}$$

where $\underset{\sim}{D}$ and $\underset{\sim}{\Omega}$ are the respective symmetric and non-symmetric parts of $\partial\underset{\sim}{\dot{x}}/\partial\underset{\sim}{x}$, and $\underset{\sim}{C}$ relates the co-rotation (or Jaumann) rate of $\underset{\sim}{\sigma}$, namely $\overset{\nabla}{\underset{\sim}{\sigma}}$, to $\underset{\sim}{D}$. Then, following the remark after (2), whenever the current and reference state are instantaneously coincident we have $nLn = n\bar{L}n$, whether (6) can be satisfied non-trivially or not, and the localization condition (6) becomes $\det(n\bar{L}n) = 0$.

Note that the localization condition is not, as commonly assumed, that the equation $\overset{\nabla}{\underset{\sim}{\sigma}} = \underset{\sim}{C}:\underset{\sim}{D}$ possess no inverse, or that some related indication of an ideally plastic response be met. Instead, it is necessary that the traction rates acting on the band be stationary with respect to some superposed combination of extension and shear within it. This can happen when the traction rates themselves are, in particular cases, either positive or negative or zero.

As Hill [3] remarks, if the localization condition $\det(nLn) = 0$ is met for some orientation $\underset{\sim}{n}$, that same orientation defines a characteristic segment for the continuing equilibrium equations, $(L_{ijk\ell}\dot{x}_{k,\ell})_{,i} = 0$. Alternatively, the onset of localization is first possible, in a program of deformation, when these equations lose ellipticity.

2.2. Acceleration waves and dynamic growth of disturbances

The interpretation of conditions for static localization as those for a vanishing propagation speed of an acceleration wave was given for elastic solids by Hadamard [1], who uses the term "stationary discontinuity" for localization, and by Hill [3] and Mandel [4] for elastic-plastic solids (although Hill limits himself to cases for which the symmetry (4) applies).

Briefly, at an acceleration wavefront (on which $\underset{\sim}{\dot{x}}, \underset{\sim}{F}$, and $\underset{\sim}{s}$ are continuous) moving with speed $c\underset{\sim}{n}$ relative to the reference configuration, the first derivatives of any continuous function f, scalar or tensor, have the representations [3]

$$\Delta(\partial f/\partial \underset{\sim}{X}) = \mu\underset{\sim}{n} \;, \qquad \Delta(\partial f/\partial t) = -c\mu$$

where now Δ denotes jumps in quantities as the wavefront sweeps by. Taking $f = \underset{\sim}{\dot{x}}$ (remembering that $\underset{\sim}{\dot{F}} = \partial\underset{\sim}{\dot{x}}/\partial\underset{\sim}{X}$) and $\underset{\sim}{s}$ successively we have the wavefront representations

$$\Delta\underset{\sim}{\dot{F}} = \underset{\sim}{g}\underset{\sim}{n} \;, \quad \Delta\underset{\sim}{\ddot{x}} = -c\underset{\sim}{g} \;, \quad \Delta(\underset{\sim}{\nabla}\underset{\sim}{s}) = -\underset{\sim}{n}\Delta\underset{\sim}{\dot{s}}/c \;, \tag{8}$$

where $\underset{\sim}{g}$ is some vector. When the last two of these are applied to the equations of motion $\underset{\sim}{\nabla}\cdot\underset{\sim}{s} + \underset{\sim}{f} = \rho\underset{\sim}{\ddot{x}}$ (ρ is mass density of the reference state), there results the condition

$$\underset{\sim}{n}\cdot\Delta\underset{\sim}{\dot{s}} = \rho c^2 \underset{\sim}{g} \tag{9}$$

where it is understood that $\Delta\underset{\sim}{\dot{s}}$ is constitutively compatible with $\Delta\underset{\sim}{\dot{F}}$, given by $(8)_1$. Indeed, comparing (1,2) with $(8)_1$ and (9), with $c = 0$ in the latter, we see the coincidence mentioned between localization and acceleration waves of vanishing speed. Further, the mode of localization coincides with the polarization vector of the wave of vanishing speed.

Of course, the real significance of acceleration wave speeds is that, if the constitutive relation for small displacements $\underset{\sim}{u}$ from a given static state (say, state A) can be written in the form

$$s_{ij} = s^A_{ij} + L^A_{ijk\ell}\, u_{k,\ell} \tag{10}$$

then the nature of the response to small disturbances is determined by the character of the eigen-solutions c^2 to (9) for state A. Specifically, displacements satisfying

$$s_{ij,i} + f_j = L^A_{ijk\ell}\, u_{k,\ell i} = \rho\ddot{u}_k \;, \tag{11}$$

and corresponding to a periodic initial disturbance of wave number $k\underset{\sim}{n}$, with (for convenience) polarity identical to that for an eigenvalue $\underset{\sim}{g}$ of $(8_1,9)$, corresponding to wavespeed c, have the form

$$\underset{\sim}{u} \sim \mathrm{Re}\{\underset{\sim}{g}\exp[ik(\underset{\sim}{n}\cdot\underset{\sim}{X} - ct)]\} \;.$$

Here Re means "real part" and i is the unit imaginary number. The c's are, of course, to be chosen so that ρc^2 is an eigenvalue of nLn in state A.

There are considerable difficulties, to be discussed, with adopting (10) for elastic-plastic solids. But proceeding with it, we see that if c^2 is real and positive there is stability to small disturbance. If c^2 is real but negative there is "divergence" growth, and if c^2 is complex, there is "flutter" growth of disturbances. Thus when (4) applies (elastic, or elastic-plastic with normality), nLn is symmetric and all roots c^2 are real, and thus there is stability or not according to whether the smallest root is positive or negative, equivalently, to whether nLn is positive definite or indefinite, and the static localization criterion marks the transition from stable to unstable. When normality does not apply, nLn is unsymmetric and there seems at least the possibility, for suitably chosen constitutive parameters, that two of the roots c^2 are complex conjugates (one root must always be real) so that "flutter" occurs.

To the writers knowledge, no specific case exhibiting this kind of divergence has been discussed. Its occurrence would, in any event, invalidate (10) which assumes that response remains in a given constitutive cone. Indeed, in the general case, it is the presence of some imposed deformation like $\dot{F}^o$ which validates a particular response cone for small perturbations from it, and then the stress at any instant is, in general, not only a function of $\partial u/\partial X$ at the current instant, but also a functional of prior $\partial u/\partial X$ values.

