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CONDITIONS FOR THE LOCALIZATION OF DEFORMATION IN PRESSURE-SENSITIVE DILATANT MATERIALS

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Summary

This paper investigates the hypothesis that localization of deformation into a shear band may be considered a result of an instability in the constitutive description of homogeneous deformation. General conditions for a bifurcation, corresponding to the localization of deformation into a planar band, are derived. Although the analysis is general and applications to other localization phenomena are noted, the constitutive relations which are examined in application of the criterion for localization are intended to model the behavior of brittle rock masses under compressive principal stresses. These relations are strongly pressure-sensitive since inelasticity results from frictional sliding on an array of fissures; the resulting inelastic response is dilatant, owing to uplift in sliding at asperities and to local tensile cracking from fissure tips. The appropriate constitutive descriptions involve non-normality of plastic strain increments to the yield hyper-surface. Also, it is argued that the subsequent yield surfaces will develop a vertex-like structure. Both of these features are shown to be destabilizing and to strongly influence the resulting predictions for localization by comparison to predictions based on classical plasticity idealizations, involving normality and smooth yield surfaces. These results seem widely applicable to discussions of the inception of rupture as a constitutive instability.

1. Introduction

Zones of localized deformation, in the form of narrow shear bands, are a common feature of brittle rock masses that have failed under compressive principal stresses, both in laboratory experiments (e.g. Brace, 1964, Wawersik and Fairhurst, 1970, and Wawersik and Brace, 1971) and, naturally, as earth faults. It is possible that such behavior can be explained only by modelling in detail the processes of growth and interaction of the many individual fissures that ultimately join together in forming the rubble-like macroscopic surface of rupture. *However, we investigate here an alternative hypothesis: That localization can be understood as an instability in the macroscopic constitutive description of inelastic deformation of the material.* Specifically, instability is understood in the sense that the constitutive relations may allow the homogeneous deformation of an initially uniform material to lead to a bifurcation point, at which non-uniform deformation can be incipient in a planar band under conditions of continuing equilibrium and continuing homogeneous deformation outside the zone of localization.

An explanation of this kind has been proposed by Berg (1970) for the inception of rupture in ductile metals owing to the nucleation and progressive growth of microscopic voids, and Rice (1973) has given a formulation for localization in connection with shear band formation in overconsolidated clay soils. The general theoretical framework for such localizations is given by HILL (1962), who investigates them in connection with the special case of a stationary acceleration wave. Indeed, our present considerations are applicable to a variety of localization phenomena in the mechanical behavior of materials. However, the specific constitutive relations that we propose in implementation of the criterion for localization are motivated by the behavior of compressed rock masses in a pressure and temperature regime appropriate to brittle behavior. In this case, the apparent macroscopic inelastic strain arises from frictional sliding on microscopic fissures accompanied by further local tensile cracking from fissure tips and local uplifts in sliding at asperities. The latter features lead to macroscopic dilatancy; also, the frictional nature of sliding causes the criterion for continued inelastic deformation to be strongly pressure-sensitive.

These circumstances are such that 'normality' (i.e. an associated flow rule) cannot reasonably be assumed for the inelastic strain increments. Also, as we shall demonstrate, the physical mechanisms dictate a vertex-like structure to the yield hyper-surface separating elastic from inelastic response. In contrast to classical elastic-plastic theories, founded on the associated flow rule and an assumption of smooth yield surfaces, both of these features are destabilizing and affect significantly the predicted conditions for onset of localization.

Although the view of the inception of rupture as a constitutive instability is not new, it has remained relatively unexplored. HILL's (1952) analysis of the initiation of localized necking in thin, ductile sheets can be considered as a prototype of such studies, in that necks can be viewed as constitutive instabilities for a two-dimensional continuum. THOMAS (1961) later derived general geometrical and kinematical conditions for moving surfaces of discontinuity and applied these to problems of instability of solids. The investigation, however, considered only specific geometries and the classical Mises model of an incompressible, non-hardening solid. HILL (1962), as part of the study of acceleration waves noted above, investigated the degenerate case of a 'stationary discontinuity' which has the character of a planar zone of localization. He considered inelastic materials characterized, for a given constitutive branch, by a linear relation between strain and conjugate stress rate. However, he limited attention to materials satisfying normality in conjugate variables and he made assumptions equivalent to the neglect of vertex-like effects. As remarked, both of these issues turn out to be pivotal to the character of predicted response. A recent paper by TOKUOKA (1972) addresses conditions for instability without making contact with this prior work and fails to take proper account of kinematical conditions on the bifurcating field.

While the general formulations of THOMAS (1961) and HILL (1962) may be used as a starting point for the present analysis, we follow (Section 2) the direct quasi-static formulation of localization conditions by RICE (1973); this gives an equivalent criterion that is immediately applied to the form in which we propose (Section 3) constitutive relations. The relations themselves are of two kinds. First, a simple isotropic-hardening model is proposed based on the first and second stress invariants. This incorporates inelastic dilatancy and hydrostatic stress dependence of the yield criterion in a form intended to model natural rock; it enables a simple assessment of the role of the most obvious kind of non-normality (pressure dependence of yield, not fully matched by inelastic dilatancy) in destabilizing the material. A more elaborate constitutive model is also proposed based on viewing compressed, brittle rock masses as elastic bodies having random arrays of fissures with frictional resistance to sliding. It is shown that this leads to a vertex-like structure of the inelastic flow law and an approximate constitutive model is proposed for 'fully active' deformation increments from it. Vertex effects are found to drastically alter predictions of localization based on similar constitutive models entailing smooth yield surfaces.

The predictions, by application of the localization criterion, are given in Section 4. Calculations are simplified through a perturbation expansion for the critical instantaneous hardening rate, in terms of the ratio *representative stress/elastic modulus*. The result given by the zero-th order terms is equivalent to replacing a proper co-rotational rate of stress in the constitutive law by an ordinary time rate; the first order correction to the predicted critical hardening rate is also derived. Apart from the specific relevance to localization processes in compressed rock masses, the conclusions drawn here would seem to be of general interest to the theories of plasticity and rupture, not least for drawing attention to the destabilizing roles of non-normality and vertex effects. One or both of these may appear in constitutive descriptions for a wide variety of materials and material behavior. For example, there are substantial grounds (e.g. Hill (1967)) for expecting similar vertex effects in metal plasticity.

2. Formulation

In this section, the method for addressing bifurcation conditions is presented, and a general criterion for the localization instability is given. Consider an initially uniform deformation field achieved by uniform stressing of a homogeneous material. Conditions are sought for which continued deformation may result in an incipient non-uniform field in which deformation rates vary with position across a planar band but remain uniform outside the band. Introduce rectangular cartesian coordinates x_i ($i = 1, 2, 3$), such that the x_2-direction is normal to the planes bounding the band (Fig. 1). Since the velocities themselves remain continuous, at least initially, derivatives of velocity in the x_1- and x_3-directions, parallel to the band, remain

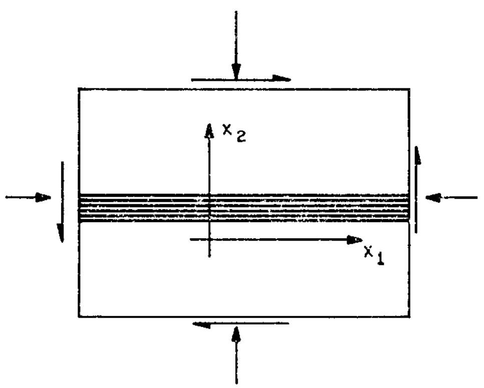

Fig. 1. Coordinate system for the band of localization.

uniform. Thus, any non-uniformities in the rate of deformation field are kinematically restricted to the form

$$\Delta(\partial v_i/\partial x_j) = g_i(x_2)\delta_{j2}, \quad (i, j) = 1, 2, 3, \tag{1}$$

where v_i is a velocity component, Δ denotes the difference between the local field at a point and the uniform field outside the band, and the functions g_i of x_2 are non-zero only within the band.

