VOL. 14, NO. 2 REVIEWS OF GEOPHYSICS AND SPACE PHYSICS MAY 1976

Some Basic Stress Diffusion Solutions for Fluid-Saturated Elastic Porous Media With Compressible Constituents

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This is a study of the formulation, some basic solutions, and applications of the Biot linearized quasi-static elasticity theory of fluid-infiltrated porous materials. Whereas most previously solved problems are based on idealizing the fluid and solid constituents as separately incompressible, full account is taken here of constituent compressibility. Previous studies are reviewed and the Biot constitutive equations relating strain and fluid mass content to stress and pore pressure are recast in terms of new material parameters, more directly open to physical interpretation as the Poisson ratio and induced pore pressure coefficient in undrained deformation. Different formulations of the coupled deformation/diffusion field equations and their analogues in coupled thermoelasticity are discussed, and a new formulation with stress and pore pressure as basic variables is presented that leads, for plane problems, to a convenient complex variable representation of solutions. The problems solved include those of the suddenly introduced edge dislocation and concentrated line force and of the suddenly pressurized cylindrical and spherical cavity. The dislocation solution is employed to represent that for quasi-static motions along a shear fault, and a discussion is given, based on fracture mechanics models for fault propagation, of phenomena involving coupled behavior between the rupturing solid and its pore fluid, which could serve to stabilize a fault against rapid spreading. Also, the solution for a pressurized cylindrical cavity leads to a time-dependent stress field near the cavity wall, and its relevance to time effects in the inception of hydraulic fractures from boreholes, or from drilled holes in laboratory specimens, is discussed. Various limiting cases are identified, and numerical values of the controlling porous media elastic parameters are given for several rocks.

INTRODUCTION

The stress-induced flow of interstitial fluid in porous solids has been suggested as accounting for a variety of phenomena observed in geophysical studies and engineering practice. In the years since *Terzaghi* [1923, 1936] proposed his 'effective stress' theory to rationalize observations of time-dependent consolidation and failure in soil masses, some significant progress has been made in developing reasonable constitutive and field equations for porous media [e.g., *Biot*, 1941, 1973; *Süklje*, 1969; *Nikolaevskii et al.*, 1970] and in applying these to explain observed behavior in geological materials.

Indeed, the linear formulation of Biot for stress diffusion fields is precisely analogous [*Biot*, 1956*b*] to the well-established theory of linear coupled thermoelasticity [e.g., *Carlson*, 1972], so an abundance of available solutions might be expected. Unfortunately, the neglect of coupling terms that is so commonly and justifiably used to simplify the equations in thermoelasticity [e.g., *Boley and Weiner*, 1960; *Boley*, 1974] is certainly not appropriate for fluid-infiltrated solids. Thus the majority of thermal stress solutions fail to apply to the seemingly analogous porous media problems. Except for some simple cases [e.g., *Boley and Tolins*, 1962] and some other very general fundamental solutions [e.g., *Nowacki*, 1964], in which coupling is rigorously retained in the equation governing the temperature distribution, resort must usually be had to one of a small number of formal procedures [e.g., *Biot*, 1956*a*, *b*; *McNamee and Gibson*, 1960] in order to solve the equations governing the response to stress of porous solids. Thus very few worked solutions of significant problems are actually available, perhaps in part because the algebraic details become rather complicated when work is done with those existing formalisms. One example is the recent paper by *Booker* [1974], which uses the formulation of *McNamee and Gibson* [1960] to obtain some features of the stress field due to a dipole pair of straight edge dislocations representing slip in an infinite body of porous material with incompressible constituents.

We show here, however, that the governing equations are not significantly more complicated when arbitrary compressibilities are assigned to the fluid and solid constituents. In fact, then, the same formulation could have been used to obtain the more general dislocation solution with such full compressibility. But we also develop and utilize here a complex potential representation for plane problems, which we have found simpler than that of *McNamee and Gibson* [1960]; effectively this is an extension of the *Muskhelishvili* [1953] formalism, in classical elasticity, to stress diffusion problems. It allows us to easily solve the dislocation and other basic problems, which have direct application to shear faulting, hydraulic fracturing, mining, and other types of disturbance in porous media.

Before doing that, we set out, using readily identifiable parameters, isotropic constitutive equations like those of *Biot and Willis* [1957]. We also summarize the governing three-dimensional equations in their simplest form in order to clarify their exact structure and their thermoelastic counterparts. We wish further to emphasize that our presentation of the equations with stress components and pore pressure as basic variables may frequently prove advantageous to the Navier (displacement type) formulation given by *Biot* [1956*a*].

LINEARIZED CONSTITUTIVE RELATIONS FOR A FLUID-SATURATED POROUS ELASTIC SOLID

The framework for a general theory of mixtures of interacting continua [*Truesdell and Toupin*, 1960; *Bowen*, 1971] has provided a popular approach to recent studies of the rheology of porous solids [e.g., *Tabaddor and Little*, 1971; *Garg*, 1971; *Morland*, 1972]. But there is no improvement to be had on the

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classical *Biot* [1941, 1955, 1956*a*, 1973] formulation in the present circumstances of quasi-static elastic deformation, certainly under conditions for which local equilibrium of the pore fluid can be assumed. It must be emphasized that Biot's formalism does not prohibit the continuum viewpoint nor imply images of an array of grains, as frequently used to derive elastic properties of the solid matrix, or even of a solid skeleton so constructed as to give a partial fluid stress proportional to porosity [e.g., *Biot*, 1956*a*; *Nikolaevskii et al.*, 1970]. The appropriate decomposition into effective stresses on the solid matrix and pressure on the fluid can be allowed to occur naturally without postulating how they make up the total stress on a material element.

The simplest rigorous approach is to define total stresses σ_{ij} and pore pressure p as basic state variables and assume that these are related in some appropriate fashion to the strains of the solid (components ϵ_{ij}, derivable from a solid displacement vector) and the mass m of pore fluid per unit reference volume. (Here i, j = 1, 2, 3; also, the summation convention applies when repeated indices are used subsequently.) The pore pressure p is most fundamentally defined as the equilibrium pressure that must be exerted on a homogeneous reservoir of pore fluid, brought into contact with a material element, so as to prevent any fluid exchange between it and the element. It is most convenient, in fact, to represent the fluid mass content of an element in terms of the apparent fluid volume fraction v, where $\rho v = m$ and ρ is the fluid density in the imagined fluid reservoir at pressure equilibrium with the element.

The linear and isotropic expression for ϵ_{ij} necessarily has the form [*Biot*, 1941] for isothermal conditions,

$$2G\epsilon_{ij} = (\sigma_{ij} + p\delta_{ij}) - \frac{\nu}{(1+\nu)}(\sigma_{kk} + 3p)\delta_{ij} + \frac{2G}{3}\left(\frac{1}{H} - \frac{1}{K}\right)p\delta_{ij} \qquad (1)$$

where G and ν are the shear modulus and Poisson ratio when the material is deformed under 'drained' conditions (i.e., p constant) and the 'drained bulk modulus' is $K = 2G(1+\nu)/3(1-2\nu)$. The constant H is that of Biot, but we have regrouped his equation in order to isolate a new material constant K_s', given by $1/K - 1/H \equiv 1/K_s'$, which can in appropriate circumstances be identified as the bulk modulus K_s of the solid phase [e.g., *Nur and Byerlee*, 1971, and discussion to follow]. The arguments used by *Biot* [1941, 1973], involving the fact that $\sigma_{ij}d\epsilon_{ij} + pdv$, and thence $\epsilon_{ij}d\sigma_{ij} + vdp$, is an exact differential, may be applied to deduce, from (1), the relation for apparent volume fraction

$$v - v_0 = \frac{1}{3H}(\sigma_{kk} + 3p) - \frac{v_0}{K_s''}p \qquad (2)$$

where v_0 is its reference value in the unstressed state. Here the grouping of terms is chosen to facilitate recognition of the modulus K_s'', which can also often be identified [e.g., *Cornet and Fairhurst*, 1974] with K_s. *Biot* [1941] simply employed a constant R, related to our K_s'' by $1/R \equiv 1/H - v_0/K_s''$; physical significance was later attached to H and R by *Biot and Willis* [1957].

Interpretation of K_s', K_s''. In the special case where all void space of any elemental volume is continuous and allows free fluid filtration, for which all points of the solid phase may be taken as elastically isotropic with the same local bulk modulus K_s, and where both fluid and solid are chemically inert, the moduli K_s' and K_s'' are indeed sensibly associated with K_s. That this is so is recognized by considering an interior uniform pore pressure change Δp and simultaneous change $\Delta\sigma_{ij} = -\Delta p\delta_{ij}$ of total stresses on the faces of an element. The observation of *Nur and Byerlee* [1971] is precisely that these macroscopic changes in variables produce a local stress alteration of $-\Delta p\delta_{ij}$ at each point of the solid phase: every linear dimension of the solid phase is thus reduced by a fraction $\Delta p/3K_s$, and thus the unique deformation pattern is a fractional reduction $\Delta p/K_s$ in every volume (including the volume of fluid-filled interstices). This observation may be stated as

$$\Delta\epsilon_{ij} = -\delta_{ij}\Delta p/3K_s \qquad \Delta v = -v_0\Delta p/K_s$$

which can be compared with (1) and (2) to yield $K_s' = K_s'' = K_s$.

More generally, however, K_s' and K_s'' must be regarded as experimental constants additional to G and ν and analogous to H and R. Conceptual and realistic appropriate tests are also described by *Biot and Willis* [1957]. Typically, K_s' and K_s'' will have the same order of magnitude as a representative bulk modulus for the 'non-fluid-infiltrated' (as opposed to 'solid') phase [e.g., *Cornet and Fairhurst*, 1974].

Undrained elastic behavior. The mass $m = \rho v$ of pore fluid per unit volume of material may be expressed from (2) in a linearized expansion to give

$$m - m_0 = (\rho - \rho_0)v_0 + \rho_0(v - v_0) = \rho_0\frac{v_0}{K_f}p + \frac{\rho_0}{3}\left(\frac{1}{K} - \frac{1}{K_s'}\right)(\sigma_{kk} + 3p) - \rho_0\frac{v_0}{K_s''}p \qquad (3)$$

where m_0 and ρ_0 obtain in the reference state and the bulk modulus of the fluid is $K_f \equiv \rho_0 p/(\rho - \rho_0)$.

By 'undrained deformation' we mean the imposition of stress alterations $\Delta\sigma_{ij}$ over a time scale that is too short to allow the loss or gain of pore fluid in an element by diffusive transport to or from neighboring elements, i.e., $\Delta m = 0$. Still, it is assumed for our present considerations that the time scale is long enough that local pressure equilibrium is attained within the various communicating pores constituting a 'point' in the continuum model of the material. This kind of local equilibrium cannot always be attained, and our present meaning of 'undrained conditions' may be contrasted with that for an even shorter time scale, as in the work of *O'Connell and Budiansky* [1974]. Indeed, they determine approximately overall elastic moduli for saturated rock under conditions for which there is no fluid loss or gain to any individual pore and hence no degree of pressure equilibrium between neighboring pores, no matter how close. The O'Connell-Budiansky short-time undrained moduli should govern the response to truly instantaneous impositions of stress, but these moduli can usually be expected to relax to the undrained moduli of the Biot theory, as based on the assumption of local pressure equilibrium, over a time that is quite short by comparison with that needed for induced D'Arcy flows to achieve global pressure equilibrium over the entire deformed region. Thus while there is indeed the need of a theory that is broad enough to contend with cases of local pressure nonequilibrium, we shall here understand 'undrained deformation' and 'instantaneous response' in the context of local equilibrium only.