2.3. Response with initial imperfections

The last topic opens the subject of analysis of localizations in presence of initial imperfections and, as suggested, the response is necessarily of an hereditary character in prior non-uniformities of $\partial u/\partial X$. Most of the analysis done for localization with imperfections has been directed to sheet materials, where the instability corresponds to localized necking.

It is well known that when such problems are analyzed with a smooth-yield-surface rigid-plastic model, localization is predicted never to occur in a sheet deformed under conditions for which both principal stretch rates are positive. Experiments suggest otherwise, and Marciniak and Kuczynski [28] showed that localization could be explained by assuming that a long rectangular slice of the sheet had properties slightly different from the material outside. Other explanations, based on vertex-like yield behavior [29] seem to be at least as suitable in explaining quantitatively the experimental results, but the point is that in this case imperfections lead to localization at a finite strain whereas the perfect sheet could never localize. Needleman [30] has recently completed a comparative study of various imperfection approaches and of vertex effects for localized (and other) necking modes in a "sheet" in the form of a pressurized spherical membrane.

The MK approach [28] can be recast in our general framework. Suppost that a thin slice of material ,of parallel surfaces having normal n in the reference state, has initial properties differing, perhaps only slightly, from the mateial outside it. If F, s refer to the material within the slice and F^o, s^o to that outside, then the kinematical relation $(1)_1$ applies, where now g is simply to be written as $\dot{q}$ and q is the deformation non-uniformity accumulated up to the present instant, $F = F^o + qn$. Eq.$(2)_1$ applies also, and if L and L^o represent the (finitely different) incremental moduli inside and outside the slice, then an equation identical to (5) applies for growth of q:

$$(n_i L_{ijk\ell} n_\ell)\dot{q}_k = n_i (L^o - L)_{ijk\ell} \dot{F}^o_{k\ell} . \qquad (12)$$

In this L^o is a functional only of the imposed outer field F^o, whereas L is a functional of q as well. The localization condition is that $\det(nLn) = 0$, and the relevance of the calculation is that for small but finite initial non-uniformities, the field within the slice can be driven to the localization condition well before the moduli of the outer field would allow localization. This approach merits wider study for a variety of constitutive relations. The critical conditions, phrased in terms of L, are the same as for the bifurcation problem as worked out in section 3, but one must additionally determine, through solving (12) for the program of imposed deformation, the relation of the field in the non-uniform zone to the nominal (or outer) deformation field.

2.4. Limiting nature of the localization instability

Experimentally, other types of bifurcation or deformation non-uniformities (e.g., necking, bulging, buckling) can precede localization, but the latter seems to be the limiting mode. This is suggested by specific calculations [30,31] and some understanding is obtained by following Hadamard's [1] example, extended by Hill [3], of a solid deformed homogeneously by imposition of displacement boundary conditions at all points of its surface (so as to rule out "geometric" modes). Let S be the surface and V the volume of the body (both in the reference configuration), and let $\dot{x} = \dot{F}^o \cdot X$ on S so that one solution of the quasi-static field equations (without body force) is that $\dot{x} = \dot{F}^o \cdot X$ in V.

For any other solution $\dot{x}$ meeting the same conditions on S, let $\Delta\dot{x} = \dot{x} - \dot{F}^o \cdot X$ and observe that this vanishes on S. Accordingly,

$$0 = \int_S n_i \Delta\dot{s}_{ij} \Delta\dot{x}_j dS = \int_V \Delta\dot{s}_{ij} \Delta\dot{F}_{ji} dV ,$$

since $\Delta\dot{s}_{ij,i} = 0$, and thus if it is assumed that both the uniform and non-uniform field correspond to the same constitutive cone, a sufficient condition for uniqueness (i.e., the absence of a non-uniform solution) is, as in Hill [3], that the functional

$$I[\Delta\dot{x}] \equiv \int_V \Delta\dot{x}_{j,i} L_{ijk\ell} \Delta\dot{x}_{k,\ell} dV \qquad (13)$$

be positive definite for all fields $\Delta\dot{x}$ vanishing on S.

In the restricted case for which the symmetry (4) applies (i.e., for elastic-plastic materials with normality), the equations of continuing equilibrium validate Hill's variational statement $\delta I[\Delta\dot{x}] = 0$, and hence the first point in a program of deformation at which $I[\Delta\dot{x}]$ becomes semi-definite is also the point at which, at least formally, conditions are met for existence of a non-uniform solution field. In this sense the Hill criterion is necessary as well, but there is no reason for coincidence of the failure of his sufficient condition and the existence of a non-uniform solution when L corresponds to non-normality.

The connection with the condition for localization can be developed through the procedure of Van Hove [32]. By extending any field $\Delta\dot{x}$ to the exterior of V as 0, the Fourier representation

$$\Delta\dot{x} = \int_{-\infty}^{+\infty} A(K) \exp[\sqrt{(-)}K\cdot X]\, d^3K$$

can be employed and thus

$$I[\Delta\dot{x}] \propto \int_{-\infty}^{-\infty} A_j^*(K_i L_{ijk\ell} K_\ell) A_k d^3K$$

where the * denotes complex conjugate. Hence, if L has the symmetry (4), $I[\Delta\dot{x}]$ will be positive definite so long as the 3×3 matrix KLK is positive definite. Thus, if the localization condition det(nLn) = 0 has not been met for any orientation n in the program of deformation up to the current instant, the displacement b.v.p. has only the homogeneous solution. Conversely, if the localization condition has been met and exceeded by any finite amount, in the sense that nLn is indefinite for some range of orientations, then it is elementary to construct a field such that $I[\Delta\dot{x}]$ is negative [1,3], implying that a bifurcation point has been passed. Such fields $\Delta\dot{x}$ are constructed through multiplying a smooth function vanishing on S by another function which generates gradients of the form $(1)_2$ within a narrow transition zone (of orientation n for which nLn is indefinite). By suitable choice of g it is then possible to generate a negative contribution to (13) from the integral over the transition zone, and by choosing a sufficiently small thickness of that zone it is always possible to make the negative contribution outweigh the possibly positive contribution to the integral from other parts of V.

The development of a corresponding theoretical framework for materials not exhibiting normality would, in view of the physical examples cited, be a worthwhile undertaking. Also, in situations with vertex-like yielding, it seems possible that in some cases the bifurcating field could take advantage of the response moduli of a different constitutive cone than that of the homogeneous field; this too merits study.