It is also required that stress equilibrium continues to be satisfied at the inception of bifurcation. In the total stress and rate forms this is expressed by

$$\partial\sigma_{ij}/\partial x_i = 0 \quad \text{and} \quad \partial(\partial\sigma_{ij}/\partial t)/\partial x_i = \partial(\dot{\sigma}_{ij} - v_k\, \partial\sigma_{ij}/\partial x_k)/\partial x_i = 0$$

respectively, where σ_{ij} is the stress field and the superposed dot denotes its material time rate. Thus, the stress rates at incipient localization from the presumed uniform field satisfy

$$\partial\dot{\sigma}_{ij}/\partial x_i = 0. \tag{2}$$

It is clear from (1) that the stress rates will be functions only of x_2 within the band, and will be uniform outside the band. Consequently, (2) will be satisfied if and only if $\dot{\sigma}_{2j}$ ($j = 1, 2, 3$) is the same both inside and outside the band. Hence, the condition for continuing equilibrium is

$$\Delta\dot{\sigma}_{2j} = 0, \quad j = 1, 2, 3, \tag{3}$$

where Δ has the same meaning as before. The stress rates will themselves be expressible in terms of velocity gradients, by the constitutive law, and localization can occur at the first point in the deformation for which non-zero g's exist, satisfying (1) and (3).

Since $\dot{\sigma}_{ij}$ is not invariant under rigid rotations, it is more convenient to introduce the Jaumann (co-rotational) stress rate for use in constitutive laws; this is defined as (PRAGER, 1961)

$$\overset{\triangledown}{\sigma}_{ij} = \dot{\sigma}_{ij} - \sigma_{ip}W_{pj} - \sigma_{jp}W_{pi},$$

where W_{ij} is the antisymmetric part of the velocity gradient tensor $\partial v_j/\partial x_i$. The ΔW's are readily expressed in terms of the g's from (1), and thus (3) becomes

$$\left.\begin{aligned} \Delta\overset{\triangledown}{\sigma}_{21} &= -\tfrac{1}{2}(\sigma_{22} - \sigma_{11})g_1 + \tfrac{1}{2}\sigma_{13}g_3, \\ \Delta\overset{\triangledown}{\sigma}_{22} &= \sigma_{21}g_1 + \sigma_{23}g_3, \\ \Delta\overset{\triangledown}{\sigma}_{23} &= -\tfrac{1}{2}(\sigma_{22} - \sigma_{33})g_3 + \tfrac{1}{2}\sigma_{31}g_1. \end{aligned}\right\} \tag{4}$$

The constitutive law will relate the $\Delta\overset{\triangledown}{\sigma}_{ij}$ to the g_i's, and, in general, to the uniform field outside the band, in such a way that the $\Delta\overset{\triangledown}{\sigma}$'s vanish when the g's vanish. With this, (4) can be viewed as a set of 3 quasi-homogeneous equations in g_1, g_2, g_3 and the conditions for bifurcation are merely those for which solutions other than $g_1 = g_2 = g_3 = 0$ exist. Some special cases are discussed by RICE (1973); for the present we suppose that the stress rate and rate of deformation

$$D_{ij} = \tfrac{1}{2}(\partial v_i/\partial x_j + \partial v_j/\partial x_i)$$

are related by

$$\overset{\triangledown}{\sigma}_{ij} = L_{ijkl}D_{kl},$$

where the modulus tensor L_{ijkl} is unspecified except that $L_{ijkl} = L_{jikl}$ and $L_{ijkl} = L_{ijlk}$. If, at the bifurcation of deformation-rates, the values L_{ijkl} remain the same inside and outside the band, the following difference can be formed:

$$\Delta\overset{\nabla}{\sigma}_{ij} = L_{ijkl}\Delta D_{kl} = L_{ijk2}\, g_k. \tag{5}$$

In this case, the g's are uncoupled to the outside field.

After using (4), equation (5) yields

$$L_{2jk2}\, g_k = R_{jk}\, g_k, \quad j = 1, 2, 3, \tag{6}$$

where $R_{jk}g_k$ represents the right sides of (4). Since (6) is a set of linear, homogeneous equations in the g's, the condition for bifurcation is that

$$\det |L_{2jk2} - R_{jk}| = 0. \tag{7}$$

This is equivalent to the condition given by HILL (1962) for a stationary discontinuity. The treatments differ in that Hill considers the constitutive relation to be phrased primitively in terms of time rates of contravariant Kirchhoff stress components on materially convected coordinates. Also, he expresses equilibrium rate conditions in terms of the nominal (or first Piola–Kirchhoff) stress s_{ij}. This leads, in place of (3), to the requirement $\Delta\dot{s}_{ij} = 0$ when the instantaneous state coincides with the reference state for nominal stress. The result is entirely equivalent to (3) given the kinematical condition (1); this may be shown formally by operating with Δ on the $2j$-component of the equation that relates the rates:

$$\left.\begin{aligned} \dot{s}_{ij} &= \dot{\sigma}_{ij} + \sigma_{ij}\, \partial v_k/\partial x_k - (\partial v_i/\partial x_k)\sigma_{kj}, \\ \Delta\dot{s}_{2j} &= \Delta\dot{\sigma}_{2j} + \sigma_{2j}\, g_k \delta_{k2} - g_2 \delta_{k2}\sigma_{kj} = \Delta\dot{\sigma}_{2j}. \end{aligned}\right\}$$

The physical origin of the equivalence lies with the vanishing of the Δ-differences in the rates of stretching and rotation of the planar elements upon which s_{2j} acts.

3. Constitutive Relations

Elastic–plastic constitutive relations are proposed in this section in a form appropriate to the behavior of brittle rock under compressive principal stresses. Rock stressed in this way is classified as brittle when the primary modes of inelastic behavior are frictional sliding on fissure surfaces and microcracking. The classification depends not only on the particular type of rock but also on the pressure–temperature regime. Ductile rock behavior, by contrast, seems to involve dislocation processes analogous to those of metal plasticity and is of less interest in the present context because deformation then tends to be stable, rather than exhibiting the unstable shear-zone localizations that can occur in brittle rock typifying crustal conditions.

A typical stress–strain curve displaying the essential features of brittle rock behavior in the triaxial compression test is shown in Fig. 2(a). The curve divides itself into four regions: (I), in which it is slightly convex upward; (II), a nearly linear portion; (III), a non-linear region of decreasing slope; and (IV), in which a maximum is reached and the curve decreases. In Fig. 2(b), the corresponding stress/volumetric strain curve is shown. The non-linearity in region (I) is due to the elastic closing of cracks, and unloading during regions (I) or (II) is essentially elastic and produces little hysteresis. Region (III) begins with the initiation of dilatant volume increase and non-linearity due to microcrack growth and frictional sliding on microcrack surfaces. The onset of (III) is dependent on the amount of hydrostatic stress, since

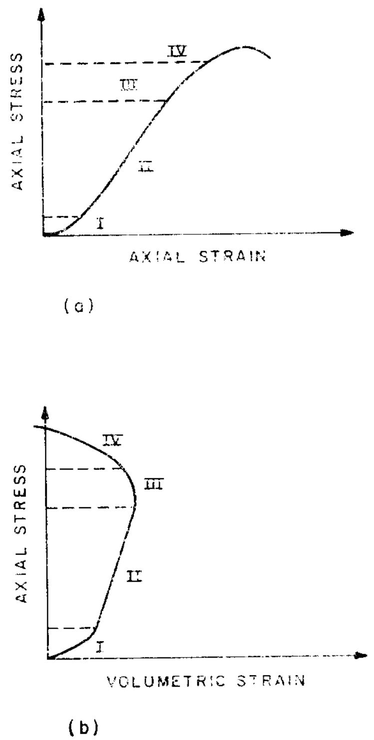

FIG. 2. Typical stress–strain curve for uniaxial compression of brittle, crystalline rock.

normal stress across crack surfaces increases resistance to frictional sliding and inhibits microcrack growth. The initiation of region (IV) is less clearly defined, but it is marked by accelerated microcrack growth and rapid increase of dilatant volume change leading to failure. The extent and details of the post-peak region are left unspecified. In (III) and (IV), the slope is decreasing and unloading produces large hysteresis. As shown in Fig. 2(b), inelastic volume change in these regions is comparable to, or exceeds, elastic volume changes in magnitude, but is opposite in sign. (See BRACE (1964), JAEGER and COOK (1969), and WALSH and BRACE (1973).)