Our undrained response may be written as $\Delta m = 0$, and (3) then gives a relation like that of *Skempton* [1954] between initial induced pore pressure and total hydrostatic stress on an element

$$\Delta p = -B\frac{\Delta\sigma_{kk}}{3}$$
$$B = \frac{1/K - 1/K_s'}{v_0/K_f + 1/K - 1/K_s' - v_0/K_s''} \tag{4}$$

As noted by Skempton, B would typically be unity for water-saturated soils ($K_s''/v_0 > K_s' > K_f >> K$) but can be substantially less for rocks, constituents of which are not effectively incompressible.

An expression for the 'undrained Poisson ratio' ν_u may be obtained by substituting from (4), for Δp, into (1) and comparing resulting coefficients with the definition of instantaneous elastic response

$$2G\Delta\epsilon_{ij} \equiv \Delta\sigma_{ij} - \frac{\nu_u}{1+\nu_u}\Delta\sigma_{kk}\delta_{ij} \tag{5}$$

The conclusion is that

$$\nu_u = \frac{3\nu + B(1-2\nu)(1-K/K_s')}{3 - B(1-2\nu)(1-K/K_s')} \tag{6}$$

The practical range of ν_u is obviously $\frac{1}{2} \geq \nu_u \geq \nu$; the upper limit is reached for separately incompressible constituents ($B = 1$, $K/K_s' = 0$), and the lower bound is achieved when pore fluid is highly compressible, $K_f << v_0K$ (then $B \simeq 0$).

Subsequently, we shall use B and ν_u instead of K_s', K_s'' (or H and R of Biot), and v_0/K_f, since these are open to such simple physical interpretations. Indeed, we may either calculate B and ν_u in terms of the other parameters or simply take them directly from the result of a single undrained test in which the Poisson effect and induced pore pressure are measured. In terms of them, (1) and (3) can be shown to take the forms

$$2G\epsilon_{ij} = \sigma_{ij} - \frac{\nu}{1+\nu}\sigma_{kk}\delta_{ij} + \frac{3(\nu_u-\nu)}{B(1+\nu)(1+\nu_u)}p\delta_{ij} \tag{7}$$

$$m - m_0 = \frac{3\rho_0(\nu_u-\nu)}{2GB(1+\nu)(1+\nu_u)}\left[\sigma_{kk} + \frac{3}{B}p\right] \tag{8}$$

which contain only four elastic constants: G, ν, B, and ν_u. This form is sensible in light of the complete similarity of the porous medium constitutive equations (7) and (8) to those for a linear isotropic thermoelastic solid [e.g., *Boley and Weiner*, 1960]. The correspondence is recognized simply by identifying pore pressure p with a multiple of temperature fluctuation and fluid mass m with some multiple of specific entropy per unit reference volume [*Biot*, 1956*b*]. Thus, the analogue of our undrained response is the isentropic deformation of a thermoelastic solid while ongoing dissipation of pore pressures matches its approach to isothermal equilibrium conditions.

The analogy with linear thermoelasticity is completed by the constitutive law governing pore fluid diffusion, namely that of D'Arcy, given here for the isotropic case

$$q_i = -\rho_0\kappa\, \partial p/\partial x_i \tag{9}$$

relating the fluid mass flow rate in the x_i direction, q_i per unit area, to gradient of pore pressure. Equation (9) is written for the absence of dynamic or other perturbations in the form of a body force field (then ρf_i would be subtracted from $\partial p/\partial x_i$, where f_i is the force per unit mass of fluid). The permeability κ is usually given as units of area k, where $\kappa \equiv k/\mu$ and μ is fluid viscosity. The corresponding thermoelastic law is Fourier's linear proportionality between the temperature gradient and heat flux.

FIELD EQUATIONS

Since our present concern is with quasi-static phenomena, it is adequate and convenient to express the governing equations in terms of σ_{ij} and p. The former must obey equilibrium conditions (neglecting body forces)

$$\partial\sigma_{ij}/\partial x_j = 0 \qquad \sigma_{ij} = \sigma_{ji} \tag{10}$$

But they must also be related, through (7), to infinitesimal solid strains ϵ_{ij}, which must, in turn, be derivable from a solid displacement vector. The appropriate compatibility conditions on ϵ_{ij} are well-known in elasticity [e.g., *Love*, 1927, article 17] and, by using (7) and (10), they reduce to six mutually independent equations in σ_{ij} and p, most conveniently chosen as

$$\nabla^2[(1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}] + \partial^2\sigma_{kk}/\partial x_i\,\partial x_j + \frac{3(\nu_u-\nu)}{B(1+\nu_u)}[\nabla^2 p\delta_{ij} + \partial^2 p/\partial x_i\,\partial x_j] = 0 \tag{11}$$

where the notation used hereafter is $\nabla^2(\ ) \equiv \partial^2(\ )/\partial x_k\partial x_k$. As a special feature of these equations, we note that contraction on i, j gives a very useful relation between σ_{kk} and p,

$$\nabla^2\left[\sigma_{kk} + \frac{6(\nu_u-\nu)}{B(1-\nu)(1+\nu_u)}p\right] = 0 \tag{12}$$

This procedure is almost completely parallel to that followed in arriving at the Beltrami-Michell formulation in classical elasticity (i.e., stresses rather than displacements taken as basic variables) with body forces proportional to the gradient of p. But a distinction does arise from the entrance of p through the constitutive rather than equilibrium equations. Equations (11) and the conventional elasticity equations become formally identical when we use $\langle\sigma_{ij}\rangle$, the 'effective stress' of *Nur and Byerlee* [1971], to rewrite (1) as

$$2G\epsilon_{ij} = \langle\sigma_{ij}\rangle - \frac{\nu}{1+\nu}\langle\sigma_{kk}\rangle\delta_{ij}$$
$$\langle\sigma_{ij}\rangle \equiv \sigma_{ij} + (1 - K/K_s')p\delta_{ij} \tag{13a}$$

because p then actually enters (formally) through the equilibrium equations

$$\partial\langle\sigma_{ij}\rangle/\partial x_j - (1 - K/K_s')\,\partial p/\partial x_i = 0 \tag{13b}$$

The Beltrami-Michell equations (equations (11)) thereby take the form

$$\nabla^2\langle\sigma_{ij}\rangle + (1+\nu)^{-1}\,\partial^2\langle\sigma_{kk}\rangle/\partial x_i\,\partial x_j - (1 - K/K_s')\cdot\left[2\partial^2 p/\partial x_i\,\partial x_j + \frac{\nu}{1-\nu}\nabla^2 p\delta_{ij}\right] = 0 \tag{13c}$$

As usual, there is no overdeterminacy involved in (10) and (11). The only essential distinction from a conventional elasticity body force problem, when the question of analytic or numerical solution techniques arises for instance, is that the 'body force' field is now coupled to the stresses in general. Indeed, this coupling is accomplished through the final governing equation, that of mass conservation for the infiltrating pore fluid,

$$\partial q_i/\partial x_i + \partial m/\partial t = 0 \tag{14}$$

This is transformed to the variables σ_{ij} and p, by (8) and (9), to get

$$\kappa\nabla^2 p = \frac{3(\nu_u-\nu)}{2GB(1+\nu)(1+\nu_u)}\frac{\partial}{\partial t}\left(\sigma_{kk} + \frac{3}{B}p\right) \tag{15}$$

We can combine this with (12) in an obvious fashion to write

$$c\nabla^2\left(\sigma_{kk} + \frac{3}{B}p\right) = \frac{\partial}{\partial t}\left(\sigma_{kk} + \frac{3}{B}p\right) \quad (16)$$

where c is the 'coefficient of consolidation' or 'diffusivity,' given by

$$c = \kappa\left[\frac{2G(1-\nu)}{(1-2\nu)}\right]\left[\frac{B^2(1+\nu_u)^2(1-2\nu)}{9(1-\nu_u)(\nu_u-\nu)}\right] \quad (17)$$

The classical consolidation coefficient, as frequently used in soil mechanics [e.g., *Süklje*, 1969, section 14.1], may be found by specializing to incompressible constituents, and then the last bracket gives unity. The first bracket gives the drained elastic modulus for one-dimensional deformation (e.g., in the oedometer test). Lastly, the various diffusion equations found by *Biot* [1941, 1955, 1956*a*] are given here in the most general common form, since (8) and (16) show that the fluid mass content m per unit volume satisfies a homogeneous diffusion equation

$$c\nabla^2 m = \partial m/\partial t \quad (18)$$

In summary, the governing equations are as follows: (i) the equilibrium equations (10), (ii) three of the compatibility equations (11) and (12), and (iii) the diffusion equation (16). But if, instead, we followed the Biot procedure of taking displacements and pressure as variables, the governing equations would be (10) and (16), with σ_{ij} expressed in terms of displacement gradients, $\partial u_k/\partial x_l$, and p.

EQUATIONS GOVERNING PLANE DEFORMATION

The kinematic constraint of 'plane strain,' say only in the x_1x_2 plane, is $\epsilon_{3i} = 0$. Thus (7) requires

$$\sigma_{33} = \nu(\sigma_{11} + \sigma_{22}) - \frac{3(\nu_u - \nu)}{B(1+\nu_u)}p \qquad \sigma_{31} = \sigma_{32} = 0 \quad (19)$$

Now the constitutive laws ((7) and (8)) may be written between the lesser number of variables, ϵ_{11}, ϵ_{22}, ϵ_{12}, m and σ_{11}, σ_{22}, σ_{12}, p:

$$2G\epsilon_{\alpha\beta} = \sigma_{\alpha\beta} - \nu(\sigma_{11} + \sigma_{22})\delta_{\alpha\beta} + \frac{3(\nu_u - \nu)}{B(1+\nu_u)}p\delta_{\alpha\beta} \quad (20a)$$

$$m - m_0 = \frac{3\rho_0(\nu_u - \nu)}{2GB(1+\nu_u)}M \quad (20b)$$

where

$$M \equiv (\sigma_{11} + \sigma_{22}) + [3p/B(1+\nu_u)] \quad (20b)$$

Greek indices have range 1, 2. The variable M has dimensions of stress but is proportional to change in fluid mass content and thus automatically satisfies $c\nabla^2 M = \partial M/\partial t$. Equation (20*a*) may be rewritten in terms of M,

$$2G\epsilon_{\alpha\beta} = \sigma_{\alpha\beta} - \nu_u(\sigma_{11} + \sigma_{22})\delta_{\alpha\beta} + (\nu_u - \nu)M\delta_{\alpha\beta} \quad (21)$$

The equations governing the four chosen independent variables σ_{11}, σ_{22}, σ_{12}, and p (or sometimes M, where convenient in the following) may be enumerated as before: we simply use (19) in (10), (12), and (16) to get

(i) $\partial\sigma_{11}/\partial x_1 + \partial\sigma_{12}/\partial x_2 = 0$

$$\partial\sigma_{12}/\partial x_1 + \partial\sigma_{22}/\partial x_2 = 0 \quad (22a)$$

(ii) $\nabla^2[\sigma_{11} + \sigma_{22} + 2\eta p] = 0$ (22*b*)

where

$$\eta \equiv 3(\nu_u - \nu)/[2B(1+\nu_u)(1-\nu)] \quad (22b)$$

(iii) $$c\nabla^2\left[\sigma_{11} + \sigma_{22} + \frac{3}{B(1+\nu_u)}p\right] = \frac{\partial}{\partial t}\left[\sigma_{11} + \sigma_{22} + \frac{3}{B(1+\nu_u)}p\right] \quad (22c)$$

The compatibility equation (22*b*) suggests that we introduce the complex variable $z = x_1 + ix_2$, $i = (-1)^{1/2}$, and represent the solution in terms of a function $\Phi(z, t)$, analytic in z, named in analogy with the first Goursat function of the *Muskhelishvili* [1953] formalism. Thus we define

$$4\,\mathrm{Re}\,[\Phi(z, t)] \equiv (\sigma_{11} + \sigma_{22}) + 2\eta p = \frac{1-\nu_u}{1-\nu}(\sigma_{11} + \sigma_{22}) + \frac{\nu_u - \nu}{1-\nu}M \quad (23)$$

Re means 'real part of,' and, as the notation implies, the function will be time-dependent in general.