3. RESULTS FOR VARIOUS MATERIAL MODELS

3.1. Rigid-plastic material (smooth yield locus and plastic potential)

For this case it is assumed that during plastic deformation

$$h\, D_{ij} = P_{ij} Q_{k\ell} \overset{\nabla}{\sigma}_{k\ell} \;. \qquad (14)$$

Here P and Q are dimensionless symmetric tensors, normalized in any convenient way, so that P gives the "direction" of plastic deformation and Q the outer normal to the "yield locus", assumed smooth. Also, h is the rate of hardening, and it may be positive or negative (i.e., strain softening allowed). When $h > 0$, plastic flow occurs only when $Q:\overset{\nabla}{\sigma} > 0$, and otherwise $D = 0$. But when $h < 0$ flow occurs under conditions for which $Q:\overset{\nabla}{\sigma} < 0$ (there is also a rigid branch of response compatible with this) and the case $Q:\overset{\nabla}{\sigma} > 0$ is then prohibited. In general, P, Q, and h depend on the history of deformation.

Equation (14) corresponds to the rigid-plastic model as classically understood, with a smooth yield surface and plastic potential (i.e., P and Q independent of the direction of $\overset{\nabla}{\sigma}$). As will be seen, it is too simple a model for our present purposes, but localization conditions can be given for it in general terms and the results are suggestive of the response of more realistic models.

When $P = Q$ in (14) we say that plastic normality applies. In fact (14) is form invariant under changes to other objective stress rate measures based on a reference state that is coincident instantaneously with the current state. Any such measure, say σ^*, relates to $\overset{\nabla}{\sigma}$ by an expression of the form (Hill [33])

$$\overset{*}{\sigma}_{ij} = \overset{\nabla}{\sigma}_{ij} + \Sigma_{ijk\ell} D_{k\ell} \;,$$

where the components of Σ are linear in components of σ. Thus one may show that (14) transforms to

$$(h + Q_{pq}\Sigma_{pqrs}P_{rs})\, D_{ij} = P_{ij}\, Q_{k\ell}\overset{*}{\sigma}_{k\ell} \;;$$

P and Q remain the same, but the hardening rate is measure dependent.

The present constitutive relation connot be put in the form (3), so the localization conditions (1) and (2) must be applied directly. Thus one seeks a non-trivial solution for g and $\Delta\dot{\sigma}$ which satisfies

$$\frac{1}{2} h(n_i g_j + n_j g_i) = P_{ij} Q_{k\ell}[\Delta\dot{\sigma}_{k\ell} - \frac{1}{2}(g_k n_r - n_k g_r)\sigma_{r\ell} + \frac{1}{2}\sigma_{kr}(g_r n_\ell - n_r g_\ell)] \qquad (15)$$

$$\text{and} \qquad n_i \Delta\dot{\sigma}_{ij} = 0$$

From the first of these it is evident that the localization can occur on a plane of normal n only if P has the representation

$$P_{ij} = \frac{1}{2}(n_i \mu_j + \mu_i n_j) \qquad (16)$$

where μ is some vector, and the bifurcation vector g must be of the form $\lambda\mu$.

The restriction on P is equivalent to stating that it correspond to shear and extension relative to the plane n but not to deformation within the plane. Hence, a first requirement for localization is that a non-deforming surface exist in the deformation field; the intermediate principal value of P must vanish. This condition is very restrictive. When one of either the major or minor values vanish as well, there is one possible plane of localization. When neither vanishes, there are two possible planes (interchange n and μ in (16)), both having normals in the plane of the greatest and least principal directions and being symmetrical relative to these directions. Let α,β index components of tensors relative to two cartesian axes lying in the plane of the band. Then, e.g., (16) requires $P_{\alpha\beta} = 0$, and because of the restriction $(15)_2$, $Q_{ij}\Delta\dot{\sigma}_{ij} = Q_{\alpha\beta}\Delta\dot{\sigma}_{\alpha\beta}$. Thus writing $\underset{\sim}{g}=\lambda\underset{\sim}{\mu}$, we have

$$\lambda\{h + \frac{1}{2} Q_{k\ell}[(\mu_k n_r - n_k\mu_r)\sigma_{r\ell} - \sigma_{kr}(\mu_r n_\ell - n_r\mu_\ell)]\} = Q_{\alpha\beta}\Delta\dot{\sigma}_{\alpha\beta} \quad (17)$$

If normality holds, $Q_{\alpha\beta} = P_{\alpha\beta} = 0$ and a non-zero solution for λ (or g) can exist only when the bracketed term vanishes. This gives the critical hardening rate h for localization. Writing $\underset{\sim}{Q} = \underset{\sim}{P}$ and using (16) for the latter, the result may be rearranged to the compact form

$$h_{crit} = \frac{1}{2} |\underset{\sim}{\mu}|^2 (\sigma_{\mu\mu} - \sigma_{nn}) \quad (18)$$

where the stresses are the respective normal stresses in the directions of $\underset{\sim}{\mu}$ and $\underset{\sim}{n}$. When the principal axes of deformation (i.e., of $\underset{\sim}{P}$) coincide with those of $\underset{\sim}{\sigma}$ we have, since $\underset{\sim}{\mu}$ and $\underset{\sim}{n}$ lie along lines making equal angles with the principal axes, $\sigma_{\mu\mu} = \sigma_{nn}$ and thus

$$h_{crit} = 0 . \quad (19)$$

On the other hand, when normality does not apply, it is possible (but not required) that some of the components $Q_{\alpha\beta}$ are non-zero. In that case, since the components $\Delta\dot{\sigma}_{\alpha\beta}$ are unrestricted, (17) permits a non-zero λ and hence localization for any value of h ! As will be seen, the inclusion of elastic effects will mitigate this strong tendency for localization in the absence of normality, but the tendency remains. Also, the strong kinematical restriction, embodied in the need for a non-deforming plane, is relaxed when more elaborate constitutive models are considered, but still localization conditions are more readily met when pre-localization deformation fields meet approximately such a restriction than when they do not.

3.2. Elastic-plastic materials
Consider the constitutive rate relation

$$\overset{\nabla}{\sigma}_{ij} = E_{ijk\ell}[D_{k\ell} - \frac{1}{h} P_{k\ell}Q_{rs}\overset{\nabla}{\sigma}_{rs}] . \quad (20)$$

By reference to (14), it is clear that the bracketed term is the difference between the total rate of deformation and a plastic part given as in (14). Thus $\underset{\sim}{E}$ is the tensor of elastic moduli, and this is symmetric under the interchanges $ij \to ji$ and $k\ell \to \ell k$. We can rearrange the equation so that the right side is expressed solely in terms of $\underset{\sim}{D}$, and then put it in the form (7) so that, in a compact notation,

$$\dot{\underset{\sim}{\sigma}} = \underset{\sim}{E}:\underset{\sim}{D} - \frac{1}{h+\underset{\sim}{Q}:\underset{\sim}{E}:\underset{\sim}{P}} \underset{\sim}{E}:\underset{\sim}{P} (\underset{\sim}{Q}:\underset{\sim}{E}:\underset{\sim}{D}) - \underset{\sim}{\sigma}\cdot\underset{\sim}{\Omega} + \underset{\sim}{\Omega}\cdot\underset{\sim}{\sigma} .$$

This form covers a very general class of materials, but it is important to realize that vertex-like yielding effects are not yet included. Examples incorporating these will be taken up later.