Although the mechanisms of inelastic deformation in brittle rock differ from those of metal plasticity, the essential macroscopic characteristics described above can be idealized satisfactorily by elastic and work-hardening (or -softening) plastic relations, if inelastic volume changes and dependence of the yield surface on the hydrostatic stress are included in the formulation.

3.1 *A simple isotropic-hardening constitutive relation*

We present here a generalization, in the spirit of the Prandtl–Reuss equations, of the elementary forms of constitutive laws typically used in soil and rock mechanics. In order to make evident the structure of the generalization and to emphasize the relationship to the more simple laws, we first consider the case of a material element under a hydrostatic stress σ (positive in compression) and a shear stress τ (Fig. 3). The heavy curve through the current stress state in Fig. 3(a) is the yield 'surface'.

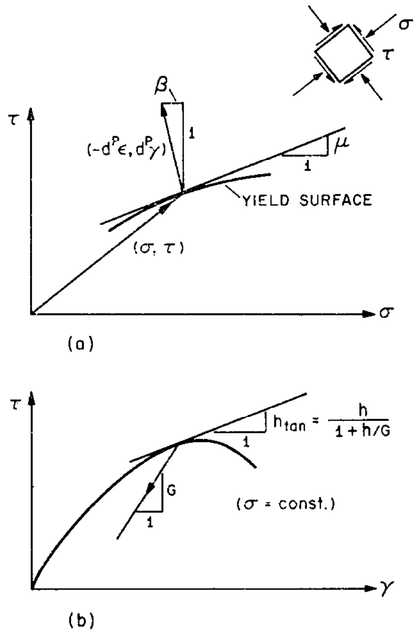

FIG. 3. (a) A portion of a yield surface showing the geometric interpretation of the coefficient of internal friction μ and the dilatancy factor β. (b) Curve of shear stress *vs.* shear strain showing the geometric interpretation of the hardening modulus h, the tangent modulus h_{tan}, and the elastic shear modulus G.

and its slope μ defines the *internal friction coefficient*. Deformation increments tending to make $d\tau > \mu\, d\sigma$ correspond to further plastic loading, whereas those with $d\tau < \mu\, d\sigma$ correspond to elastic unloading. Let the resulting shear and volumetric strain increments be $d\gamma$, $d\varepsilon$, and let the elastic shear and bulk moduli governing elastic unloading be G, K, respectively. Then, the plastic portions of the strain increments are defined by

$$d\varepsilon = -(d\sigma/K) + d^p\varepsilon, \qquad d\gamma = (d\tau/G) + d^p\gamma.$$

Ratios of the plastic strain increments are taken to be fixed by the current state, and we write

$$d^p\varepsilon = \beta d^p\gamma. \tag{8}$$

β is the *dilatancy factor*, identified in Fig. 3(a) in which the plastic strain increment appears as a vector. Figure 3(b) shows a representative τ *vs.* γ curve at fixed σ. During plastic response, $d\tau = h\, d^p\gamma$ for constant σ, and $d\tau - \mu\, d\sigma = h\, d^p\gamma$ in general, where the plastic hardening modulus is h. This modulus is related to the tangent modulus as shown, and differs from it only slightly when $h/G \ll 1$.

Thus, the relations between stress and strain increments are taken to be

$$d\gamma = (d\tau/G) + (d\tau - \mu\, d\sigma)/h, \qquad d\varepsilon = -(d\sigma/K) + \beta(d\tau - \mu\, d\sigma)/h \tag{9}$$

during plastic response, whereas the latter terms are dropped during elastic unloading. It may be observed that the normality rule of classical plasticity corresponds to $\beta = \mu$ but, owing to the frictional origin of the deformation resistance, this would be too restrictive for describing the inelastic response of rocks.

We generalize the foregoing to arbitrary stress-states by making the identifications $\sigma = -\frac{1}{3}\sigma_{kk}$ and $\tau = \bar{\tau}$, where

$$\bar{\tau} = (\tfrac{1}{2}\sigma'_{ij}\sigma'_{ij})^{\frac{1}{2}}$$

(a prime $'$ on a tensor denotes its deviatoric part), and by assuming that ratios of the plastic parts of the components D'_{ij} are equal to ratios of the corresponding components of σ'_{ij}. Further, the elasticity is treated as isotropic and the spin-invariant Jaumann stress rate is employed. Thus, the plastic shear loading parameter is

$$\dot{\tau} - \mu\dot{\sigma} = \frac{\sigma'_{kl}}{2\bar{\tau}}\overset{\triangledown}{\sigma}_{kl} + \frac{\mu}{3}\overset{\triangledown}{\sigma}_{kk}$$

and (9) becomes

$$\left.\begin{aligned} 2D'_{ij} &= \frac{\overset{\triangledown}{\sigma}{}'_{ij}}{G} + \frac{1}{h}\frac{\sigma'_{ij}}{\bar{\tau}}\left[\frac{\sigma'_{kl}}{2\bar{\tau}}\overset{\triangledown}{\sigma}_{kl} + \mu\frac{\overset{\triangledown}{\sigma}_{kk}}{3}\right], \\ D_{kk} &= \frac{\overset{\triangledown}{\sigma}_{kk}}{3K} + \frac{\beta}{h}\left[\frac{\sigma'_{kl}}{2\bar{\tau}}\overset{\triangledown}{\sigma}_{kl} + \mu\frac{\overset{\triangledown}{\sigma}_{kk}}{3}\right], \end{aligned}\right\} \tag{10}$$

during plastic response.

In order to estimate the frictional and dilational parameters, μ and β, equation (10) can be fitted to the special case of axially-symmetric compression specimens, with lateral confining pressures, as normally used in rock and soil stress–strain experiments. In this way we find from the data of BRACE, PAULDING and SCHOLZ (1966) on Westerly granite and aplite, and of BIENIAWSKI (1967) on norite and quartzite, that the pressure dependence of the onset of inelastic response (region III, Fig. 2) is described by μ-values ranging from 0·4 to 0·9. These seem consistent with sliding friction coefficients in mineral-to-mineral contact. Fitting the pressure dependence of the peak stress observed in the tests suggests a higher range of μ (0·9 to 1·3); this is somewhat suspect because it does not represent the pressure dependence of yield after a *single* deformation history, but rather a pressure dependence among a group of specimens, each brought to maximum deviatoric stress conditions under different confining pressure levels. BRACE *et al.* (1966, Fig. 6) plot dilatant strain against a measure of deviatoric strain, and from this a representative range of β is inferred as 0·2 to 0·4. The β-value seems to increase by approximately a factor of two from the inception of yield to conditions near failure; also, the value diminishes somewhat with increasing confining pressure.

As for the incremental elastic moduli G and K, these diminish progressively with inelastic deformation. For example, the stiff-machine unloading tests of WAWERSIK and FAIRHURST (1970) suggest the Young's modulus in the axially-symmetric compression test diminishes slightly, to approximately 0·96 of its initial value, near maximum load and to fractions ranging from 0·9 to 0·5 in the regime of general failure on the descending portion of the stress–strain curve of Fig. 2.

Although the development of the constitutive law (10) has been from the viewpoint of describing rock behavior, it also represents other types of elastic–plastic behavior. For example, if $\beta = \mu$, (10) could be used to describe dilatant metal plasticity in the form considered by BERG (1970) for a model of ductile rupture. In this case, dilatancy arises from plastic hole growth on the microscale and instability is interpreted as ductile fracture initiation. For $\beta = \mu = 0$, the usual form of the Prandtl–Reuss elastic–plastic relation is recovered.