Equation (22*c*) may now be recast in the form

$$c\nabla^2 p = \frac{\partial p}{\partial t} + \frac{2(\nu_u - \nu)}{\eta(1-\nu_u)}\mathrm{Re}\left[\frac{\partial\Phi}{\partial t}\right] \quad (24)$$

which shows that pore pressure is governed by a homogeneous diffusion equation only in the special cases where Φ is time-independent. Such cases will have central attention in this paper because it is obvious that the solution technique is then, at least formally, very straightforward: (1) Φ is deduced from either the initial undrained or final drained elasticity solutions, (2) p is determined by solving a simple diffusion equation subject to appropriate boundary and initial conditions on some region. To deduce the remaining stresses, we must find a second stress function, called $\Psi(z, t)$ in the complex variable formulation to be given next. Although this function may be hard to extract for many complex problems, it emerges simply for the problems considered later.

In reference to case 1 above, we may identify the nature of $\Phi(z, t)$ at $t = 0$ and $t = \infty$. Immediately after ($t = 0^+$) a sudden (but quasi-static) disturbance the classical elasticity solution applies based on the 'undrained' elastic constants G and ν_u. Since no fluid transfer can have occurred then, $M_{0^+} = 0$, and (20*b*) gives

$$P_{0^+} = -\frac{B(1+\nu_u)}{3}(\sigma_{11} + \sigma_{22})_{0^+} \quad (25)$$

In general, we will be discussing problems where pore pressures dissipate to zero after long times ($t = \infty$), so (23) has the twin consequences

$$(\sigma_{11} + \sigma_{22})_{0^+} = 4\left(\frac{1-\nu}{1-\nu_u}\right)\mathrm{Re}\,[\Phi(z, 0^+)]$$
$$(\sigma_{11} + \sigma_{22})_\infty = 4\,\mathrm{Re}\,[\Phi(z, \infty)] \quad (26)$$

Now, a minimal requirement for time independence of Φ is that $\Phi(z, 0^+) = \Phi(z, \infty)$. This immediately excludes plane strain problems for which boundary conditions are given solely as applied tractions, since these are well-known to have stress solutions with no dependence on elastic moduli. Equations (26) show that $(\sigma_{11} + \sigma_{22})$ should depend on the effective Poisson ratio ($\equiv\nu_e$, which is ν_u at $t = 0$ and ν at $t = \infty$) and should be specifically proportional to $(1 - \nu_e)^{-1}$, with no other dependence on ν_e, if the minimal requirement is to be met. This is exactly the feature of some basic singular elasticity solutions, such as the classical plane strain solutions for an isolated edge dislocation and an isolated line force, which led us to extract their counterparts for a porous solid. *Booker* [1974] found the corresponding time dependence in his dislocation solution for the special case of incompressible constituents, but his formalism does not permit tracing its source to the structure of the classical elasticity solutions.

COMPLEX VARIABLE REPRESENTATIONS OF SOLUTIONS IN PLANE STRAIN

We present a complex variable representation in this section for the solution of the equations governing states of plane strain. It is motivated by the success of the corresponding formalism in classical elasticity [e.g., *Muskhelishvili*, 1953]. Stresses and displacements will be expressible in terms of two analytic functions, $\Phi(z, t)$ and $\Psi(z, t)$, together with p or M. The last two are not analytic but may be written $p(z, \bar{z}, t)$ or $M(z, \bar{z}, t)$ to show explicit dependence on $\bar{z} = x_1 - ix_2$. It will be convenient to regard z and its complex conjugate $\bar{z}$ as being formally independent so that any function $h(x_1, x_2, t)$ may be alternately expressed as $h(z, \bar{z}, t)$. This is the process familiar in formally assigning characteristics to elliptic partial differential equations, and we may convert partial derivatives according to

$$\partial h/\partial z = \partial h/\partial x_1 - i\partial h/\partial x_2$$

$$\partial h/\partial \bar{z} = \partial h/\partial x_1 + i\partial h/\partial x_2$$

Now the equilibrium equations (22*a*) become a single equation

$$\partial \sigma/\partial z = \partial \tau/\partial \bar{z} \tag{27}$$

where $\sigma = (\sigma_{11} + \sigma_{22})/2$, $\tau = (\sigma_{22} - \sigma_{11})/2 + i\sigma_{12}$, and this result prompts us to rewrite (23) in the forms

$$\sigma = \Phi(z, t) + \bar{\Phi}(\bar{z}, t) - \eta p(z, \bar{z}, t) \tag{28a}$$

$$\sigma = \left(\frac{1-\nu}{1-\nu_u}\right)[\Phi(z, t) + \bar{\Phi}(\bar{z}, t)] - \frac{(\nu_u - \nu)}{2(1-\nu_u)} M(z, \bar{z}, t) \tag{28b}$$

where $\bar{\Phi}(\bar{z}, t) \equiv \overline{\Phi(z, t)}$ is simply a complex conjugate, so written to emphasize that $\bar{\Phi}$ is analytic around $\bar{z}$ when Φ is analytic around z. Equation (27) now implies, on simple integration, that

$$\tau = \bar{z}\,\frac{\partial\Phi(z, t)}{\partial z} + \Psi_p(z, t) - \eta \int_{f(z,t)}^{\bar{z}} \frac{\partial p(z, \zeta, t)}{\partial z}\, d\zeta \tag{29a}$$

$$\tau = \left(\frac{1-\nu}{1-\nu_u}\right)\bar{z}\,\frac{\partial\Phi(z, t)}{\partial z} + \Psi_M(z, t) - \frac{(\nu_u - \nu)}{2(1-\nu_u)} \int_{g(z,t)}^{\bar{z}} \frac{\partial M(z, \zeta, t)}{\partial z}\, d\zeta \tag{29b}$$

Here the analytic functions $f(z, t)$ and $g(z, t)$ arise as arbitrary integration limits and will be chosen to suit the region, boundary conditions, etc. The second Goursat function $\Psi(z, t)$ is also an arbitrary function of integration on which the subscripts p and M imply its dependence on the representation chosen. Obviously, arbitrariness in the choice of the lower integration limits f or g is interchangeable with that in the choice of the Ψ.

Formulae for the displacements u_1 and u_2 may be obtained by observing that

$$\partial(u_1 - iu_2)/\partial z = -[(\epsilon_{22} - \epsilon_{11})/2 + i\epsilon_{12}] = -\tau/2G$$

$$\text{Re}\,[\partial(u_1 - iu_2)/\partial \bar{z}] = \epsilon_{11} + \epsilon_{22} = (1 - 2\nu)\sigma/2G + 2\eta(1-\nu)p/G = (1 - 2\nu_u)\sigma/2G + (\nu_u - \nu)M/G$$

which may be combined with (28) and (29) to get $u \equiv u_1 + iu_2$:

$$2G\bar{u} = (3 - 4\nu)\bar{\phi}(\bar{z}, t) - \bar{z}\Phi(z, t) - \psi_p(z, t) + \eta \int_{f(z,t)}^{\bar{z}} p(z, \zeta, t)\, d\zeta \tag{30a}$$

$$2G\bar{u} = \left(\frac{1-\nu}{1-\nu_u}\right)[(3 - 4\nu_u)\bar{\phi}(\bar{z}, t) - \bar{z}\Phi(z, t)] - \psi_M(z, t) + \frac{\nu_u - \nu}{2(1-\nu_u)} \int_{g(z,t)}^{\bar{z}} M(z, \zeta, t)\, d\zeta \tag{30b}$$

where

$$\partial\phi(z, t)/\partial z = \Phi(z, t)$$

but

$$\partial\psi_p(z, t)/\partial z = \Psi_p(z, t) - \eta p[z, f(z, t), t]\, \partial f(z, t)/\partial z$$

and

$$\partial\psi_M(z, t)/\partial z = \Psi_M(z, t) - \frac{\nu_u - \nu}{2(1-\nu_u)} M[z, g(z, t), t]\, \partial g(z, t)/\partial z$$

Finally, it will be necessary later to have formulae for the force vector $F = F_1 + iF_2$ resulting from integration of the traction vector $T_1 + iT_2$ along any contour in the plane. With reference to Figure 1, elementary equilibrium considerations suffice to show that

$$-i(F_1 - iF_2) \equiv -i\int_{s_0}^{s} (T_1 - iT_2)\, ds = \int_{s_0}^{s} (\sigma\, d\bar{z} + \tau\, dz)$$

and the latter integral, path independent in any simply connected region containing no singularities, is easily performed to get

$$-i(F_1 - iF_2) = \bar{\phi}(\bar{z}, t) + \bar{z}\Phi(z, t) + \psi_p(z, t) - \eta \int_{f(z,t)}^{\bar{z}} p(z, \zeta, t)\, d\zeta \tag{31a}$$

$$-i(F_1 - iF_2) = \left(\frac{1-\nu}{1-\nu_u}\right)[\bar{\phi}(\bar{z}, t) + \bar{z}\Phi(z, t)] + \psi_M(z, t) - \frac{\nu_u - \nu}{2(1-\nu_u)} \int_{g(z,t)}^{\bar{z}} M(z, \zeta, t)\, d\zeta \tag{31b}$$

FUNDAMENTAL SINGULAR SOLUTIONS FOR STRAIGHT DISLOCATIONS AND LINE FORCES

As an application of the foregoing formulation, we consider the sudden (but quasi-static) introduction, at $t = 0$, of an edge dislocation with Burgers components b_1 and b_2 (denoted by $b \equiv b_1 + ib_2$), together with a point force $P \equiv P_1 + iP_2$, acting on the solid phase, at the origin of an infinite porous medium. The singularity is to be maintained for all time $t > 0$. The appropriate classical elasticity Goursat functions, $\Phi(z)$ and $\Psi(z)$, are

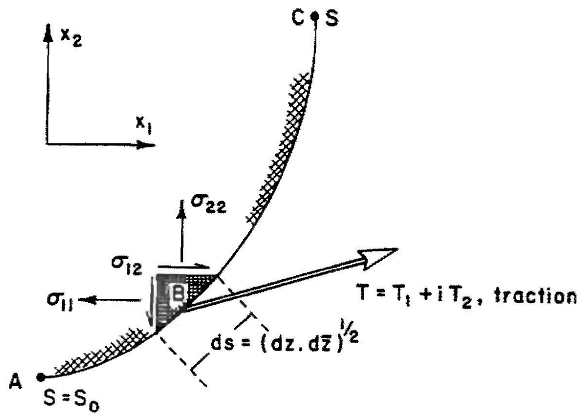

Fig. 1. Tractions on any arbitrary contour in the plane.

defined by (28) and (29) when we set $\nu \equiv \nu_u \equiv \nu_e$; their form is well-known to be

$$\Phi(z) = \frac{2Gb - iP}{8\pi i(1 - \nu_e)}\frac{1}{z} \equiv \frac{D}{1 - \nu_e}\frac{1}{z}$$
$$\Psi(z) = \frac{-2G\bar{b} - (3 - 4\nu_e)i\bar{P}}{8\pi i(1 - \nu_e)}\frac{1}{z} \quad (32)$$

Since hydrostatic stress in these limiting elasticity solutions is simply a multiple of Re $[\Phi]$,

$$(\sigma_{11} + \sigma_{22})_{0^+} = \frac{4}{(1 - \nu_u)}\text{Re}\left[\frac{D}{z}\right]$$
$$(\sigma_{11} + \sigma_{22})_\infty = \frac{4}{(1 - \nu)}\text{Re}\left[\frac{D}{z}\right] \quad (33)$$

We observe that these satisfy the minimal requirement for time independence, and we now proceed to find a solution that indeed verifies that $\Phi(z, t)$ is time-independent.