The conditions (1) and (2), implemented with the current state taken as reference state, require that for localization on a plane of normal $\underset{\sim}{n}$, a non-trivial solution $\underset{\sim}{g}$ exists to

$$0 = (\underset{\sim}{n}\cdot\underset{\sim}{E}\cdot\underset{\sim}{n})\cdot g - \frac{1}{h+\underset{\sim}{Q}:\underset{\sim}{E}:\underset{\sim}{P}} (\underset{\sim}{n}\cdot\underset{\sim}{E}:\underset{\sim}{P})(\underset{\sim}{Q}:\underset{\sim}{E}\cdot\underset{\sim}{n})\cdot g + \underset{\sim}{A}\cdot g , \quad (21)$$

where $\underset{\sim}{A} = \frac{1}{2} [-\underset{\sim}{n}(\underset{\sim}{n}\cdot\underset{\sim}{\sigma}) + (\underset{\sim}{n}\cdot\underset{\sim}{\sigma}\cdot\underset{\sim}{n})\underset{\sim}{I} + (\underset{\sim}{n}\cdot\underset{\sim}{\sigma})\underset{\sim}{n} - \underset{\sim}{\sigma}]$

and $(\underset{\sim}{I})_{ij} = \delta_{ij}$.

Now, unless the elastic moduli $\underset{\sim}{E}$ themselves allow localization $\underset{\sim}{n}\cdot\underset{\sim}{E}\cdot\underset{\sim}{n}$ (considered as a 3×3 matrix) has an inverse, which will be denoted by $(\underset{\sim}{n}\cdot\underset{\sim}{E}\cdot\underset{\sim}{n})^{-1}$. Indeed, if the incremental elastic response is isotropic, $\underset{\sim}{E}$ is such that

$$E_{ijk\ell}\varepsilon_{k\ell} = \Lambda\, \delta_{ij}\varepsilon_{kk} + 2G\, \varepsilon_{ij} ,$$

where Λ and G are the Lamé moduli, and

$$\underset{\sim}{n}\cdot\underset{\sim}{E}\cdot\underset{\sim}{n} = (\Lambda+G)\underset{\sim}{n}\underset{\sim}{n} + G\underset{\sim}{I} ,$$

$$(\underset{\sim}{n}\cdot\underset{\sim}{E}\cdot\underset{\sim}{n})^{-1} = -\frac{\Lambda+G}{G(\Lambda+2G)} \underset{\sim}{n}\underset{\sim}{n} + \frac{1}{G} \underset{\sim}{I} .$$

Thus (21) can be rewritten as

$$\underset{\sim}{g} = \frac{1}{\lambda} \underset{\sim}{a} (\underset{\sim}{b}\cdot\underset{\sim}{g}) + \underset{\sim}{B}\cdot\underset{\sim}{g} , \quad \text{where} \quad (22)$$

$$\underset{\sim}{a} = (\underset{\sim}{n}\cdot\underset{\sim}{E}\cdot\underset{\sim}{n})^{-1}\cdot(\underset{\sim}{n}\cdot\underset{\sim}{E}:\underset{\sim}{P}) , \quad \underset{\sim}{b} = \underset{\sim}{Q}:\underset{\sim}{E}\cdot\underset{\sim}{n} ,$$

$$\lambda = h + \underset{\sim}{Q}:\underset{\sim}{E}:\underset{\sim}{P} , \quad \text{and} \quad \underset{\sim}{B} = -(\underset{\sim}{n}\cdot\underset{\sim}{E}\cdot\underset{\sim}{n})^{-1}\cdot\underset{\sim}{A} .$$

Now, to find the value of h (or λ) which allows a non-trivial solution, let us first observe that $\underset{\sim}{B}$ contains terms which are all of the order of stress divided by an elastic modulus. Such terms are very much smaller than the other terms multiplying $\underset{\sim}{g}$ in (22), which are all of order unity at least when h is of the same order or smaller than elastic moduli. Thus we first consider the solution when we set $\underset{\sim}{B} = 0$; this is equivalent to neglecting rotational effects on the stress rate, i.e. to writing $\dot{\sigma}$ in place of $\overset{\nabla}{\sigma}$ in (20). In that case $(22)_1$ has, by inspection, a non-trivial solution with $\underset{\sim}{g} \propto \underset{\sim}{a}$ when $\lambda = \underset{\sim}{b}\cdot\underset{\sim}{a}$. Thus the critical value of h for localization on a plane of normal $\underset{\sim}{n}$ is, from $(22)_4$,

$$h = -\underset{\sim}{Q}:\underset{\sim}{E}:\underset{\sim}{P} + (\underset{\sim}{Q}:\underset{\sim}{E}\cdot\underset{\sim}{n})\cdot(\underset{\sim}{n}\cdot\underset{\sim}{E}\cdot\underset{\sim}{n})^{-1}\cdot(\underset{\sim}{n}\cdot\underset{\sim}{E}:\underset{\sim}{P}) , \quad (23)$$

and the corresponding mode of localization $\underset{\sim}{g}$ has a form such that

$$(\underset{\sim}{n}\cdot\underset{\sim}{E}\cdot\underset{\sim}{n})\cdot\underset{\sim}{g} \propto \underset{\sim}{n}\cdot\underset{\sim}{E}:\underset{\sim}{P} . \quad (24)$$

This last result has a simple interpretation. The term on the left is the incremental traction vector that would be created on the surface of normal $\underset{\sim}{n}$ if the material responded elastically to an incremental deformation in the form of the bifurcation, namely $2\underset{\sim}{D} = \underset{\sim}{n}\underset{\sim}{g} + \underset{\sim}{g}\underset{\sim}{n}$. That on the right is the vector created in the same way by a deformation increment in the form of the plastic field, namely $\underset{\sim}{D} \propto \underset{\sim}{P}$. The two vectors must be co-axial.