In describing the inelastic deformation of *fluid saturated* rocks, the stress in (10) (and the normal stress σ in the simplified version of the constitutive law) is to be the *effective* stress, defined as

$$\bar{\sigma}_{ij} = \sigma_{ij} + p\delta_{ij},$$

where p is the pore fluid pressure. Under *undrained* deformation conditions (no change of fluid mass content) the tendency to dilation induces suction in the pore fluid, thereby causing $\bar{\sigma}$ to increase in a compressive sense even when σ is constant. This results in an *apparent* hardening modulus significantly greater than that for *drained* deformation (p = constant). The phenomenon has been analyzed by RICE (1975) for a special deformation state, simple shear, based on constitutive relations similar to (9). He shows also that such *dilatantly hardened* deformation becomes unstable, in the sense that locally induced D'Arcy pore fluid fluxes cause initial non-uniformities in the amount of shear to grow exponentially with time, when the *underlying drained response* has become unstable in the sense discussed here. Thus, our present considerations for non-fluid-infiltrated materials may be regarded as applicable as well to the inception of rupture in the undrained deformation of fluid-saturated, dilatantly hardened rock provided that any induced suctions are viewed as an added hydrostatic stress variation in the stressing program prior to localization. Indeed, the characterization of the state of an inelastically deforming, fluid-saturated rock mass just prior to localization is central to the interpretation of premonitory events in the Earth's crust prior to faulting (e.g. NUR (1972), SCHOLZ *et al.* (1973), and ANDERSON and WHITCOMB (1975)).

It should be recognized that the constitutive law (10), and other simple generalizations of (9) to arbitrary stress-states, will be most suitable under sustained stressing without abrupt alterations in the 'direction' of deformation. Indeed, as is well known in the corresponding isotropic-hardening formulation for metal plasticity, the stiffness of response may be substantially overestimated for stress increments directed 'tangential' to what is taken as the current yield surface (i.e. increments $d\sigma_{ij}$ for which $d\bar{\tau} = 0$), at least by comparison to more elaborate crystalline-slip-based constitutive models (HILL (1967) and HUTCHINSON (1970)). In Section 3.2, this deficiency is examined more closely and more elaborate constitutive models based on corresponding 'yield vertex' effects for fissured rock masses are discussed.

3.2 *Yield-vertex constitutive model for fissured rock masses*

The simple constitutive model just described is adequate for studying the effect of non-normality on the criterion for localization. But, as has been suggested above, the predictions of localization are very sensitive to the exact structure of the (D_{ij} *vs.* $\overset{\triangledown}{\sigma}_{ij}$)-relationship, and particularly to whether a vertex-like structure exists. In this section a randomly oriented fissure model, similar in spirit to the simple slip model of BATDORF and BUDIANSKY (1949) for metal plasticity, is introduced to motivate a more elaborate constitutive description. As will be seen, this leads to the formation of a yield vertex, and the resulting prediction of conditions for localization differs substantially from that for the isotropic-hardening model, even when the two constitutive models are made to correspond exactly for proportional stressing programs.

We idealize brittle rock as containing a collection of randomly oriented fissures of the type shown in Fig. 4. The orientation is given by the unit normal $\underset{\sim}{n}$; slip directions in the plane are given by $\underset{\sim}{m}$, where, in general, $\underset{\sim}{m}$ will be the direction of greatest shear stress on the fissure. It is assumed that the hydrostatic stress is initially great enough to have closed all fissures and that inelastic deformation results from frictional sliding on the fissure surfaces. Sliding causes dilation by opening the fissure

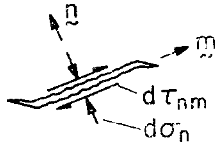

FIG. 4. Model of a fissure. Normal $\underset{\sim}{n}$, slip direction $\underset{\sim}{m}$. Resolved shear stress $d\tau_{nm}$, normal stress $d\sigma_n$.

at asperities and by inducing local tensile fractures at some angle to the fissure. Thus, it is reasonable to assume that the dilation arising from sliding on a given fissure bears some fixed relation, expressed again by the parameter β, to the inelastic shear strain. The stress increment governing sliding is $d\tau_{nm}-\mu\, d\sigma_n$, where σ_n is the normal stress (positive in compression), τ_{nm} is the resolved shear stress in the slip direction on the plane of the fissure, and μ is the friction coefficient. These local stress increments are related to the macroscopic stress increments by

$$d\tau_{nm} = n_i m_j\, d\sigma_{ij} = n_i m_j\, d\sigma'_{ij}, \qquad d\sigma_n = -n_i n_j\, d\sigma_{ij} = -n_i n_j\, d\sigma'_{ij} - \tfrac{1}{3} d\sigma_{kk},$$

and thus the increment governing sliding on a given fissure of parameters $\underset{\sim}{m}$, $\underset{\sim}{n}$ is

$$d\tau_{nm} - \mu\, d\sigma_n = (n_i m_j + \mu n_i n_j)\, d\sigma'_{ij} + (\mu/3)\, d\sigma_{kk}.$$

Thus, an individual yield surface in stress space may be associated with each fissure. This passes through stress states which are sufficient to initiate sliding on that fissure and has the equation $d\tau_{nm} - \mu\, d\sigma_n = 0$; it will be a plane in stress space if, as in the slip theory of metal plasticity, $\underset{\sim}{m}$ is fixed in direction, but it will be somewhat curved in the present case since $\underset{\sim}{m}$ itself will be dependent on the direction of greatest resolved shear stress. Hence, at each stage of the deformation, the current macroscopic yield surface is the envelope of the essentially infinite number of individual current yield surfaces for fissures of all possible orientations (BATDORF and BUDIANSKY, 1949, and HILL, 1967).

It is straightforward to see how such a model leads to the formation of a vertex on the current macroscopic yield surface after some amount of inelastic deformation. Consider a given macroscopic deviatoric ($d\sigma_{kk} = 0$) stress increment. The value of the fissure loading parameter $d\tau_{nm}$ will vary according to orientation, being greatest for the most favorably oriented fissure. Continued stressing in the same direction will cause continued sliding on already activated fissure surfaces and the initiation of sliding for a progressively greater number of orientations. A stress increment in another direction will cause a different orientation to be most favorable and consequently a different preferential activation of other fissures. Schematically, this can be represented by depicting the stress history as a series of vectors in stress space. Sliding is initiated for each orientation for which the vector passes through the asso-

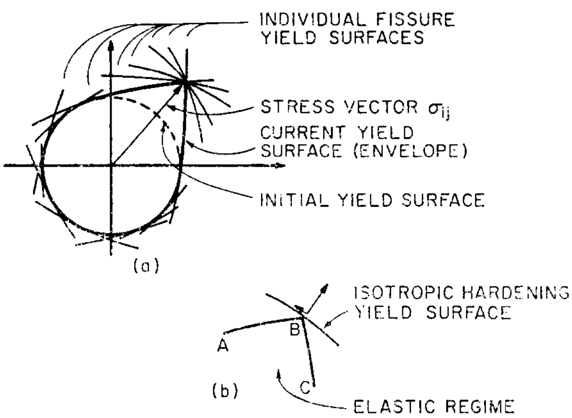

FIG. 5. (a) Depicted in stress space is a macroscopic yield surface formed as an envelope of individual fissure yield surfaces. The formation of a vertex on the current yield surface, due to sliding on favorably oriented fissure surfaces, is also shown. (b) A stress increment from a vertex on the yield surface. Components of the stress increment directed tangential and normal to the corresponding isotropic-hardening yield surface are shown.

ciated yield surface (Fig. 5). Because it is this preferential activation of the fissures, with respect to orientation, which leads to the formation of a vertex, no vertex can result from hydrostatic stressing, at least if μ is the same for all fissures. An increment of hydrostatic stress ($d\sigma'_{ij} = 0$) has the same effect on every fissure orientation. Thus, the yield surface associated with each fissure is moved to or from the origin by the same amount, causing only smooth changes in the envelope.