The fluid content parameter M (equations (20b)) always satisfies $c\nabla^2 M = \partial M/\partial t$, presently in the infinite medium, subject to $M(t = 0^+) = 0$. By setting $p(t = \infty) = 0$ in (20c), we get $M(t = \infty) = (\sigma_{11} + \sigma_{22})_\infty$. The solution giving appropriate decay at infinity is

$$M = \frac{2}{(1 - \nu)}\left[\frac{D}{z} + \frac{\bar{D}}{\bar{z}}\right]\exp(-z\bar{z}/4ct) \quad (34)$$

In order to use (30b) and (31b), we require the integral

$$I \equiv \int_{z_0}^{\bar{z}}\left(\frac{D}{z} + \frac{\bar{D}}{\zeta}\right)\exp(-z\zeta/4ct)\,d\zeta$$
$$= \frac{4ctD}{z^2}[\exp(-zz_0/4ct) - \exp(-z\bar{z}/4ct)]$$
$$+ \int_{z_0}^{\bar{z}}\frac{\bar{D}}{\zeta}\left[1 - \frac{z\zeta}{4ct} + \frac{1}{2}\left(\frac{z\zeta}{4ct}\right)^2 - \cdots\right]d\zeta \quad (35)$$

Here we have set $g(z, t) = z_0$, a real constant. We shall find that z_0 must be 0 in order to comply with the classical elasticity stress solution at $t = \infty$. This value seems to generate a divergence in evaluating displacement, but such anomalies are familiar, and we consider only the contribution, from the integral, which generates a function with a branch-cut, since only the jump in u or F will be specified. It is thus adequate to write

$$I = \frac{4ctD}{z^2}[1 - \exp(-z\bar{z}/4ct)] + \bar{D}\ln\bar{z}$$
$$+ \int_0^{\bar{z}}[\exp(-z\zeta/4ct) - 1]\frac{\bar{D}}{\zeta}\,d\zeta \quad (36)$$

and comment that u diverges like ln z near the origin, as in classical elasticity. Thus the jump generated by I is constant in time. We now propose that

$$\phi(z, t) = A(t)\ln z \qquad \psi_M(z, t) = B(t)\ln z \quad (37)$$

so that the jump conditions, deduced by using (36) and (37) in (30b) and (31b), take the form

$$2G\bar{b} = \left(\frac{1 - \nu}{1 - \nu_u}\right)(3 - 4\nu_u)(-2\pi i)\bar{A}(t) - 2\pi i B(t)$$
$$- \frac{\nu_u - \nu}{(1 - \nu)(1 - \nu_u)}(2\pi i\bar{D})$$
$$i\bar{P} = \left(\frac{1 - \nu}{1 - \nu_u}\right)(-2\pi i)\bar{A}(t) + 2\pi i B(t)$$
$$+ \frac{\nu_u - \nu}{(1 - \nu)(1 - \nu_u)}(2\pi i\bar{D}) \quad (38)$$

The solution of (38) is a time-independent $A(t)$ and $B(t)$, namely,

$$A(t) \equiv A = \frac{2Gb - iP}{8\pi i(1 - \nu)} \equiv \frac{D}{1 - \nu} \qquad \Phi(z, t) = \frac{A}{z} \quad (39)$$
$$B(t) \equiv B = \frac{-2G\bar{b} - (3 - 4\nu)i\bar{P}}{8\pi i(1 - \nu)} \qquad \Psi(z, t) = \frac{B}{z} \quad (40)$$

The time-dependent stress field may now be obtained from (28) and (29). For this purpose, we need one last integral:

$$\int_{z_0}^{\bar{z}}\frac{\partial M(z, \zeta, t)}{\partial z}\,d\zeta$$
$$= 4A\left(\frac{4ct}{z^3}\right)[\exp(-z\bar{z}/4ct) - \exp(-zz_0/4ct)]$$
$$+ \frac{2A}{z^2}[\bar{z}\exp(-z\bar{z}/4ct) - z_0\exp(-zz_0/4ct)]$$
$$+ \frac{2\bar{A}}{z}[\exp(-z\bar{z}/4ct) - \exp(-zz_0/4ct)] \quad (41)$$

By imposing the known elasticity solution at $t = \infty$ in (29), we deduce $z_0 = 0$, as already mentioned.

Stress field of an edge dislocation. As an example implementing the expressions just derived, we obtain the stress field for the isolated edge dislocation shown in Figure 2; for this, $A = -B$, and thus

$$\Phi(z) = -\Psi(z) = \frac{A}{z} = \frac{Gb_1}{4\pi i(1 - \nu)}\frac{e^{-i\theta}}{r} \quad (42)$$

whereas (34) takes the form

$$M = \frac{-Gb_1}{\pi(1 - \nu)}\frac{\sin\theta}{r}\exp(-r^2/4ct) \quad (43)$$

Now (28b) may be used to find

$$\tfrac{1}{2}(\sigma_{rr} + \sigma_{\theta\theta}) = \tfrac{1}{2}(\sigma_{11} + \sigma_{22})$$
$$= -\frac{Gb_1}{2\pi r}\left[\frac{(1 - \nu) - (\nu_u - \nu)\exp(-r^2/4ct)}{(1 - \nu)(1 - \nu_u)}\right]\sin\theta \quad (44)$$

The polar coordinate version of the deviator stress is

$$\tfrac{1}{2}(\sigma_{\theta\theta} - \sigma_{rr}) + i\sigma_{r\theta} = \exp(2i\theta)[\tfrac{1}{2}(\sigma_{22} - \sigma_{11}) + i\sigma_{12}] \quad (45)$$

and, when (41) and (42) are inserted in (29*b*), we find

$$\tfrac{1}{2}(\sigma_{\theta\theta} - \sigma_{rr}) + i\sigma_{r\theta} = \frac{iGb_1}{2\pi r}\left\{\frac{\cos\theta}{(1-\nu_u)} - \frac{(\nu_u - \nu)}{(1-\nu_u)(1-\nu)}\left[i\sin\theta\exp(-r^2/4ct) + \frac{4ct}{r^2}(1 - \exp(-r^2/4ct))(\cos\theta - i\sin\theta)\right]\right\} \quad (46)$$

The pore pressure is obtained by substituting (43) and (44) into (20*b*), and the complete set of field variables is then given by

$$\begin{Bmatrix}\sigma_{\theta\theta}\\ \sigma_{r\theta}\\ \sigma_{rr}\\ p\end{Bmatrix} = \frac{Gb_1(\nu_u - \nu)}{2\pi r(1-\nu_u)(1-\nu)}\begin{Bmatrix}\sin\theta\left\{2\exp(-r^2/4ct) - \dfrac{1-\nu}{\nu_u - \nu} - \dfrac{4ct}{r^2}[1 - \exp(-r^2/4ct)]\right\}\\ \cos\theta\left\{\dfrac{1-\nu}{\nu_u-\nu} - \dfrac{4ct}{r^2}[1 - \exp(-r^2/4ct)]\right\}\\ \sin\theta\left\{\dfrac{4ct}{r^2}[1 - \exp(-r^2/4ct)] - \dfrac{1-\nu}{\nu_u - \nu}\right\}\\ \sin\theta\,\eta^{-1}[1 - \exp(-r^2/4ct)]\end{Bmatrix} \quad (47)$$

In particular, the solution for p and $\sigma_{r\theta}$ may be specialized to the case of incompressible constituents ($\nu_u = \tfrac{1}{2}$, $B = 1$), and then the solution obtained by *Booker* [1974], by using integral transforms on the *McNamee and Gibson* [1960] equations, for a fault or slip represented by a dipole pair of edge dislocations can be written down directly. This is an elementary application of (47); in the next section we take (47) as the starting point for a discussion of shear faults in fluid-infiltrated materials and of the manner in which interactions between the rupturing solid phase and its pore fluid may affect the rate or time dependence of fault spreading.

APPLICATION TO SHEAR FAULT MOTION IN FLUID-SATURATED MATERIALS

The use of dislocations, either discretely or continuously distributed, to simulate the introduction or propagation of slip in masses of rock or soil is well-exemplified in the literature [e.g., as summarized by *Cleary*, 1976] for the situations where the material can be approximated as linear elastic. These examples and applications, such as mining settlements, induction of large single and network fractures, slip-surface propagation in progressive land sliding, and aseismic earth faulting, often have time dependence associated with them. A partial account of this dependence may possibly be taken simply by replacing elastic influence functions for the dislocations by the time-dependent stress field just derived. Indeed, since porous media effects have been cited recently as possible contributors to aftershock activity [*Nur and Booker*, 1972; *Booker*, 1974] and to the stabilization of shear rupture zones against rapid propagation [*Palmer and Rice*, 1973; *Rice*, 1973], we here consider the representation of a time-dependent shear faulting process by a continuous array of dislocations.

Let the fault lie along the x_1 axis and let the relative sliding or slip on the fault be defined from the displacement field $u_i(x_1, x_2, t)$ as

$$\delta = \delta(x_1, t) = u_1(x_1, 0^+, t) - u_1(x_1, 0^-, t) \quad (48)$$

The single dislocation solution just presented corresponds to a slip function $\delta = b_1[1 - U(x_1)]U(t)$, where U is the unit step function. For a continuous slip function, the infinitesimal dislocation accumulated in time dt within an element of length dx_1 can be identified as

$$b_1 = -\frac{\partial^2\delta(x_1, t)}{\partial x_1\,\partial t}\,dx_1\,dt \quad (49)$$

which induces stresses and pressure given by (47) with origin at x_1 and time measured from the instant t of accumulation. By summing (i.e., integrating) all such infinitesimal contributions along the fault line over all preceding duration of slip we get the stress field at any time due to whatever slip $\delta(x_1, t)$ has occurred.