For an elastically isotropic material, the above results take the form

$$h/G = 4\, \underset{\sim}{n}\cdot\underset{\sim}{P}\cdot\underset{\sim}{Q}\cdot\underset{\sim}{n} - 2(\underset{\sim}{n}\cdot\underset{\sim}{P}\cdot\underset{\sim}{n})(\underset{\sim}{n}\cdot\underset{\sim}{Q}\cdot\underset{\sim}{n}) - 2\, \underset{\sim}{P}:\underset{\sim}{Q}$$

$$-[2\Lambda/(\Lambda+2G)][(\underset{\sim}{n}\cdot\underset{\sim}{P}\cdot\underset{\sim}{n})-\mathrm{tr}(\underset{\sim}{P})][(\underset{\sim}{n}\cdot\underset{\sim}{Q}\cdot\underset{\sim}{n})-\mathrm{tr}(\underset{\sim}{Q})]\ , \quad (25)$$

$$g \propto 2\, \underset{\sim}{n}\cdot\underset{\sim}{P} - \underset{\sim}{n}[2(\Lambda+G)(\underset{\sim}{n}\cdot\underset{\sim}{P}\cdot\underset{\sim}{n}) - \Lambda\, \mathrm{tr}(\underset{\sim}{P})]/(\Lambda+2G)\ .$$

These formulae are useful for the models developed in the next two sections. That for h has a simple form when written with α,β denoting components on cartesian axes in the plane of localization:

$$h/G = -2\, P_{\alpha\beta}Q_{\alpha\beta} - 2\Lambda P_{\alpha\alpha}Q_{\beta\beta}/(\Lambda+2G)\ .$$

From this it is clear that when normality holds (i.e., $P = Q$), the value of h at localization can never be positive, at least when the rotational effects in the stress rate are neglected.

To incorporate these effects, i.e. to use $\overset{\nabla}{\underset{\sim}{\sigma}}$ rather than $\dot{\underset{\sim}{\sigma}}$ in (20), we must return to the general case, represented concisely by (20), and solve the problem for $\underset{\sim}{B} \neq 0$. Remembering that $\underset{\sim}{B}$ has components that are small compared to unity in practical cases (order stress divided by elastic modulus), we may assume the inverse $(\underset{\sim}{I}-\underset{\sim}{B})^{-1}$ exists and, if so desired, represent it to any desired order in the series

$$(\underset{\sim}{I}-\underset{\sim}{B})^{-1} = \underset{\sim}{I} + \underset{\sim}{B} + \underset{\sim}{B}\cdot\underset{\sim}{B} + \underset{\sim}{B}\cdot\underset{\sim}{B}\cdot\underset{\sim}{B} + \dots\ .$$

In terms of it the solution for λ (and thus, from $(22)_4$, for h) is

$$\lambda = \underset{\sim}{b}\cdot(\underset{\sim}{I}-\underset{\sim}{B})^{-1}\cdot\underset{\sim}{a}\ . \qquad [\underset{\sim}{g} \propto (\underset{\sim}{I}-\underset{\sim}{B})^{-1}\cdot\underset{\sim}{a}]$$

3.3. Plastically dilatant material with pressure-sensitive yielding

Rudnicki and Rice [22] derived conditions for localization in a material which was assumed to exhibit an isotropic elastic response and to have, in the present notation for plastic response,

$$P_{ij} = \sigma'_{ij}/2\tau + (\beta/3)\,\delta_{ij}\ ,$$
$$Q_{ij} = \sigma'_{ij}/2\tau + (\mu/3)\,\delta_{ij}\ , \quad (26)$$

where $2\tau^2 \equiv \underset{\sim}{\sigma}':\underset{\sim}{\sigma}'$ and $\underset{\sim}{\sigma}'$ is the deviatoric part of $\underset{\sim}{\sigma}$. If $\mu = \beta = 0$, this coincides with the classical Prandtl-Reuss material. If $\mu \neq 0$, the model can also represent a material for which there is a "frictional" effect in yielding: μ is the rate of increase with pressure of the "equivalent shear stress" τ required for yield. This kind of effect is pres- in fissured rock masses and in granular materials under compression. It seems also to account for the "stress-differential effect" [34], by which the yield stress levels of certain high-strength steels are higher in compression than in tension. The parameter β gives the ratio of plastic dilation to plastic shear. If $\beta = \mu$, normality applies. But representative values for geological materials [22] suggest that β is usually less than μ (representative values for fissured rocks are $\beta = .1$ to $.5$, $\mu = .3$ to 1.0). Also, measurement of plastic volume change in the strength-differential experiments [34] suggests, as expected, that β is negligible by comparison to μ and can be taken as zero. Hence, in neither of the cases cited does normality apply precisely. The parameter h is the rate of plastic hardening observed in a pure shear test, e.g. $2h\, D^P_{12} = \dot{\tau}$ for an element subjected to a state of hydrostatic stress plus a pure shear stress $\sigma_{12} = \sigma_{12} = \tau$, with $\dot{\sigma}_{kk} = 0$.

The present model has an interpretation also for void-containing plastic materials undergoing ductile rupture. The porosity increase through void growth requires $\beta > 0$ and , as Berg [15] remarks, if the material of the solid matrix is modelled locally by continuum plasticity of a kind for which the normality rule holds, then the rule applies to the aggregate also and thus $\mu=\beta$. Gurson [35] gives specific forms for β and μ based on a rigid-plastic model of a voided material, and remarks that the inclusion in the constitutive relations of a hydrostatic stress dependent criterion for void <u>nucleation</u> (say, by the brittle cracking or decohering of inclusions) leads to deviations from normality with $\mu > \beta$.

Let us first analyze the material described by (26) according to the <u>rigid-plastic</u> model. In that case localization <u>is possible only</u> if the intermediate principal value of $\underset{\sim}{P}$ vanishes, i.e. if

$$\sigma'_{II} = -(2\beta/3)\,\tau\ ,$$

and then localization is possible no matter how large or small the value of h if $\beta \neq \mu$, whereas localization is possible only when $h = 0$ if $\beta = \mu$. Rudnicki and Rice [22] obtained directly the result corresponding to (25) for the <u>elastic-plastic</u> model and, searching out the plane which first allows localization (that corresponding to the maximum h over all orientations $\underset{\sim}{n}$), they showed that provided μ and β are not too large (see p. 384 of [22]), the critical value is

$$h_{crit}/G = \frac{1+\nu}{9(1-\nu)}(\mu-\beta)^2 - \frac{1+\nu}{2}\left(\frac{\sigma'_{II}}{\tau} + \frac{\mu+\beta}{3}\right)^2 + O\left(\frac{\tau}{G}\right)\ . \quad (27)$$

Here ν is the elastic Poisson ratio.