The discussion makes clear the reason that the isotropic-hardening idealization overestimates the stiffness of response to stress increments directed tangential to what is taken as the current yield surface in that idealization. In the fissure model, such stressing at a pointed vertex will initiate or cause continued plastic loading for some orientations. Hence, the response is not purely elastic, as predicted by the isotropic-hardening model. Rather, it is intermediate between the soft plastic response to 'straight ahead' stressing, governed by h in (10), and the stiff elastic response, governed by G and K.

Guided by the predictions of the randomly-oriented fissure model, we seek to modify the isotropic-hardening constitutive law (10) in a manner which preserves, as much as possible, its mathematical simplicity, but which more accurately represents the behavior at a vertex. In a recent paper, SEWELL (1974) has considered the hardening effects due to a pyramidal vertex. We take a different approach and introduce, in addition to the plastic hardening modulus h in (10) governing 'straight ahead' stressing, a second hardening modulus h_1 ($h < h_1$) which governs the response to that part of a stress increment directed tangentially to what is taken as the yield surface in the isotropic-hardening idealization. Since it has been argued that no vertex is associated with hydrostatic stress increments, these tangential increments are taken as purely deviatoric, and can be expressed in rate form as

$$\overset{\triangledown}{\sigma}{}'_{ij} - \frac{\sigma'_{ij}}{\bar{\tau}} \left[\frac{\sigma'_{kl}}{2\bar{\tau}} \overset{\triangledown}{\sigma}_{kl}\right],$$

where the second term merely subtracts off the normal component of the stress increment. Note that the inner product formed from this expression and $\sigma'_{ij}/\bar{\tau}$ (the portion of (10_1) governed by h) is zero. Dividing the above expression by h_1 and affixing it to (10_1) gives the deviatoric component of the rate of deformation, modified to approximate behavior at a vertex:

$$2D'_{ij} = \frac{\overset{\nabla}{\sigma}{}'_{ij}}{G} + \frac{1}{h}\frac{\sigma'_{ij}}{\bar{\tau}}\left[\frac{\sigma'_{kl}}{2\bar{\tau}}\overset{\nabla}{\sigma}_{kl} + \mu\frac{\overset{\nabla}{\sigma}_{kk}}{3}\right] + \frac{1}{h_1}\left[\overset{\nabla}{\sigma}{}'_{ij} - \frac{\sigma'_{ij}\sigma'_{kl}}{2\bar{\tau}^2}\overset{\nabla}{\sigma}_{kl}\right]. \tag{11}$$

This can be used to model the response to stress increments differing by only a small amount from the 'straight ahead' direction; the corresponding relations cannot be linear (*vs.* homogeneous of degree one) for the full range of directions.

In discussing the fissure model, it was argued that the dilation was equal to the shear strain caused by sliding multiplied by the parameter β. In formulating (10), a similar relation was generalized to

$$D^p_{kk} = \beta(2(D'_{ij})^p(D'_{ij})^p)^{\frac{1}{2}}, \tag{12}$$

where the superscript p denotes the inelastic portion of the rate-of-deformation components. However, because of the additional term in (11) due to the inelastic deformation induced by the tangential increment of stressing, this expression becomes exceedingly unwieldy. At this point we assume, consistently with the use of (11), that the tangential component of the stress increment is small. This enables the dilation to be expressed by a more convenient incrementally linear form. To obtain a linear expression for the dilation, substitute for $(D'_{ij})^p$ in (12) from (11) and use the binomial expansion. Retaining the first two terms gives

$$D^p_{kk} = \frac{\beta}{h}\left[\frac{\sigma'_{kl}}{2\bar{\tau}}\overset{\nabla}{\sigma}_{kl} + \mu\frac{\overset{\nabla}{\sigma}_{kk}}{3}\right]\left\{1 + \frac{1}{2}\left(\frac{h}{h_1}\right)^2\left[\overset{\nabla}{\sigma}{}'_{ij} - \frac{\sigma'_{ij}\sigma'_{kl}}{2\bar{\tau}^2}\overset{\nabla}{\sigma}_{kl}\right]^2\Big/\left[\frac{\sigma'_{kl}}{2\bar{\tau}}\overset{\nabla}{\sigma}_{kl} + \mu\frac{\overset{\nabla}{\sigma}_{kk}}{3}\right]^2\right\}.$$

The second term in parenthesis contains the square of the ratio of tangential to normal component of the stress increment, the square of the former being understood as an inner product. Since this ratio has been assumed to be small and it is multiplied by $(h/h_1)^2$, also a small fraction of unity, it follows that this term may be neglected with respect to unity. Hence, the dilation induced by the tangential stress increment is assumed to be negligible, and the modified constitutive law may be written

$$\left.\begin{aligned} 2D'_{ij} &= \frac{\overset{\nabla}{\sigma}{}'_{ij}}{G} + \frac{1}{h}\frac{\sigma'_{ij}}{\bar{\tau}}\left[\frac{\sigma'_{kl}}{2\bar{\tau}}\overset{\nabla}{\sigma}_{kl} + \mu\frac{\overset{\nabla}{\sigma}_{kk}}{3}\right] + \frac{1}{h_1}\left[\overset{\nabla}{\sigma}{}'_{ij} - \frac{\sigma'_{ij}\sigma'_{kl}}{2\bar{\tau}^2}\overset{\nabla}{\sigma}_{kl}\right], \\ D_{kk} &= \frac{\overset{\nabla}{\sigma}_{kk}}{3K} + \frac{\beta}{h}\left[\frac{\sigma'_{kl}}{2\bar{\tau}}\overset{\nabla}{\sigma}_{kl} + \mu\frac{\overset{\nabla}{\sigma}_{kk}}{3}\right], \end{aligned}\right\} \tag{13}$$

The effect of incorporating the modulus h_1 is illustrated by noting that (13) may be obtained from (10) by making the following identifications:

$$\left.\begin{aligned} &(1/G) \to (1/G)+(1/h_1), && \beta \to \beta(1-(h/h_1))^{-1}, \\ &(1/h) \to (1/h)-(1/h_1), && \mu \to \mu(1-(h/h_1))^{-1}. \\ &(1/K) \to (1/K)-(\beta\mu/h_1)(1-(h/h_1))^{-1}, && \end{aligned}\right\}$$

4. Conditions for Localization

4.1 *Based on isotropic-hardening constitutive model*

In this section, the condition for instability (7) is used with the constitutive laws (10) and (13) to derive expressions for the critical hardening modulus and for the plane of localization. Results are obtained first for the simple isotropic-hardening model (10), and then compared with those obtained by using the more realistic constitutive law (13).

Equation (10) can be inverted to yield

$$\overset{\triangledown}{\sigma}_{kl} = \left\{ G(\delta_{mk}\delta_{nl}+\delta_{ml}\delta_{kn}) + (K-\tfrac{2}{3}G)\delta_{kl}\delta_{mn} - \frac{\left(\frac{G}{\bar{\tau}}\sigma'_{kl}+\beta K\delta_{kl}\right)\left(\frac{G}{\bar{\tau}}\sigma'_{mn}+K\mu\delta_{mn}\right)}{h+G+\mu K\beta} \right\} D_{mn}. \quad (14)$$

Since this is of the form (5), the condition for localization can be written as (7):

$$\det |M_{ij}| = 0, \quad (15)$$

where

$$M_{ij} \equiv L_{2ij2} - R_{ij},$$

and

$$L_{2ij2} = G(\delta_{i2}\delta_{2j}+\delta_{ij})+(K-\tfrac{2}{3}G)\delta_{2i}\delta_{j2} - \frac{\left(\frac{G}{\bar{\tau}}\sigma'_{2i}+\beta K\delta_{2i}\right)\left(\frac{G}{\bar{\tau}}\sigma'_{j2}+K\mu\delta_{j2}\right)}{h+G+\mu K\beta}.$$

In order to simplify the expressions which will be derived from (15), we adopt, at this point, the assumption that the R_{ij}, the terms in M_{ij} introduced by the co-rotational stress rate, are negligible, and, as a result, $\overset{\triangledown}{\sigma}_{ij} \approx \dot{\sigma}_{ij}$. More explicitly, this involves neglecting terms of magnitude *stress divided by shear modulus* in an expression for the critical value of h/G. But it is expected that, under certain circumstances, the predicted value of h/G for localization may itself be near zero, and the neglected terms could then be important. Thus, in a later section, this assumption will be made more precise by developing the critical hardening modulus in terms of a perturbation expansion in the *stress/elastic modulus* ratio; the development here is of the zero-th order term.