For example, the shear stress τ ($\equiv\sigma_{12}$) induced along the x_1 axis, owing to a slip dislocation b_1 introduced at position x_1' and time t' is, by (47),

$$\tau(x_1, t) = \frac{Gb_1}{2\pi(1-\nu_u)}\frac{1}{x_1 - x_1'}\cdot\mathcal{L}[(x_1 - x_1')^2/4c(t - t')] \qquad t > t' \quad (50)$$

where

$$\mathcal{L}[\xi] = 1 - [(\nu_u - \nu)/(1 - \nu)]\xi^{-1}(1 - e^{-\xi})$$

Note the decrease of $\mathcal{L}$ from unity at short times (or great distances) to the value $(1 - \nu_u)/(1 - \nu)$ at long times (or short distances). Thus if the slippage δ is prescribed over some region $L = L(t)$ of the x_1 axis, starting at $t = t_0$, and if the applied stress distribution (i.e., that which would be present at any point and time if slip had not taken place) is $\tau_{\text{appl}}(x_1, t)$, then the stress $\tau(x_1, t)$, as altered by the slip, is

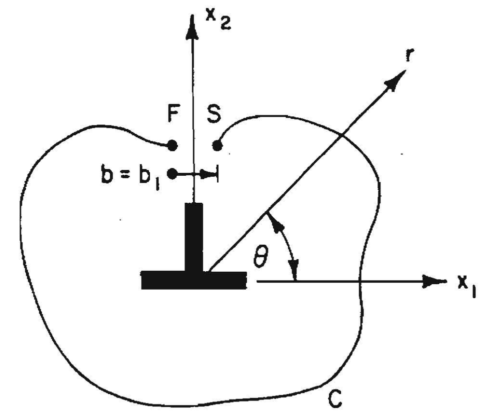

Fig. 2. An edge dislocation at the origin of coordinates in the plane.

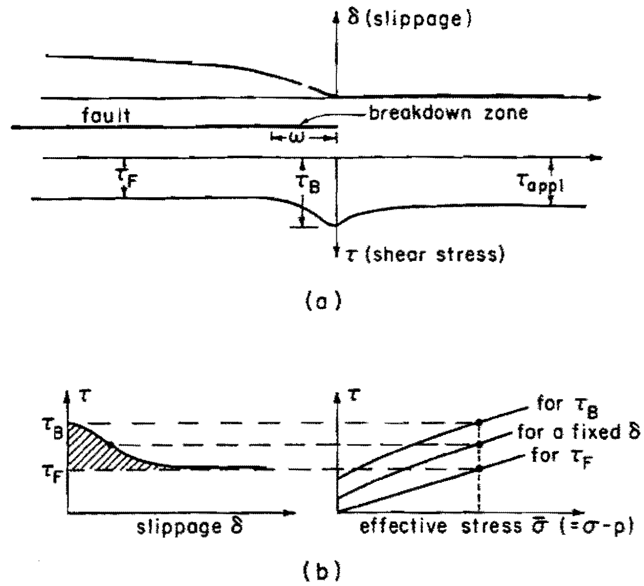

Fig. 3. (*a*) Schematic of a shear fault, showing relative slip δ and shear stresses τ on the line of the fault. (*b*) A simple failure criterion for material on the fault. Endurable shear stress is related to effective normal compressive stress and amount of sliding. Note that reductions in p, induced transiently by prerupture dilatancy, increase the resistance to fault spreading.

$$\tau(x_1, t) = \tau_{\text{appl}}(x_1, t) - \frac{G}{2\pi(1-\nu_u)} \int_{t_0}^{t} \int_{L(t')} \frac{\partial^2 \delta(x_1', t')}{\partial x_1' \, \partial t'} \cdot \mathcal{L}\left[\frac{(x_1 - x_1')^2}{4c(t-t')}\right] \frac{dx_1'}{x_1 - x_1'} \, dt' \tag{51}$$

where the integral on x_1' is taken in the Cauchy principal value sense.

It is, of course, seldom the case that δ is given and τ is merely to be calculated by integration. Rather, the integral relation is to be understood as a first step toward an integral equation governing the slippage when certain information is given concerning its left-hand side; this formulation is analogous to the representation of cracks as dislocation arrays in classical elasticity, the exact density in the array being chosen as that satisfying an integral equation that expresses, for example, the condition that the crack surfaces be stress free [e.g., *Bilby and Eshelby*, 1968; *Cleary*, 1976].

The simplest condition on τ is that analogous to the freely slipping shear crack, namely, that τ has some reduced value, say a frictional sliding stress τ_F, everywhere along the region $L(t)$ of slippage, where $L(t)$ is given a priori, and the friction stress τ_F ($<\tau_{\text{appl}}$, at least in some average sense) is that residual resistance remaining after completion of the 'breakdown' process at the ends of the spreading fault zone. In that case, stress singularities will result at the spreading ends of the fault, and the condition for fault propagation can be phrased in fracture mechanics terminology [e.g., *Rice*, 1968] as the requirement that a critical energy release rate, expressible in terms of the strength of the singularity, be achieved for propagation. Indeed, this kind of characterization has been proposed for shear faults in overconsolidated soils by *Palmer and Rice* [1973] and for earthquake faults by *Husseini et al.* [1976].

Models can also be formulated that include a more detailed account of the breakdown process: following *Palmer and Rice* [1973] and with reference to Figure 3, the shear stress τ can be considered to be some prescribed function of the slippage δ near the tip of the fault, decreasing from a breakdown stress level τ_B, sufficient to initiate slippage, to the residual friction level τ_F after large amounts of sliding; τ_B, τ_F, and the values of τ at intermediate values of δ increase with the local 'effective' compressive stress $\bar{\sigma}$ ($=\sigma - p$, where σ is the total compressive stress) acting on the fault, as illustrated. This formulation removes the point stress singularity at the tip in favor of a direct (if oversimplified) model of the breakdown process. With it, $\tau(x_1, t)$ on the left side of (51) is expressible as a function of $\delta(x_1, t)$ at all points that have previously been brought to the breakdown stress level, and hence (52) becomes a nonlinear integral equation for δ.

Solutions for similar models in the classical elasticity context have been given by *Palmer and Rice* [1973], and a general numerical scheme has been presented by *Cleary* [1976]. In cases for which the size ω of the end region (Figure 3), over which strength degradation takes place, is small in comparison with overall fault length, the propagation criterion from this model accords with the fracture mechanics point singularity approach, and the effective fracture energy is equal to the shaded area in the τ versus δ plot of Figure 3. It is thus possible to give simple estimates of how the aforementioned features affect progress of a fault.

Pore fluid effects in the stabilization of fault spreading. Two distinct mechanisms have been proposed by which the coupling between pore fluids and deformation can stabilize a shear fault against rapid growth [*Palmer and Rice*, 1973; *Rice*, 1973], and these may be important for explaining observed fault creep events [e.g., *King et al.*, 1973]. The first is that, for a given set of stresses exerted on a fault, the amount of energy that can flow to its tip in a unit advance will be different according to whether the surrounding material responds in an undrained or drained fashion (or, in general, in a way intermediate between these short and long time extremes). More energy is released under drained than undrained conditions (see below), and hence the magnitude of the applied stresses necessary to deliver some fixed energy to the breakdown process must increase with the speed of fault spreading, the implication thus being that a stable creeplike rupturing process exists, at least over some range of driving stress.

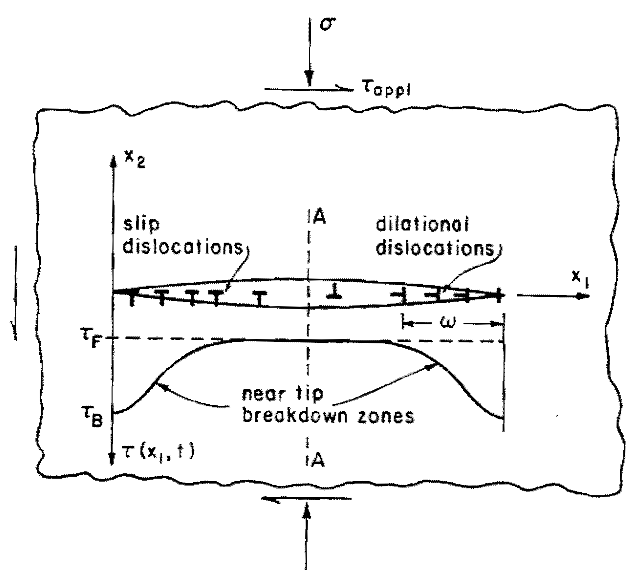

Fig. 4. Simulation of relative sliding and dilation, on the line of the shear fault, by a continuous density of dislocations.

The response for the two limiting cases may be ascertained directly from (51), for which $\mathcal{L}$ may be given its short and long time values, 1.0 and $(1-\nu_u)/(1-\nu)$, respectively. Then the time integration is trivial, and the equation reduces to the corresponding crack integral equation of classical elasticity [e.g., *Bilby and Eshelby,* 1968], with the elastic properties entering in the form $G/(1-\nu_u)$ for the short time (rapid fault creep) response and $G/(1-\nu)$ for the long time (slow creep) response. Of course, these are the same forms in which elastic properties enter the expression for energy flow to the crack tip: for the simplest case of a plane strain fault of length l with uniform applied stress and constant resistance τ_F everywhere except near a small end zone, the energy flow to the breakdown process per unit of newly created fault surface [e.g., *Rice,* 1968; *Palmer and Rice,* 1973] is

$$\mathcal{G}' = \pi(1-\nu_e)(\tau_{appl}-\tau_F)^2 l/4G \tag{52}$$

Hence if the fracture energy can be taken as essentially independent of the speed of fault creep, the ratio of the driving stress required for slow (s), as opposed to fast (f), completely undrained, but still quasi-static, fault spreading is

$$(\tau_{appl}-\tau_F)_f/(\tau_{appl}-\tau_F)_s = [(1-\nu)/(1-\nu_u)]^{1/2} \equiv \beta^{1/2} \tag{53}$$

This is 1.13 for data representative of sandstone ($\nu = 0.12$, $\nu_u = 0.31$) and may be somewhat higher for heavily fissured rock masses or for soils, in which cases $\nu_u = 0.5$; for example, $\beta^{1/2}$ is then 1.30 if $\nu = 0.15$. Hence no fault spreading can occur on this basis if the driving stress is less than $(\tau_{appl}-\tau_F)_s$, and it must exceed $(\tau_{appl}-\tau_F)_f$ for seismic spreading to ensue.

(Since preparation of the original version of this manuscript, a full analysis of a shear fault advancing at steady speed in a fluid-infiltrated porous material has been given by *Rice and Simons* [1976]. Remarkably, they find an effect even greater than that indicated by the above comparison of the completely drained and completely undrained cases. Indeed, the greatest resistance to fault spreading is found to occur at an intermediate speed, and at this speed the ratio corresponding to the left side of (53) has a value lying between $\beta^{1/2}$ and β. The higher value is approached when the size of the end region is an extremely small fraction of fault length, the lower value when the end region is large. By using a field diffusivity $c \approx 10^4$ cm^2/s [*Anderson and Whitcomb,* 1975], Rice and Simons report that the range of fault lengths and spreading speeds given by *King et al.* [1973] as representative of San Andreas creep events are, when they are compared with the theoretical predictions, supportive of the notion that the fluid interaction effects under discussion could indeed be active in fault stabilization.)

The second means by which porous media effects can stabilize fault propagation is complementary to the first and involves the fact that the rupture process in the breakdown zone may entail nonlinear dilatant deformation of the rock, owing to local propping at asperities and to the opening of new or existing fissures. Hence pore fluid suctions are induced when the time scale is insufficient for their diffusive alleviation, and by the effective stress principle, the material in the breakdown zone is 'dilatantly strengthened' over the resistance to deformation that it would show under less rapid, drained conditions. The extent of the strength increase for completely undrained deformation of a material element can be estimated from a formulation of its inelastic stress-strain relations in accord with the effective stress principle [*Rice,* 1975].