The result is interesting in several respects. First note that to neglect of terms of $O(\tau/G)$, arising from the distinction between $\overset{\nabla}{\underset{\sim}{\sigma}}$ and $\dot{\underset{\sim}{\sigma}}$, localization can never occur with positive h if normality applies (i.e., if $\mu = \beta$). On the other hand, if normality does not apply, it is possible for localization to occur with positive h, depending on the value of σ'_{II}/τ. Indeed, replacing this quantity in favor of P_{II} (recall that according to the rigid-plastic model, localization can occur only when $P_{II} = 0$), the critical condition is

$$h_{crit}/G = \frac{1+\nu}{9(1-\nu)}(\mu-\beta)^2 - \frac{1+\nu}{2}(2P_{II} + \frac{\mu-\beta}{3})^2 . \quad (28)$$

Hence, when $P_{II} = 0$, localization can occur with the positive hardening rate

$$h_{crit}/G = \frac{(1+\nu)^2}{18(1-\nu)}(\mu-\beta)^2 ,$$

but in contrast to the rigid-plastic case, localization can also occur for other values of P_{II} (the most critical case is $P_{II} = -(\mu-\beta)/6$). If the intermediate plastic strain rate departs too much from zero, the critical h value turns negative. This is consistent with what appears to be a greater tendency for localization in plane strain than under axially symmetric conditions, tensile ductility being less in the former case (Clausing [36]).

Several numerical tabulations of these results are given in [22], in which it is also pointed out that vertex-like yield effects have a strong influence on the predicted conditions for localization (see section 3.5 to follow).

3.4. Localization in a single crystal

Consider a ductile single crystal undergoing plastic flow by slip on a single system of planes having normal in the x_2 direction and slip direction in the x_1 direction. For brevity, we analyze the problem by neglecting the distinction between $\overset{\nabla}{\sigma}$ and $\dot{\sigma}$ and otherwise neglecting terms of order stress divided by elastic modulus; a full analysis is to be presented in the near future by the author in collaboration with R. J. Asaro. If the Schmid law of resolved shear stress governing slip is followed, the plastic response is given by

$$D^P_{12} = \dot{\sigma}_{12}/2h ,$$

where h is the hardening rate. This law corresponds to $\underset{\sim}{P} = \underset{\sim}{Q}$, hence normality, with $P_{12} = P_{21} = 1/2$ and all other $P_{ij} = 0$. A non-deforming surface always exists in this case (namely the x_1, x_3 plane) so that when studied as a rigid-plastic problem, the critical condition is $h = 0$. Sometimes this fits the experimental facts at onset of coarsened slip (Jackson and Basinski [8]). However, it is interesting for other cases to study sources of deviations from Schmid's law, because the corresponding non-normality can be destabilizing.

One of the most promising candidates is cross slip. When screw segments of dislocations surmount local obstacles by this mode of slip the incremental plastic deformation should depend not only on the increment of σ_{12}, but also of the stress resolved onto the cross-slip plane and of that which serves to coalesce dislocation stacking faults, so as to make the local change of slip plane possible. For effects of numerically comparable size, that due to the resolved stress on the cross-slip plane is by far the most important for localization (it corresponds to non-zero $Q_{\alpha\beta}$ in (17)), and hence for simplicity we rewrite the plastic constitutive relation to include only this effect:

$$D^P_{12} = (1/2h)(\dot{\sigma}_{12} + \mu\dot{\sigma}_{13}) \quad (29)$$

For this relation, $\underset{\sim}{P}$ is as given above, but $Q_{12} = Q_{21} = 1/2$, $Q_{13} = Q_{31} = \mu/2$ and other $Q_{ij} = 0$.

Following the general solution (25) for h at localization on a plane of normal $\underset{\sim}{n}$, these values of $\underset{\sim}{P}$ and $\underset{\sim}{Q}$ lead to

$$\begin{aligned} h/G &= -1 + n_1^2 + n_2^2 + \mu n_2 n_3 - \chi n_1^2 n_2 (n_2 + \mu n_3) \\ &= -\sin^2\phi \sin^2\theta + \mu \sin\phi \cos\phi \sin\theta \\ &\quad - \chi \sin^2\phi \cos^2\theta \cos\phi (\cos\phi + \mu \sin\phi \sin\theta) \end{aligned}$$

where $\chi = 4(\Lambda+G)/(\Lambda+2G)$ and where, in the last form, ϕ is the angle between $\underset{\sim}{n}$ and the x_2 axis and θ is the angle made with the x_1 axis by the projection of $\underset{\sim}{n}$ onto the x_1, x_3 plane; θ increases by $\pi/2$ in a rotation from the x_1 to x_3 direction. If μ is small as expected, so also will be the deviation of ϕ for the most critical plane from zero. Thus, expanding the result to quadratic order in ϕ

$$h/G \approx \phi \mu \sin\theta - \phi^2(\sin^2\theta + \chi \cos^2\theta) .$$

This is easily shown to take on its maximum value when $\theta = \pi/2$ and $\phi = \mu/2$, corresponding to

$$h_{crit} \approx \mu^2 G/4 \quad (30)$$

This shows again the destabilizing influence of deviations from normality. A value of μ on the order of 1/10 would give localization according to (30) at what would have to be judged as a significantly strain-hardening state.

As remarked earlier, it remains an open question for many ductile metals as to whether the present "coarse slip" type of localization serves to concentrate strain and lead to the nucleation and growth to coalescence of voids in fracture, or whether it is instead the incipient nucleation and growth of voids which leads to a localization, as described in the previous section.

3.5. Yield vertex effects

When the plastic portion of the pre-localized deformation rate field contains no non-deforming plane, the criterion for localization is controlled by those constitutive parameters which mark the stiffness of response to abrupt, though perhaps small, changes in the "direction" of the deformation rate. These changes correspond to the superposition of the localized shear band mode on the given homogeneous field. For "smooth yield surface and plastic potential" models, the relevant response moduli are of the same order as elastic moduli, since superposed deformation increments that are orthogonal to the plastic flow direction $\underset{\sim}{P}$ induce only an elastic response. This feature is at root of the strongly negative values predicted for h in (28) when $\underset{\sim}{P}$ departs greatly from conditions for existence of a non-deforming plane.

However, physical models of rate-insensitive plastic flow based on microstructural slip, either of a Schmid (metal plasticity) or frictional (geologic materials) type, lead universally to the prediction of vertex development at the current stress point on "zero-offset" yield and plastic potential surfaces [22,24,26,27]. Such models entail a considerably softer elastic-plastic response to small superposed deformation increments, orthogonal to the prevailing plastic flow direction (see, e.g., fig. 2 of Hutchinson [27] for crystalline slip), and this has a strong, generally destabilizing effect on predicted conditions for localization.