The condition (15) relates the hardening modulus at localization to the incremental constitutive parameters G, K, μ, β and to the prevailing stress-state. Neglecting the R_{ij} in M_{ij}, evaluating (15), and solving for the hardening modulus yields

$$\frac{h}{G+\mu K\beta} = \frac{(G\sigma'_{22}+\beta K\bar{\tau})(G\sigma'_{22}+\mu K\bar{\tau})+(\tfrac{4}{3}G+K)G(\sigma_{21}^2+\sigma_{23}^2)}{\bar{\tau}^2(\tfrac{4}{3}G+K)(G+\mu K\beta)} - 1. \quad (16)$$

This value of h is a function of the orientation of the potential plane of localization, and we wish to seek, for a fixed state of stress, the orientation of the plane for which the localization criterion is first met. This can be done precisely only when the detailed variation of each constitutive parameter with the deformation is specified. But it is expected that h will vary in a substantially more rapid fashion than G, K, μ

or β, and thus, because h is a decreasing function of the amount of strain, we seek the orientation for which the value of h is a maximum.

To establish a frame of reference, axes corresponding to the principal stresses $\sigma_{\rm I}$, $\sigma_{\rm II}$ and $\sigma_{\rm III}$ are introduced, with $\sigma_{\rm I} \geq \sigma_{\rm II} \geq \sigma_{\rm III}$. Let n_K (K = I, II, III) denote the corresponding components in the principal directions of the unit normal to the plane of localization. Then, the stresses in (16) can be rewritten in terms of the principal stresses as

$$\left.\begin{aligned} \sigma'_{22} &= n_{\rm I}^2\sigma'_{\rm I} + n_{\rm II}^2\sigma'_{\rm II} + n_{\rm III}^2\sigma'_{\rm III}, \\ \sigma_{21}^2 + \sigma_{23}^2 &= n_{\rm I}^2\sigma_{\rm I}'^2 + n_{\rm II}^2\sigma_{\rm II}'^2 + n_{\rm III}^2\sigma_{\rm III}'^2 - \sigma_{22}'^2. \end{aligned}\right\} \tag{17}$$

After substituting into (16), the orientation of the plane of localization can be obtained by requiring h to be a maximum with respect to the n_K (some details of this calculation are given in Appendix I).

If the principal stresses are distinct, the normal to the critical plane of localization is perpendicular to the direction of $\sigma_{\rm II}$ if the inequality

$$2\sigma'_{\rm I} - \sigma'_{\rm II} - \sigma'_{\rm III} > 2\bar{\tau}(\beta+\mu) \tag{18}$$

is satisfied and to that of $\sigma_{\rm III}$ if it is not.

The deviatoric stresses occurring in this and other formulae may be expressed in terms of $\bar{\tau}$ and a *single* stress-state parameter N defined by $\sigma'_{\rm II} = N\bar{\tau}$. This is particularly attractive since this parameter is zero for pure shear and takes on its maximum and minimum values ($N = \pm 1/\sqrt{3}$) for axially-symmetric compression ($\sigma_{\rm I} = \sigma_{\rm II} > \sigma_{\rm III}$) and extension ($\sigma_{\rm I} > \sigma_{\rm II} = \sigma_{\rm III}$), respectively. With N, the inequality above becomes

$$3(1-3N^2/4)^{\frac{1}{2}} - 3N/2 > 2(\beta+\mu),$$

and hence a conservative bound assuring that $\mathbf{n}$ is perpendicular to the $\sigma_{\rm II}$-direction is

$$\sqrt{3}/2 > \beta+\mu,$$

whereas the condition

$$\sqrt{3} < \beta+\mu$$

assures that $\mathbf{n}$ is perpendicular to the $\sigma_{\rm III}$-direction. Given the range of β- and μ-values noted earlier, this latter case would seem to be exceptional. Indeed, the necessity for condition (18) may solely be an artifact of the isotropic-hardening idealization. The approximate calculation with the constitutive law (13) indicates that $\mathbf{n}$ is normal to the intermediate principal direction for all values of β and μ which were considered, except when h_1/G takes on extreme values (see Appendix II).

Because the normal to the critical plane is always perpendicular to a principal direction, there exists a direction in the localized band for which there is no shear and for which the associated g_i (see (1)) is zero. If two of the principal stresses are equal, only the angle between the normal to the critical plane and the third of the principal axes is uniquely determined. The direction of the projection of the normal on the plane of the two equal principal stresses is arbitrary. Therefore, the result that there exists a direction of no shear in the plane of localization is unchanged.

Although the possibility that (18) may be violated has not been ruled out, this will occur for relatively large values of β and μ. For definiteness, it will be assumed, in the following, that (18) is satisfied and that the normal to the plane of localization is perpendicular to the $\sigma_{\rm II}$-direction. If the normal is perpendicular to $\sigma_{\rm III}$, then it is only necessary to exchange the subscripts II and III in the relevant formulae. With this assumption, we may write

$$n_{\mathrm{I}} = \sin\theta, \qquad n_{\mathrm{II}} = 0, \qquad n_{\mathrm{III}} = \cos\theta,$$

where θ is the angle between the normal and the σ_{III}-direction. The angle θ_0 which maximizes h, and hence defines the plane of localization, is given by

$$\tan\theta_0 = ((\xi - N_{\min})/(N_{\max} - \xi))^{\frac{1}{2}}, \tag{19}$$

where $\xi = (1+\nu)(\beta+\mu)/3 - N(1-\nu)$, ν is Poisson's ratio, and $N_{\max}$, N, and $N_{\min}$ are $\sigma'_{\mathrm{I}}/\bar{\tau}$, $\sigma'_{\mathrm{II}}/\bar{\tau}$, and $\sigma'_{\mathrm{III}}/\bar{\tau}$, respectively. The corresponding value of the hardening modulus then takes a simple form:

$$\frac{h_{\mathrm{cr}}}{G} = \frac{1+\nu}{9(1-\nu)}(\beta-\mu)^2 - \frac{1+\nu}{2}\left(N + \frac{\beta+\mu}{3}\right)^2, \tag{20}$$

from which the role of the constitutive parameters and stress state are readily discerned in determining the critical modulus.

4.2 *Discussion of predicted hardening-rate at localization*

The expressions (19) and (20) are most easily interpreted for deviatoric states of pure shear stress ($N = 0$). For $\beta = \mu = 0$, they yield the expected result: h_{cr} is zero and localization occurs on the plane whose normal is 45° from the principal stress directions. For all cases in which $\beta = \mu$, h_{cr} is *non-positive*. However, if normality is *not* obeyed, the value of the hardening modulus at localization can, in general, be *positive*.

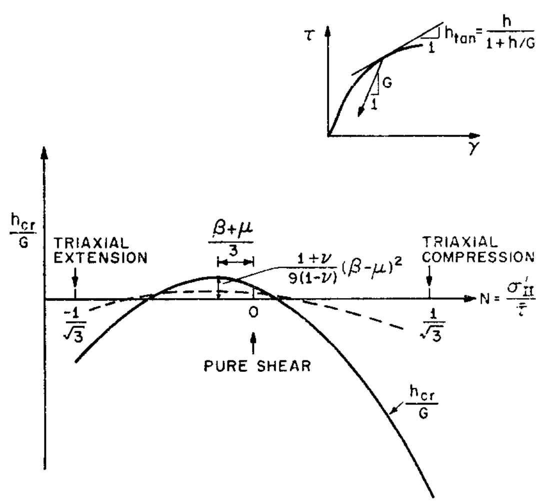

FIG. 6. Schematic showing the variation of h_{cr}/G with N, and (dashed line) the estimate from the modified constitutive law. (Not to scale.)