Within our present linear elastic context, in which the breakdown zone is represented as a prolongation of the fault plane (Figure 3), the dilatancy during rupture can be simulated as a continuous array of opening dislocations within the end region, as on the right in Figure 4; these are superposed on the main array of slip dislocations. The effect of the pore fluid suction distribution thus induced along the fault plane is to increase the effective compressive stress $\bar{\sigma}$, and as illustrated on the right in Figure 3*b*, this will raise the level of the τ versus δ curve and hence result in an increase in the shaded area, which, as has been remarked, is a measure of the fracture energy required for fault propagation. Thus the effect again serves to stabilize the fault against rapid spreading, for the induced suctions will be greater the greater the speed of fault creep.

Rice [1973] provided an approximate estimate of these induced suctions through a treatment that regards the solid and fluid phases as separately incompressible (soil mechanics case); assumes that the dilation is equivalent to the induction of a flow of fluid into the end zone ω at a uniform rate, so that a net height h of fluid is indrawn per unit area of newly created fault plane; and treats the diffusive flow as that corresponding locally to one-dimensional consolidation in the x_2 direction under a constant total compressive stress (i.e., p is assumed to satisfy the equation $c\partial^2 p/\partial x_2^2 = \partial p/\partial t$, where the net volumetric flow rate, $2\kappa(\partial p/\partial x_2)$, is given on $x_2 = 0$ as Vh/ω when a material point is within the end region). This results in a suction distribution that is at a maximum at the trailing end of the breakdown zone and has there the value

$$-(\Delta p)_{max} \approx (\Delta\bar{\sigma})_{max} \approx (Vh/2\kappa\omega)(4c\omega/\pi V)^{1/2}$$
$$= [4(1-\nu)/\pi(1-2\nu)](Gh/\omega)(\pi\omega V/4c)^{1/2} \tag{54}$$

where, in the last rearrangement, (17) for c has been used in the form appropriate to incompressible constituents ($B = 1$, $\nu_u = \frac{1}{2}$), and V enters in the dimensionless combination $\omega V/c$. There seems, unfortunately, to be inadequate data from which to deduce numerical values of the parameters, although estimates have been made by *Rice* [1973] for shear faults in clay soils as part of a discussion of time effects in progressive failure of slopes.

By contrast, the porous media effects discussed by *Nur and Booker* [1972] and *Booker* [1974] entail a partial destabilization of a recently slipped fault. As is clear from (51), the stress alterations in faulting will have their greatest values immediately after a sudden slip ($\mathcal{L} = 1.0$), whereas they will relax by the factor $(1-\nu_u)/(1-\nu)$ after a long time. Hence the shear stress builds up gradually on the part of the fault where the stress was relaxed by the sudden slip, while it decays in the more highly stressed regions bordering the zone of spreading. This sequence has been proposed as consistent with limited subsequent faulting, in the form of aftershocks, along the region that slipped in the sudden faulting.

STRESSES NEAR A PRESSURIZED CYLINDRICAL CAVITY

As another example of a fundamental plane strain solution, derivable within the complex variable formalism now available to us, we consider a circular cylindrical hole of radius a in a body of porous saturated material with concentric circular outer boundary at radius b (Figure 5). The body is to be stressed so that the resultant field depends only on the radial coordinate r: for instance, by pressurizing the test fluid filling the interior of the hole. Despite its simplicity, the problem has direct relevance for initiation of hydraulic fractures in deep boreholes [e.g., *Haimson and Fairhurst,* 1970], and interior

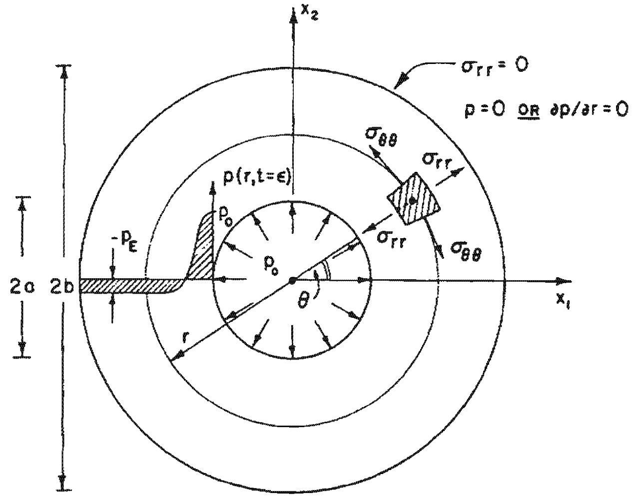

Fig. 5. Typical annular specimen employed for determining fracture strength of rock specimens by internal hydraulic pressurization.

fluid pressure is increasingly being used for laboratory tests to determine the tensile strength of cylindrical rock specimens with drilled central holes [e.g., *Johnson et al.*, 1973].

Since $p = p(r, t)$ and $r^2 = z\bar{z}$, we may use $p(\rho, t)$ and $\rho^2 \equiv z\zeta$ in the integrals of (29*a*), (30*a*), and (31*a*). By noting that

$$d\zeta = 2(\rho/z)d\rho \qquad \partial p(z, \zeta, t)/\partial z = (\tfrac{1}{2})(\zeta/z)^{1/2}\,\partial p(\rho, t)/\partial\rho$$

we can convert the integrals to simpler radial dependence only:

$$\int_{a^2/z}^{\bar{z}} p(z, \zeta, t)\, d\zeta = \frac{2}{z}\int_a^r \rho p(\rho, t)\, d\rho$$
$$\int_{a^2/z}^{\bar{z}} [\partial p(z, \zeta, t)/\partial z]\, d\zeta = \frac{1}{z^2}\int_a^r \rho^2[\partial p(\rho, t)/\partial\rho]\, d\rho \qquad (55)$$

We have chosen $f(z, t) \equiv a^2/z$ so that the integral entails only real values of ρ.

By recalling (44) and (45) and recognizing $e^{2i\theta} = z^2/r^2$, we can transform (28*a*) and (29*a*) to polar coordinates:

$$\tfrac{1}{2}(\sigma_{rr} + \sigma_{\theta\theta}) = \Phi(z, t) + \bar{\Phi}(\bar{z}, t) - \eta p(r, t)$$
$$\tfrac{1}{2}(\sigma_{\theta\theta} - \sigma_{rr}) + i\sigma_{r\theta} = \frac{1}{r^2}\left[\bar{z}z^2\frac{\partial\Phi(z, t)}{\partial z} + z^2\Psi(z, t) - \eta\int_a^r \rho^2\frac{\partial p(\rho, t)}{\partial\rho}\, d\rho\right] \qquad (56)$$

where we have dropped the p subscript on $\Psi_p(z, t)$, since we will not use the M representation. Also, we recall that the material property η is defined in (22*b*); it will prove an important parameter in discussions of hydraulic fracture, and numerical values are given subsequently.

Symmetry demands that the above stresses be independent of θ and that $\sigma_{r\theta} = 0$; only one possible solution has these features, namely,

$$\Phi(z, t) = N(t)\ln z + C(t) \qquad \Psi(z, t) = S(t)/z^2 \qquad (57)$$

where, in the most general case, N, S, and C are unknown functions of time. But if $N \ln z$ and $Nz \ln z - Nz$ are entered for Φ and ϕ, respectively, in (30), we find a jump of the type Nr in the value of u_θ (the circumferential displacement) as we traverse a closed circular contour of radius $r > a$, namely, we find a wedgelike dislocation. By specifying that such dislocations are not present, we deduce $N \equiv 0$. Then the stresses simplify, after using integration by parts once, to

$$\sigma_{\theta\theta} = 4C(t) - 2\eta p(r, t) - \sigma_{rr} \qquad (58a)$$

$$\sigma_{rr} = 2C(t) - \frac{1}{r^2}\left[S(t) + \eta a^2 p(a, t) + 2\eta\int_a^r \rho p(\rho, t)\, d\rho\right] \qquad (58b)$$

where p is to be obtained from the polar coordinate version of (24), namely,

$$c\left[\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r}\right] = \frac{\partial p}{\partial t} + \frac{2(\nu_u - \nu)}{\eta(1 - \nu_u)}\frac{dC(t)}{dt} \qquad (59)$$

We note that *Geertsma* [1957] has given the more usual body force approach to the same general problem of radial symmetry in a porous medium. But he erroneously disposes of the derivative dC/dt in (59) and so concludes that p satisfies a homogeneous diffusion equation: this step will be seen justified only when $b/a \to \infty$, and then only because $C(t) \to 0$.

A variety of boundary conditions may be imposed; for instance, either the total radial stress or the radial displacement and the pore pressure, or the rate of fluid mass exchange with surroundings may be specified as functions of time at either boundary. A problem of practical application, to be studied here, is that of a fluid-filled cavity in which the fluid is suddenly (at $t = 0$) subjected to a pressure p_0, equal total stress and pore pressure at the boundary thereby being induced,

$$\sigma_{rr}(a, t) = -p_0 \qquad t > 0 \qquad (60a)$$

$$p(a, t) = p_0 \qquad t > 0 \qquad (60b)$$

We suppose, for the moment, that the outer boundary is free of stress and fluid pressure

$$\sigma_{rr}(b, t) = p(b, t) = 0 \qquad t > 0 \qquad (61)$$

By using (60), we may solve for $S(t)$ in (58*b*)

$$S(t) = a^2[2C(t) + (1 - \eta)p_0] \qquad (62)$$

and then σ_{rr} is given by

$$\sigma_{rr} = 2(1 - a^2/r^2)C(t) - \frac{1}{r^2}\left[a^2 p_0 + 2\eta\int_a^r \rho p(\rho, t)\, d\rho\right] \qquad (63)$$

By making σ_{rr} vanish (equation (61)) at $r = b$ we find an implicit equation for $C(t)$.

$$C(t) = (b^2 - a^2)^{-1}\left[\frac{a^2}{2}p_0 + \eta\int_a^b \rho p(\rho, t)\, d\rho\right] \qquad (64)$$

which is, in fact, a linear integral equation to be solved after p has first been determined, as a functional of $C(t)$, from (59) with (60*b*) and (61) as boundary conditions. The process is not simple to carry through, so here we just study the special cases of short and very long times.

Short-time solutions. Immediately after the loading has been applied, the classical elasticity solution (with $\nu_e = \nu_u$) applies in all of the region $a < r < b$. Equations (25) and (58) may be used in (24) to find a relation between $p(r, 0^+)$ and $C(0^+)$ attained immediately after loading. Alternately, (59) may be integrated from $t = 0^-$ to $t = 0^+$; in any case, the result is

$$p(r, 0^+) = -\frac{2(\nu_u - \nu)}{\eta(1 - \nu_u)} C(0^+) \qquad a < r < b \qquad (65)$$

and, when this is substituted into (64), we may solve to get

$$C(0^+) = \frac{(1 - \nu_u)}{2(1 - \nu)} \frac{a^2 p_0}{(b^2 - a^2)} \qquad (66a)$$

$$p(r, 0^+) = -\frac{(\nu_u - \nu)}{\eta(1 - \nu)} \frac{a^2 p_0}{(b^2 - a^2)} \equiv p_E \qquad (66b)$$

The instantaneous stresses ($t = 0^+$) are now obtained from (63) and (58) as

$$\sigma_{rr} = p_0\left(\frac{a^2}{b^2 - a^2}\right)(1 - a^2/r^2) - p_0 a^2/r^2 \qquad (67a)$$

$$\sigma_{\theta\theta} = p_0\left(\frac{a^2}{b^2 - a^2}\right)(1 + a^2/r^2) + p_0 a^2/r^2 \qquad (67b)$$

We emphasize that (66) and (67) are valid only for $a < r < b$ and are independent of boundary conditions on pore pressure: especially, they do not apply right at the boundary, $r = a$, where $p = p_0$ by definition. The physical implication is that the short-time ($t = \epsilon$ in Figure 5) pore pressure distribution has a steep gradient from p_0 at the wall to the negative p_E in (66b).