The matter has been studied at length by Rudnicki and Rice [22], who add vertex effects to the constitutive model of section 3.3 and demonstrate both separate and combined destabilizing effects that arise from vertex yielding and non-normality. Also, Støren and Rice [29] show that a vertex yield model leads to predictions of localized necking in thin sheets, under positive in-plane principal extensions, at conditions that compare favorably with experimental results whereas, in the same circumstances, the smooth-yield-surface rigid plastic model (without imperfections [28]) predicts unlimited ductility.

For brevity we consider here only the non-dilatant, pressure-insensitive version of the vertex constitutive relation studied in [22], first specializing it as in [29] to a <u>rigid-plastic</u> model. The relation is

$$2\,\underset{\sim}{D} = \frac{1}{h}\,\underset{\sim}{\sigma}'\frac{\dot{\tau}}{\tau} + \frac{1}{h_1}\left(\overset{\nabla}{\underset{\sim}{\sigma}}{}' - \underset{\sim}{\sigma}'\frac{\dot{\tau}}{\tau}\right) \qquad (31)$$

where $2\tau^2 = \underset{\sim}{\sigma}':\underset{\sim}{\sigma}'$ and $\underset{\sim}{\sigma}'$ is the deviatoric stress. The relation is intended to model response on deformation paths that differ only modestly from fixed-principal-axis deformation with $\overset{\nabla}{\underset{\sim}{\sigma}}{}' \propto \underset{\sim}{\sigma}'$. When deformations comply precisely with this, the last term of (31) vanishes and we are left only with the first, corresponding to classical Mises rigid-plastic response at a hardening modulus h (in shear), with $\underset{\sim}{P} = \underset{\sim}{Q} = \underset{\sim}{\sigma}'/2\tau$. The second term of (31) represents the vertex yield effect, and h_1 is the modulus of vertex response (defined analogously to an elastic shear modulus) for small superposed rates $\overset{\nabla}{\underset{\sim}{\sigma}}{}'$ that are not coaxial with $\underset{\sim}{\sigma}'$. To the extent that approximately path-independent relations between suitably defined stress and deformation measures [29] result for such only moderately non-proportional deformation paths, the "deformation theory" of plasticity applies and h_1 can be identified as the secant modulus to the stress-strain relation in shear (see [27] for a comparison with response moduli of "incremental" models). In any event, one may assume that $h_1 > h$.

By applying the theory of section 3.1 to the relation (31), we see that in the absence of a vertex effect ($h_1 = \infty$) localization can occur only if $\sigma'_{II} = 0$ and then, since (31) as written involves plastic normality, only when the tangent modulus $h = 0$. To see how the vertex effect modifies these results, first invert (31) to

$$\dot{\underset{\sim}{\sigma}} = \dot{\sigma}\underset{\sim}{I} + \underset{\sim}{\Omega}\cdot\underset{\sim}{\sigma} - \underset{\sim}{\sigma}\cdot\underset{\sim}{\Omega} + 2h_1\underset{\sim}{D} - (h_1-h)\underset{\sim}{\sigma}'(\underset{\sim}{\sigma}':\underset{\sim}{D})/\tau^2 \qquad (32)$$

where $\sigma = tr(\underset{\sim}{\sigma})/3$. Now, since the material is incompressible the bifurcation vector $\underset{\sim}{g}$ of (1) must take the form $g\underset{\sim}{m}$ where $\underset{\sim}{m}$ is a unit vector perpendicular to $\underset{\sim}{n}$, i.e. lying in the plane of localization. We take the reference state as coincident instantaneously with the current state, operate with Δ in (32), and dot with $\underset{\sim}{n}$ to obtain

$$0 = \underset{\sim}{n}\cdot\Delta\dot{\underset{\sim}{\sigma}} = \underset{\sim}{n}\Delta\dot{\sigma} + g\{-\tfrac{1}{2}\underset{\sim}{m}\cdot\underset{\sim}{\sigma} - \tfrac{1}{2}(\underset{\sim}{n}\cdot\underset{\sim}{\sigma}\cdot\underset{\sim}{m})\underset{\sim}{n} + \tfrac{1}{2}(\underset{\sim}{n}\cdot\underset{\sim}{\sigma}\cdot\underset{\sim}{n})\underset{\sim}{m}$$
$$+ h_1\underset{\sim}{m} - (h_1-h)(\underset{\sim}{n}\cdot\underset{\sim}{\sigma}')(\underset{\sim}{n}\cdot\underset{\sim}{\sigma}'\cdot\underset{\sim}{m})/\tau^2\}$$

By dotting this equation successively with unit vectors $\underset{\sim}{m}$ and $\underset{\sim}{z}$, the latter being perpendicular to $\underset{\sim}{n}$ and $\underset{\sim}{m}$ and hence also in the plane of localization, we obtain two simultaneous conditions for the existence of a non-zero bifurcation amplitude g. These are

$$(\sigma_{nn}-\sigma_{mm})/2 + h_1 - (h_1-h)\sigma_{mn}^2/\tau^2 = 0 \qquad (33)$$

$$\sigma_{mz}/2 + (h_1-h)\sigma_{nz}\sigma_{mn}/\tau^2 = 0 \qquad (34)$$

where there is no summation implied and indices denote components of $\underset{\sim}{\sigma}$ on the axes $\underset{\sim}{n},\underset{\sim}{m},\underset{\sim}{z}$.

We may view (33) as an equation giving the critical value of h for a given h_1, $\underset{\sim}{\sigma}$, and $\underset{\sim}{n}$, and view (34) as a constraint on the corresponding direction $\underset{\sim}{m}$. The most critical plane $\underset{\sim}{n}$, for a given h_1 and $\underset{\sim}{\sigma}$, is that which maximizes the corresponding h value since, for applications, h is generally a non-increasing function of the amount of deformation imposed on the material.