The variation with N of h_{cr} and θ_0 is shown in Figs. 6 and 7 for representative values of β and μ. Also, in Table 1, h_{cr}/G at instability and θ_0 are tabulated for various stress-states and for various values of β and μ. Since β and μ appear symmetrically in (19)–(21), only a few values of each parameter are shown.

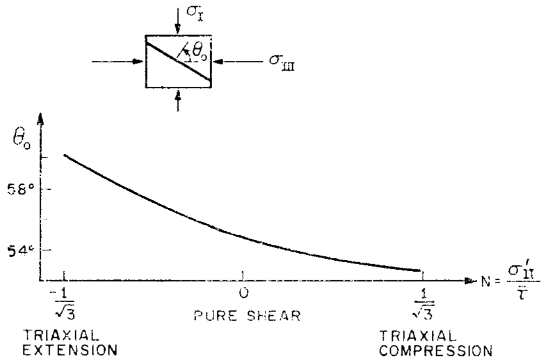

FIG. 7. Variation of θ_0 with N. (Actual graph for $\beta = 0{\cdot}3$, $\mu = 0{\cdot}5$, $\nu = 0{\cdot}3$.)

TABLE 1. *Values of* h_{cr}/G *at instability for various stress-states* (θ_0 *in degrees is given in parenthesis*). *Based on the constitutive law* (10). $h_{cr}/G = (1+\nu)(\beta-\mu)^2/9(1-\nu) - (1+\nu)[N+(\beta+\mu)/3]^2/2$, *except as noted below.* $N = \sigma'_{II}/\bar{\tau}$, $\nu = 0{\cdot}3$

μ	β	Axially-symmetric extension $N = -1/\sqrt{3}$	Maximum value $N = -(\mu+\beta)/3$	Pure shear $N = 0$	Axially-symmetric compression $N = 1/\sqrt{3}$
0	0	−0·216(48·7)	0 (45·0)	0 (45·0)	−0·216(41·2)
0·3	0	−0·129(53·1)	0·018(49·2)	0·012(48·7)	−0·281(45·3)
0·3	0·15	−0·115(55·4)	0·005(51·6)	−0·010(50·7)	−0·339(47·5)
0·3	0·3	−0·092(57·8)	0 (53·9)	−0·026(52·4)	−0·392(49·7)
0·6	0	−0·018(57·8)	0·074(53·9)	0·048(52·4)	−0·318(49·7)
0·6	0·15	−0·028(60·3)	0·042(56·3)	0·001(54·3)	−0·405(51·8)
0·6	0·3	−0·032(62·7)	0·018(58·6)	−0·040(56·3)	−0·456(90·0)†
0·6	0·45	−0·029(65·6)	0·005(61·6)	−0·075(58·5)	−0·417(90·0)†
0·6	0·6	−0·020(68·7)	0 (64·7)	−0·104(60·5)	−0·370(90·0)†
0·9	0	0·116(62·8)	0·166(58·9)	0·107(56·4)	−0·380(90·0)†
0·9	0·15	0·082(65·6)	0·115(61·7)	0·035(58·5)	−0·305(90·0)†
0·9	0·3	0·053(68·6)	0·074(66·5)	−0·030(60·6)	−0·295(90·0)†
0·9	0·45	0·031(72·0)	0·042(68·5)	−0·090(62·8)	−0·280(90·0)†
0·9	0·6	0·015(76·0)	0·018(73·1)	−0·144(65·2)	−0·259(90·0)†

† For these cases, h_{cr} is given with $N_{min} = \sigma'_{III}/\tau$ substituted for N.

From (20), it is obvious that the maximum value of the critical hardening modulus, over all stress states, occurs for $N = -(\beta+\mu)/3$, and is given by

$$h_{max} = \frac{G}{9}\frac{(1+\nu)}{(1-\nu)}(\beta-\mu)^2. \tag{21}$$

This illustrates a fundamental difference in stability characteristics between materials which obey normality and those which do not. If normality is satisfied ($\beta = \mu$), then localization is possible only at values of the hardening modulus which are non-positive. On the other hand, if normality is *not* satisfied, and if the stress state is sufficiently close to that given by $N = -(\beta+\mu)/3$, then localization can occur with positive hardening, under circumstances for which the pre-bifurcation stress history corresponds to steadily rising loads. A basic conclusion is that fissured rock and other inelastic solids not satisfying normality are more inclined to instability, by localization of deformation, than are materials for which the normality rule holds. Further, while there has been a long association in the plasticity literature between normality and some forms of uniqueness for hardening materials, this seems to be the only case in which the converse is examined, in that non-normality is shown specifically to allow non-uniqueness with positive hardening.

As has been noted, however, the isotropic-hardening model is not without its defects. For example, although the present writers have been unable to find any precise measurement of conditions at the onset of localization in brittle rock or soils, the strongly negative values of h_{cr} predicted for localization in axially-symmetric (or 'triaxial') compression (Fig. 6 and Table 1) must be regarded as suspect. While it has been reported that localization ('faulting') in triaxial specimens can occur after maximum load (WAWERSIK and BRACE, 1971, and WAWERSIK and FAIRHURST, 1970), the slopes seem to be considerably less negative than predicted. Certainly, the decrease of the incremental elastic shear modulus G owing to progressive fissuring reduces accordingly the actual negative slope associated with the calculated negative values of h_{cr}/G, but nevertheless the situation of the triaxial test must be regarded as one for which inadequacies of the isotropic-hardening model are likely to dominate. Indeed, the over-prediction of material stiffness in response to abrupt changes in the direction of stressing, as discussed earlier, is apparently the central factor in prediction of the strongly negative values of h at the extreme values of N.

4.3 *Localization criterion based on yield–vertex constitutive model*

The same analysis has been performed using (13) to illustrate that more realistic predictions for h_{cr} result for triaxial compression if a constitutive law is used for which stiffnesses for tangential deformation directions are more comparable to those for sustained straining in the original deformation direction. Because the resulting formulae, unlike (19)–(21), are decidedly opaque, they are relegated to Appendix II, in which the calculation is outlined. Table 2 gives h_{cr}/G and θ_0 for various states of stress and values of β and μ (β and μ again appear symmetrically), and a schematic for the approximate variation of h_{cr}/G with N is sketched in Fig. 6. For the calculations reported in Table 2, h_1/G was taken to be 0·1; Table 3 records the variation of h_{cr}/G for h_1/G ranging from 0·05 to 1·0. Inspection of this Table and the formulae of Appendix II indicates that h_{cr}/G varies roughly linearly with h_1/G for small values of the latter and it approaches the values predicted using the isotropic-hardening model for large values of h_1/G.

We again remark that h_1 could be determined more precisely from a slip theory analysis. In particular, in the Batdorf–Budiansky slip model for metal plasticity, h_1 is the plastic secant modulus in pure shear ($h_1 = \tau/\gamma^p$), whereas the modulus $h = d\tau/d\gamma^p$. Thus, h_1/G is the ratio of the elastic to plastic parts of the shear strain and becomes very small with continuing inelastic deformation. Given the similarity in motivation of the constitutive model, an *approximately* similar interpretation might be assumed here; its precise specification or estimation will require a great deal of further study.