Even after a very short time ϵ, there are points (at $r = R$) sufficiently close to the wall that the applied pore pressure $p = p_0$ has penetrated (e.g., choose $(R - a) << (4c\epsilon)^{1/2} << a$); σ_{rr} retains its value, $-p_0$, as does $C(0^+)$ in (66a). All of these, using (58), give

$$\sigma_{\theta\theta} = \left[\frac{2(1 - \nu_u)}{(1 - \nu)} \frac{a^2}{(b^2 - a^2)} + (1 - 2\eta)\right] p_0$$

$$\sigma_{rr} \equiv -p_0 \qquad (68)$$

As is familiar in thermoelastic problems of heat application to a surface, there is also a steep gradient in the total circumferential stress near the wall, from the value in (68) to that in (67b). In all these cases we have omitted σ_{33}: if this is needed, for instance in examining a generalized Coulomb hypothesis for failure at the wall, it is trivially computed from (19).

Steady flow, after 'very long' time. The situation here is that p and $C(t)$ become time independent in (59), of which the solution subject to (60b) and (61) then is

$$p(r, \infty) = p_0[\log (b/r)]/\log (b/a) \qquad a \le r \le b \qquad (69)$$

The value of $C(\infty)$ is now obtained from (64), after integration, as

$$C(\infty) = p_0(b^2 - a^2)^{-1} \cdot \left[(1 - \eta)a^2/2 + \frac{\eta}{4}(b^2 - a^2)/\log (b/a)\right] \qquad (70)$$

The complete stress field may be obtained from (63) and (58), but we have particular interest in the region near the inner wall, where

$$\sigma_{\theta\theta} = \left[\frac{2a^2(1 - \eta)}{(b^2 - a^2)} + \frac{\eta}{\log (b/a)} + (1 - 2\eta)\right] p_0$$

$$\sigma_{rr} = -p_0 \qquad (71)$$

Another possibility is that the outside ($r = b$) is jacketed, and thus no flow is allowed. If zero total radial stress is still applied there (experimentally, a constant jacketing pressure in the triaxial apparatus, so that for superposition of the effects of loading, the alteration in total radial stress is zero), then the final state must be $p = p_0$ everywhere, and (64) gives

$$C(\infty) = p_0[\eta + (b^2/a^2 - 1)^{-1}]/2 \qquad (72)$$

from which, again at the wall,

$$\sigma_{\theta\theta} = [1 + 2(b^2/a^2 - 1)^{-1}]p_0 \qquad \sigma_{rr} = -p_0 \qquad p = p_0 \qquad (73)$$

Complete solution for infinite outer radius. The special case $b/a \to \infty$ allows us, formally at least, to write the whole time and space variation of the variables σ_{rr}, $\sigma_{\theta\theta}$, and p because $C(t) = 0$ from conditions at infinity, and so (59), subject to boundary condition (60b), has the solution [*Carslaw and Jaegar*, 1960, section 127]

$$p = p_0 + \frac{2p_0}{\pi} \int_0^\infty \exp(-cu^2 t) \cdot \frac{J_0(ur) Y_0(ua) - J_0(ua) Y_0(ur)}{J_0^2(ua) + Y_0^2(ua)} \frac{du}{u} \qquad (74)$$

In the region $r > a$ such that $(r - a) << a$, the solution has the asymptotic expansion

$$p \simeq p_0(a/r)^{1/2} \operatorname{erfc} [(r - a)/(4ct)^{1/2}] \qquad (75)$$

where

$$\operatorname{erfc}(x) \equiv 1 - (2/\pi^{1/2}) \int_0^x \exp(-\rho^2)\, d\rho$$

This solution allows us to quantify and trace in time the progress of the zone of diffusing pore pressures as it penetrates inward from the boundary, replacing the pressure in (66b) by the imposed value p_0. As a first estimate, from (75), we may expect that after time $t = T$, the pore pressure will be $0.9p_0$ or greater up to a depth $(R - a) \simeq 0.2(cT)^{1/2}$. Incidentally, the complete stress field is obtained from inserting (74) into (63) and (58), with $C(t) \equiv 0$.

MATERIAL PARAMETERS AND INCEPTION OF HYDRAULIC FRACTURING

Laboratory tests on specimens of various rock types show that the elastic moduli G and ν vary strongly at high values of the hydrostatic effective stress [e.g., *Nur and Byerlee*, 1971] and, naturally, with increasing deviatoric stress [e.g., *Rummel*, 1974]; the porosity v_0 and the permeability k also vary with substantial changes in effective stress [e.g., *Zoback and Byerlee*, 1975]. However, it is appropriate for present purposes to list (Table 1) some typical rock properties at low to moderate effective stresses: these have been culled mainly from the work by *Rummel* [1974], *Nur and Byerlee* [1971], *Zoback and Byerlee* [1975], and *Haimson and Fairhurst* [1970]. Modifications from other sources were made when available, so that the numbers given may be considered average rather than applicable to a specific sample. Table 1 merits some comments:

1. The bulk modulus K_f of the pore fluid is representative of liquid, water or oil. The bulk modulus of the solid phase K_s is that of quartz for the sandstones; it has been measured [*Nur and Byerlee*, 1971] for the granites but is simply guessed for Tennessee marble. These particular rocks have been chosen either for their occurrence in earthquake test regions or for their use in hydraulic fracturing experiments.

2. The second section of the table contains the derived parameters, B, ν_u, η, and c (from equations (4), (6), (22), and (17), respectively), and it also contains an expression, κ_w, of the permeability measure commonly used in soil mechanics,

TABLE 1. Typical Rock Properties, Measured and Computed

Property	Ruhr Sandstone	Tennessee Marble	Charcoal Granite	Berea Sandstone	Westerly Granite	Weber Sandstone
	Section 1					
G, kbar	133	240	187	60	150	122
ν	0.12	0.25	0.27	0.20	0.25	0.15
v_0	0.02	0.02	0.02	0.19	0.01	0.06
k, md (10^{-11} cm^2)	0.2	(10^{-4})	(10^{-4})	190	4×10^{-4}	1.0
K_s, kbar	360	(500)	(454)	360	454	360
K_f, kbar	33	33	33	33	33	33
	Section 2					
B	0.88	0.51	0.55	0.62	0.85	0.73
ν_u	0.31	0.27	0.30	0.33	0.34	0.29
η	0.28	0.08	0.08	0.30	0.16	0.26
κ_w, cm/s	2×10^{-7}	10^{-10}	10^{-10}	2×10^{-4}	4×10^{-10}	10^{-6}
$(\mu/\mu_w)c$, cm^2/s	53	0.13	0.07	1.6×10^4	0.22	207

The viscosity of water is taken as $\mu_w = 0.01$ P and 1.0 bar ~ 10^6 dyn/cm^2.

namely, the velocity of flow of water under a pressure gradient of unit head drop of water per unit linear distance. For comparison, we note that clays of low permeability have $\kappa_w < 10^{-7}$ cm/s, so that the term 'impermeable' may seem appropriate to the granites and marble. However, what is really important, certainly for the fracturing and slip phenomena considered in this paper, is the rapidity with which pore pressure perturbations in some regions are damped out or transmitted to adjacent regions. It is clear from (47) and (75), or from any elementary diffusion solution, that this relation between distance (X, measured from source or perturbation) and the time elapsed (T) is given by $X = 0.2(cT)^{1/2}$, where the factor (0.2) arbitrarily arises from the requirement that the intensity of the change at X be 90% or more of the initial perturbation. If we, then, inspect the values for c in Table 1, we observe that even the marble gives values of $X \simeq 0.9$ mm for the passage of 1 s of time, when $\mu = \mu_w$. In a hydraulic fracturing experiment lasting at least 10 s, and even after allowing for an oily fracturing fluid penetrating dry rock, we shall suggest that sufficient penetration occurs to affect the pressures required for fracture.

3. A variety of earthquake-associated phenomena display a common value of $c \simeq 10^4$ cm^2/s [e.g., *Anderson and Whitcomb*, 1975]; the Weber sandstone, typical of the Rangely test region, shows only 2% of this value, and so joints or fissure networks (and some degree of dilatancy) must account for much of the diffusivity. Previously, computations have typically been based on diffusion through a rigid rock matrix: it is interesting to note that the limiting form of (17), in that event, is $c \rightarrow \kappa K_f/v_0$ (*Zoback and Byerlee* [1975] seem inadvertently to have used κK_f) and that this yields a value $c \simeq 550$ cm^2/s for Weber sandstone (with water as pore fluid).

4. The values of G and ν, listed in Table 1, are mainly those for dry rock. Ideally, they should be obtained from completely drained quasi-static tests on a sample saturated with the appropriate pore fluid, but experimentalists regard these as slow tests. In light of the values of c and the discussion in 2, it is hard to understand why induced pore pressures, due to loading a sample of maximum dimension 5 cm (say), do not effectively damp out in a matter of minutes.

Inception of hydraulic fracture from boreholes. It is frequently observed, when fracturing cylindrical rock specimens by hydraulically pressurizing a drilled central hole [e.g., *Haimson and Fairhurst*, 1970; C. B. Raleigh, personal communication, 1975], that very rapid pressurization (or a jacketed cavity wall) leads to a higher fracture pressure than that needed when pressurized fluid is allowed sufficient time to penetrate the walls of the cavity. We limit ourselves here to a preliminary simple explanation of this effect by proposing that fracture occurs when the maximum effective tensile stress, in the vicinity of the wall, reaches a so-called tensile strength σ_0. However, the expression for effective stresses is not necessarily the same for such an ultimate strength criterion as it is for deformation computations (equation (13)) but seems to be most accurately described [e.g., *Cornet and Fairhurst*, 1974] by the classical effective stress law, namely,

$$\sigma_{ij}' = \sigma_{ij} + p\delta_{ij} \tag{76a}$$

$$(\sigma_i')^{\max} = \sigma_0 \tag{76b}$$

The effective stresses are σ_{ij}', of which the principal values are σ_i', and (76b) is the chosen tensile stress fracture criterion. Such an empirical criterion is appropriate only if all flaws in the material are sufficiently small in comparison with distances over which the predicted stresses change appreciably, although the critical value σ_0 will itself have a statistical distribution according to the statistics of flaw sites, sizes, and the linking-up process.

The test configuration in Figure 5 is adopted as reference, and we note that the effects of any exterior confining pressure σ_{rr} ($r = b$) can be superposed in an obvious fashion, so we consider zero confining pressure. We now compute the fracture pressures for each of three different time scales of interior pressure application:

1. The pressure is brought up so rapidly to the fracture value p_F^i that the fluid does not penetrate into the cavity wall (equivalently, the wall $r = a$ may be jacketed). Then the elastic stress field in (67) and the pore pressure in (66b) may be used in (76) to obtain the maximum effective tensile stress, in the circumferential direction, near the wall; when fracture conditions are reached, this is

$$\sigma_{\theta\theta}' \equiv \sigma_{\theta\theta} + p$$
$$= \left[\left(\frac{b^2 + a^2}{b^2 - a^2}\right) - \frac{(\nu_u - \nu)}{\eta(1 - \nu)}\frac{a^2}{(b^2 - a^2)}\right]p_F^i = \sigma_0 \tag{77}$$

or

$$p_F{}^i \simeq 0$$

The last approximation is made for $b^2 \gg a^2$ (e.g., the specimen used by Raleigh had $b = 1.5$ cm, $a = 0.1$ cm), but there is naturally a dependence on b/a in the exact expression. This variation of $p_F{}^i$ with the size of hole used (for given b) seems to render unnecessary the explanation of *Haimson and Fairhurst* [1970], whose data agree well with the predictions of (77), that the variation is related to changes in tensile strength.