By setting the variation $\delta h = 0$ for $\delta\underset{\sim}{n}$ in the direction of $\underset{\sim}{z}$, one finds that (34) as well as the equation

$$\sigma_{nz}/2 - (h_1-h)\sigma_{mz}\sigma_{mn}/\tau^2 = 0$$

must be satisfied simultaneously. This can be done <u>only</u> if the $\underset{\sim}{z}$ direction is <u>principal</u>, $\sigma_{nz} = \sigma_{mz} = 0$. That is, $\underset{\sim}{n}$ and $\underset{\sim}{m}$ must lie in a plane formed by two of the principal directions. Next, setting $\delta h = 0$ when $\delta\underset{\sim}{n}$ is in the direction of $\underset{\sim}{m}$, and recognizing that the associated $\delta\underset{\sim}{m}$ has direction $-\underset{\sim}{n}$, one obtains

$$2\sigma_{mn} + 2(h_1-h)\sigma_{mn}(\sigma_{nn}-\sigma_{mm})/\tau^2 = 0 \qquad (35)$$

Assuming that the critical plane does not correspond to one of vanishing shear stress the critical orientation is that for which

$$(\sigma_{mm}-\sigma_{nn})/\tau = \tau/(h_1-h) , \tag{36}$$

and it can be shown that a plane exists meeting this condition if

$$\tau \leq 2(h_1-h)(1-3\sigma'^2_{zz}/4\tau^2)^{1/2} \tag{37}$$

Thus, observing that

$$\tau^2 \equiv \underset{\sim}{\sigma}':\underset{\sim}{\sigma}'/2 = (\sigma_{mm}-\sigma_{nn})^2/4 + \sigma^2_{mn} + 3\sigma'^2_{zz}/4 ,$$

and using (36), one may substitute into (33) to obtain the critical condition

$$h/(h_1-h) + 3\sigma'^2_{zz}/4\tau^2 - \tau^2/4(h_1-h)^2 = 0 . \tag{38}$$

Hence, to summarize, the critical condition is given by (38) where $\underset{\sim}{z}$ is one of the principal directions, provided that the associated critical condition allows the inequality (37) to be satisfied. It is easy to show that $\underset{\sim}{z}$ must be chosen as the intermediate principal direction (denoted by II) to give the maximum of possible solutions for h, and it is convenient at this point to introduce the dimensionless stress state parameter

$$u = 3\sigma'^2_{II}/4\tau^2 , \tag{39}$$

noting that $u = 0$ for pure shear and that u takes on its maximum value, $1/4$, for axi-symmetric extension or compression.

The condition (38) can then be written

$$\tau^2 = 4u(h_1-h)^2 + 4h(h_1-h) , \tag{40}$$

and this is the critical condition provided the inequality (37) is met. That inequality can now be rearranged to the requirement that the rate of hardening satisfies

$$h \leq (1-2u)h_1/2(1-u) \quad (\leq h_1/3) , \tag{41}$$

which does not seem restrictive in terms of the physical interpretation of h_1. Equation (40) reduces to that given by Hill and Hutchinson [31] for states of plane strain, $u = 0$, and this state allows localization at a smaller equivalent stress τ than does any other.

If τ/h_1 is regarded as a small parameter, (38) may be solved for the critical hardening modulus at localization and to the order of the terms retained the result is

$$h = -uh_1/(1-u) + \tau^2/4h_1 + \ldots \tag{42}$$

Thus for small τ/h_1 unless u is close to zero (states approaching plane strain conditions) the localization condition requires strain softening behavior, $h < 0$.

For the corresponding elastic-plastic material with a yield vertex, but treated as elastically impressible as well, one need only add a term $\overset{\nabla}{\underset{\sim}{\sigma}}/G$ to the right side of (31), where G is the shear modulus, and then the same analysis as given here applies provided that the replacements

$$h \rightarrow hG/(G+h) , \quad h_1 \rightarrow h_1G/(G+h_1)$$

are made in all formulae. In particular, when the critical condition analogous to (38) is solved for h, and the result expanded to the order of terms as in (42), one obtains

$$h_{crit} = \frac{h_1G}{h_1+(1-u)G}\left\{-u + \frac{(1-u+u^2)h_1+(1-u)G}{h_1+(1-u)G}\,\frac{(h_1+G)^2}{4\,h_1^2}\,\frac{\tau^2}{G^2} + \ldots\right\}$$

Retention of only the first term in the brackets, namely $-u$, corresponds to writing $\dot{\underset{\sim}{\sigma}}$ for $\overset{\nabla}{\underset{\sim}{\sigma}}$ in the constitutive relation and then the result may be compared with (27), setting $\mu = \beta = 0$ and $\nu = 1/2$ in the latter. Evidently, when the vertex modulus h_1 is much less than the elastic shear modulus, the critical "hardening" modulus for localization at states other than plane strain is considerably less negative than would be the case in absence of a vertex $(h_1 = \infty)$. Numerical results and comparisons are given by Rudnicki and Rice [22].

For example, in axi-symmetric extension or compression, $u = 1/4$ and the result is

$$h_{crit} = -\frac{h_1G}{4h_1+3G}$$

when both τ/G and τ/h_1 are small compared to unity. With h_1/G in the range .1 to .01, which might be regarded as representative for heavily deformed metals based on a secant modulus interpretation, this formula gives critical h/G values of $-.03$ to $-.003$ (compared to $-.25$ when vertex effects are neglected). Strain softening effects of this magnitude might well result in the necked regions of some ductile metal tensile specimens prior to fracture [16,17], with the softening resulting from progressive cavitation at inclusions. Of course, imperfection effects could also be very important in such cases, as could the sources of non-normality discussed earlier.

4. CONCLUSION

The localization of plastic flow is a fascinating and widely observed phenomenon, which seems important in setting a limit to the achievable ductility of a solid. Yet the topic has remained outside the mainstream of work on the mechanics of inelastic solids, save for the elucidation of general principles in the spirit of Hadamard by Thomas, Hill, and Mandel. The present study shows that conditions for localization relate closely to subtle and not well understood features of the constitutive description of plastic flow. Localization is favored by a low plastic hardening modulus, but the matters of how low and whether strain softening is required are determined by the nature of the pre-localized deformation, those states with non-deforming planes being highly susceptible, and by deviations from plastic normality. The latter may, for example, arise from Coulomb frictional effects in yielding or from non-Schmid effects in crystals as by cross-slip or other triggered processes, where stresses other than the resolved shear stress contribute to flow on a given slip system. Vertex yielding effects are predicted on physical grounds and these too have a strong influence on localization conditions, for example, in mitigating predictions of strongly negative hardening for localization in axisymmetrically deformed solids and like cases.

While the constitutive modelling of these features needs to be improved in relation to the detailed mechanisms of deformation, so also is there need for a fuller assessment of the role of imperfections or initial non-uniformities in material properties in promoting localization. Indeed, the latter approach seems mandatory for rate-dependent plastic flow models and these, as well as the range of thermomechanically coupled localization phenomena would seem to merit further study.

The basic theory of uniqueness in relation to localizations and stationary waves is also not yet adequately developed for materials deviating from normality, and neither can the case of vertex yielding be handled in full generality within the existing framework. Yet both are features which seem inherent to much of plastic constitutive behavior and the examples and analysis of the present study suggest that both are important destabilizing features for the process of localization.

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