2. The fluid has enough time to penetrate the rock near $r = a$ to a depth sufficiently great that a fracture of that length results in a stress concentration adequate to continue propagating. Then the required fracture pressure ($p_F{}^s$) may be obtained by adopting $p_F{}^s$ as the pore pressure in the permeated region and using (68) to get

$$\sigma_{\theta\theta}' \equiv \sigma_{\theta\theta} + p = \left[\frac{2(1-\nu_u)}{(1-\nu)}\frac{a^2}{(b^2-a^2)} + 2(1-\eta)\right]p_F{}^s = \sigma_0 \quad (78)$$

or

$$2(1-\eta)p_F{}^s \simeq \sigma_0$$

Again this last approximation is for $b^2 \gg a^2$, and by setting the maximum effective tensile stress equal to the strength σ_0, we obtain a smaller fracture pressure, $p_F{}^s \simeq \sigma_0/(2-2\eta) \simeq p_F{}^i/(2-2\eta)$.

Actually, there will be a time dependence in the pressure required for fracture (as shown by the data of *Haimson and Fairhurst* [1970]), but the estimate just obtained constitutes a lower limit for $p_F{}^s$: the data of C. B. Raleigh (personal communication, 1975) support this assertion. Table 1 may be used to assess the factor $(2-2\eta)$, but (22*b*) gives $\eta = (1 - K/K_s')(1-2\nu)/(2-2\nu)$ when (4) and (6) are employed. This last expression shows that the very stiff rock matrices (e.g., marble and granite) will show a larger effect of fluid penetration than the more loosely structured sandstones. This expression for η is precisely that used by *Haimson and Fairhurst* [1970], but they assert that Tennessee marble and charcoal granite are too impermeable to show any effect of fluid penetration. Nevertheless, their data for fracturing pressures distinctly show an effect that is time-dependent in a manner consistent with a dependence on depth of fluid penetration at the wall $r = a$.

3. *Haimson and Fairhurst* [1970] show tests on Berea sandstone where the time to fracture was so long and the rock so permeable that the fluid pressure had penetrated the whole way to the outer boundary. Even where the sample has been initially dry, it is sensible to apply our 'very long time' solutions to these tests. Suppose, as is probably most appropriate, that $p(r = b)$ does not change during the test: then (71) may be used to find the maximum effective stress, induced by the fracture pressure $p_F{}^l$, which is also the pore pressure, at the inner wall $r = a$. We set this maximum tensile stress equal to the tensile strength σ_0 to find $p_F{}^l$:

$$\sigma_{\theta\theta} \equiv \sigma_{\theta\theta} + p \simeq \left[\frac{\eta}{\log(b/a)} + 2(1-\eta)\right]p_F{}^l \equiv \sigma_0 \quad (79)$$

$$b^2 \gg a^2$$

This gives a $p_F{}^l$ slightly lower than $p_F{}^s$, but the lowest possible $p_F{}^l$ actually arises in the (perhaps unrealistic) situation where an outer jacket prevents escape of pore fluid: then (73), with a pore pressure $p_F{}^l$ everywhere, gives a maximum effective stress

$$\sigma_{\theta\theta}' \equiv \sigma_{\theta\theta} + p \simeq 2p_F{}^l \equiv \sigma_0 \quad b^2 \gg a^2 \quad (80)$$

In summary, the pressures required to initiate fracture from central boreholes in relatively intact specimens of a variety of rocks have been shown to depend strongly on the length of time during which the hydraulic pressure in the fluid-filled cavity is raised to the fracture pressure. A more careful analysis of the exact nature of time-dependence is postponed, but the limits of two separate time scales have been studied: it is clear that the 'tensile strength' σ_0 may be modified to include initial compressive stresses on the prospective line of fracture and that initial pore pressures may be taken as reference so that our results extend directly to hydraulic fracturing endeavors (and we take $b^2 \gg a^2$, as appropriate to these). For instantaneous fracturing the pressure is $p_F{}^i \simeq \sigma_0$, but this decreases to $p_F{}^s \simeq \sigma_0/(2-2\eta)$ if the fluid can penetrate a sufficient distance that a fracture of that length will continue to propagate (see Table 1 for $\eta = (1-K/K_s')(1-2\nu)/(2-2\nu)$). Over a longer time scale, the fluid may penetrate very deeply into the specimen and (if b/a is not infinite) almost reach a steady state at which the fracture pressure $p_F{}^l$ is slightly less than $p_F{}^s$ (equation (79)); it seems, however, that the fraction $(2-2\eta)^{-1}$, which is $(1-\nu)$ if $K/K_s' \ll 1$, provides a reasonable lower limit for the drop in fracturing pressures to be expected with increasing time to fracture. The data of both *Haimson and Fairhurst* [1970] and C. B. Raleigh (personal communication, 1975) support our conclusions, although we do not dispense with mechanisms like stress corrosion as further factors influencing time dependence of fracture pressures.

SPHERICAL CAVITY IN A POROUS SOLID

An especially simple application of our equations to a three-dimensional problem arises in spherically symmetric idealizations of underground perturbations [e.g., *Anderson and Whitcomb*, 1975; *Johnson et al.*, 1973]. Consider a spherical cavity of radius a subjected to a total radial stress $\sigma_{rr} = -\sigma_R$ and fluid pressure $p = p_0$ on its boundary $r = a$. If stressing is due to pressurizing fluid that fills the cavity, then $\sigma_R = p_0$. Stress equilibrium requires (in obvious notation)

$$\partial\sigma_{rr}/\partial r + 2(\sigma_{rr} - \sigma_{\theta\theta})/r = 0 \quad (81)$$

while the only independent compatibility equation (12) reduces to

$$\frac{\partial}{\partial r}[(\sigma_{rr} + 2\sigma_{\theta\theta}) + 4\eta p] = Nr^{-2} \quad (82)$$

If we insist that strains are derivable from a purely radial displacement u_r, thereby excluding conical dislocations from which N derives, we can write $\epsilon_{rr} = \partial u/\partial r$ and $\epsilon_{\theta\theta} = u/r$ and deduce (82) with $N = 0$. Hence

$$(\sigma_{rr} + 2\sigma_{\theta\theta}) + 4\eta p = C_1(t) \quad (83)$$

By using this result in (16), that diffusion process for p simplifies to

$$c\left[\frac{\partial^2 p}{\partial r^2} + \frac{2}{r}\frac{\partial p}{\partial r}\right] = \frac{\partial p}{\partial t} + \frac{2(\nu_u - \nu)}{\eta(1-\nu_u)(1+\nu)}\frac{dC_1(t)}{dt} \quad (84)$$

In a manner now familiar, we insert (83) into (81) and integrate, subject to $\sigma_{rr}\ (r = a) = -\sigma_R$, to get

$$\sigma_{rr} = \tfrac{1}{3}(1 - a^3/r^3)C_1(t) - \sigma_R a^3/r^3 - (4\eta/r^3)\int_a^r \rho^2 p(\rho, t)\,d\rho \quad (85)$$

To finish, just as for the cylindrical cavity, (84) subject to boundary conditions must be solved for $p(r, t; C_1(t))$, and then the condition on σ_{rr} at the outer boundary produces, from (85), an integral equation for $C_1(t)$.

By specializing to $b/a \to \infty$, we find $C_1(t) = 0$, since the inner pressurization will lead to vanishing p, σ_{rr}, and $\sigma_{\theta\theta}$ at $r = \infty$. Equation (84) then has the simple solution [e.g., *Carslaw and Jaegar*, 1960]:

$$p = p_0(a/r) \operatorname{erfc} [(r - a)/(4ct)^{1/2}] \tag{86}$$

derived with initial condition $p(r > a, t = 0^+) = 0$, since $m = m_0$ in (8) and $C_1(t) = 0$ in (83). Equations (85) and (83) now give the stresses

$$\sigma_{rr} = -\sigma_R a^3/r^3 - 4\eta p_0(a/r^3) \int_a^r \rho \operatorname{erfc} [(\rho - a)/(4ct)^{1/2}] \, d\rho \tag{87}$$

$$\sigma_{\theta\theta} = -\tfrac{1}{2}\sigma_{rr} - 2\eta p_0(a/r) \operatorname{erfc} [(r - a)/(4ct)^{1/2}]$$

In the region near the wall of the cavity where the applied fluid pressure has penetrated (i.e., $(r - a)^2 << ct$, $a/r \simeq 1$), the pore pressure is p_0, but the radial total stress is still that applied, so we can use (83) to write the whole field:

$$p = p_0 \qquad \sigma_{rr} = -\sigma_R \qquad \sigma_{\theta\theta} = \sigma_R/2 - 2\eta p_0$$

and, for the purposes of fracture analysis, (76) gives for this region

$$\sigma_{rr}' = p_0 - \sigma_R \qquad \sigma_{\theta\theta}' = \sigma_R/2 + (1 - 2\eta)p_0$$

On the other hand, outside the zone affected by boundary pore pressure but still at sufficiently short times that such points can be close to the wall, the stresses are ($p = 0$)

$$\sigma_{rr} = \sigma_{rr}' = -\sigma_R \qquad \sigma_{\theta\theta} = \sigma_{\theta\theta}' = \sigma_R/2$$

For the case $\sigma_R = p_0$, the internal pressure required to cause fracture will fall by 45% (for $\eta = 0.30$) to 63% ($\eta = 0.08$) when sufficient infiltration of fluid occurs to a sufficient depth that a fracture of that length can propagate.

CONCLUDING DISCUSSION

There were at the outset of this study few significant elasticity solutions available for the deformation of fluid-infiltrated porous solids, especially in the case of fully compressible fluid and solid constituents. The present work provides some such solutions and may also prove useful, through the formalisms developed, in the pursuit of further basic solutions. Indeed, there is a wide range of porous media problems involving, e.g., constitutive nonlinearities [*Biot*, 1973; *Rice*, 1975], local pore pressure nonequilibrium [*O'Connell and Budiansky*, 1974], anisotropy [*Biot*, 1955, 1956*b*], etc., but the simple linear isotropic models seem by no means exhausted either as to identification, even of primary aspects, of porous media effects or as to availability of convenient analytical formulations.

When inhomogeneities, complex geometries, etc. are considered unavoidable, then a numerical approach may be possible. For instance, a finite element scheme based on a variational principle analogous to that of *Biot* [1956*b*] may solve quite awkward problems and does not limit one to simplified constitutive representations. Attempts have been made to implement such a method with incompressible constituents [e.g., *Valliappan et al.*, 1973]. But when discontinuities and infinite regions are present and one is concerned with isolating dominant characteristics only, it is preferable to have analytic solutions. Thus our dislocation and shear fault solutions reveal at least three ways in which pore fluid flow can control shear fault motion and give the potential to trace the time-dependent progress of frictional faults. The radially symmetric solutions in plane strain give new evidence on hydraulic fracturing processes and suggest some simple experiments related to conventional annular specimens used in tensile strength tests. All solutions can be adapted to complicated loading history by a standard superposition integral.

Acknowledgments. Our study was supported by the Geophysics Program of the National Science Foundation, under grant GA-43380, and by the NSF Materials Research Laboratory at Brown University. We are grateful to Donald A. Simons and John W. Rudnicki for useful discussions.

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(Received October 14, 1975;
accepted December 5, 1975.)