TABLE 2. *Values of* h_{cr}/G *at instability for various stress-states* (θ_0 *in degrees is given in parenthesis*). *Based on the constitutive law* (13). $N = \sigma'_{II}/\bar{\tau}$, $\nu = 0{\cdot}3$, $h_1/G = 0{\cdot}1$

μ	β	Axially-symmetric extension $N = -1/\sqrt{3}$	Pure shear $N = 0$	Axially-symmetric compression $N = 1/\sqrt{3}$
0	0	−0·029(45·4)	0 (45·0)	−0·029(44·6)
0·3	0	−0·017(49·6)	0·002(49·3)	−0·037(48·2)
0·3	0·15	−0·015(51·6)	−0·001(51·4)	−0·045(50·0)
0·3	0·3	−0·013(53·8)	−0·003(53·6)	−0·052(51·7)
0·6	0·0	0·001(55·5)	0·008(54·4)	−0·042(51·6)
0·6	0·15	−0·001(57·4)	0·002(56·4)	−0·054(53·3)
0·6	0·30	−0·003(59·3)	−0·003(58·5)	−0·064(55·0)
0·6	0·5	−0·003(61·9)	−0·007(60·7)	−0·074(56·6)
0·6	0·6	−0·003(64·9)	−0·011(63·1)	−0·083(58·2)
0·9	0·0	0·035(71·9)	0·022(62·3)	−0·043(55·0)
0·9	0·15	0·028(73·5)	0·013(64·2)	−0·058(56·8)
0·9	0·30	0·021(75·2)	0·004(66·0)	−0·072(58·4)
0·9	0·45	0·015(77·3)	−0·005(68·0)	−0·086(59·9)
0·9	0·6	0·009(79·9)	−0·012(70·1)	−0·098(61·5)

TABLE 3. *Variation of* h_{cr}/G *with* h_1/G *for various stress-states* (θ_0 *in degrees is given in parenthesis*). $N = \sigma'_{II}/\bar{\tau}$, $\nu = 0{\cdot}3$

μ	β	h_1/G	Axially-symmetric extension $N = -1/\sqrt{3}$	Pure shear $N = 0$	Axially-symmetric compression $N = 1/\sqrt{3}$
0·6	0·0	0·05	0·001(55·4)	0·005(54·6)	−0·022(51·6)
		0·10	0·001(55·5)	0·008(54·4)	−0·042(51·6)
		0·50	−0·001(56·1)	0·025(53·7)	−0·137(51·0)
		1·0	−0·004(56·5)	0·033(53·3)	−0·192(50·7)
0·6	0·3	0·05	−0·001(59·3)	−0·002(58·6)	−0·034(55·0)
		0·10	−0·003(59·5)	−0·003(58·5)	−0·064(55·0)
		0·50	−0·011(60·5)	−0·012(57·8)	−0·209(54·8)
		1·0	−0·017(61·1)	−0·019(57·5)	−0·292(54·6)
0·6	0·6	0·05	−0·001(64·7)	−0·006(63·3)	−0·044(58·1)
		0·10	−0·011(63·1)	−0·011(63·1)	−0·083(58·2)
		0·50	−0·009(66·1)	−0·038(62·4)	−0·270(58·4)
		1·0	−0·012(66·8)	−0·056(61·9)	−0·376(58·5)

Comparing the results from Tables 2 and 3 with those of Table 1 reveals that the values of h_{cr} are reduced in magnitude by a factor on the order of h_1/G. This reduction does not change the essential character of the results for pure shear in that h_{cr} is still a small (or very small) fraction of G and may be positive for certain combinations of μ and β. On the other hand, the results in Tables 2 and 3 for triaxial compression and, to a lesser extent, triaxial extension, represent substantial changes from the large negative values in Table 1. Thus, it does seem that the isotropic-hardening idealization may adequately predict the critical hardening modulus for states of stress near pure shear, but that it grossly underestimates the value for triaxial compression and extension. It should be remembered that (13) was formulated under the assumption of small tangential stress increments, and that no fully general model of constitutive response at a vertex is yet available. We note also, from the tabulated results in Table 2, that localization occurs with a positive hardening-rate only when normality does not apply ($\mu \neq \beta$) and then again only for a limited range of stress states.

4.4 *Corrections for the co-rotational terms*

In the previous sections it was assumed that $\overset{\nabla}{\sigma}_{ij} \approx \dot{\sigma}_{ij}$. It will be shown that this assumption amounts to retaining the first term in an expansion in the *stress/elastic modulus* ratio. Additional terms may be retained as necessary. Explicit expressions for the first order term are developed. The procedure is carried out in detail only for the isotropic-hardening law (10).

Since it has been shown that there exists a direction in the plane of localization for which one of the g_i's is zero, without loss of generality, we can take $g_3 \equiv 0$, and consider the reduced matrix

$$M'_{ij} = \begin{bmatrix} 1+\dfrac{\sigma_{22}-\sigma_{11}}{2G}-\dfrac{G\sigma_{21}^2}{\bar{\tau}^2(h+G+\mu K\beta)} & \dfrac{-\sigma_{21}(G\sigma'_{22}+\mu K\bar{\tau})}{\bar{\tau}^2(h+G+\mu K\beta)} \\ \dfrac{-\sigma_{21}(G\sigma'_{22}+\beta K\bar{\tau})}{\bar{\tau}^2(h+G+\mu K\beta)}-\dfrac{\sigma_{21}}{G} & \dfrac{(K+\frac{4}{3}G)}{G}-\dfrac{(G\sigma'_{22}+\beta K\bar{\tau})(G\sigma'_{22}+\mu K\bar{\tau})}{G\bar{\tau}^2(h+G+\mu K\beta)} \end{bmatrix}.$$

M'_{ij} has been formed from M_{ij} by deleting the third row and the third column. Using the condition

$$\det |M'_{ij}| = 0$$

and solving for the hardening modulus yields

$$h = h_0 + h_*, \tag{22}$$

where

$$h_* = \frac{\sigma_{21}^2}{\bar{\tau}^2}\,\frac{1}{\left[1+\dfrac{\sigma_{22}-\sigma_{11}}{2G}\right]}\left[\frac{\sigma_{11}-\sigma_{22}}{2}+\frac{1-2\nu}{2(1-\nu)}\sigma'_{22}+\mu\frac{\bar{\tau}}{3}\frac{(1+\nu)}{(1-\nu)}\right],$$

and h_0 is the value of the hardening modulus computed by neglecting the co-rotational terms and given by (16). We can write $h_0 = h_0(\theta)$, $h_* = h_*(\theta)$ where θ is the angle between the normal to a prospective plane of localization and the σ_{III}-axis, as earlier.

To determine the change in orientation of the plane of localization, a linear expansion of the condition $h'(\theta) = 0$ was used, where the prime $'$ denotes the derivative with respect to θ. Expanding the function $h'(\theta)$ about θ_0 of (19), and noting that $h'_0(\theta_0) = 0$, the condition, to first order, can be written as

$$h'(\theta) = h''_0(\theta_0)(\theta-\theta_0)+h'_*(\theta_0)+h''_*(\theta_0)(\theta-\theta_0) = 0.$$

This results in

$$\theta = \theta_0 + \frac{\bar{\tau}}{G}\left[(\mu-\beta)\frac{1+\nu}{6}\frac{\cos 2\theta_0}{\sin 2\theta_0} - \tfrac{1}{4}(1-\tfrac{3}{4}N^2)^{\frac{1}{2}}\sin 2\theta_0\right] + O\left(\frac{\bar{\tau}}{G}\right)^2. \tag{23}$$

The critical hardening-rate can be written by expanding (22) in similar fashion. Because $h_0'(\theta_0) = 0$ and both h_*/G and $\theta-\theta_0$ are order $\bar{\tau}/G$, we can express h_{cr} as $h_0(\theta_0)+h_*(\theta_0)$, where the terms are of respective order G and $\bar{\tau}$, and where the neglected terms are order $(\bar{\tau}/G)^2$ and smaller. Thus, one finds that (20) is amended to

$$\frac{h_{cr}}{G} = \frac{1+\nu}{9(1-\nu)}(\beta-\mu)^2 - \frac{1+\nu}{2}\left(N+\frac{\beta+\mu}{3}\right)^2 + \\ + \frac{(4-3N^2)(1+\nu)}{24(1-\nu)}(\mu-\beta)\sin^2 2\theta_0 \frac{\bar{\tau}}{G} + O\left(\frac{\bar{\tau}}{G}\right)^2. \quad (24)$$

It is interesting to note that inclusion of the co-rotational stress-rate terms now causes β and μ to enter in an unsymmetrical manner.

It is seen that in the case of normality, $\beta = \mu$, the effect of the co-rotational terms on both θ and h_{cr} is zero to first order. However, in general, the inclusion of the co-rotational terms introduces a correction in the critical hardening modulus on the order of a typical stress component and a correction in the orientation of the most probable plane of localization on the order of *stress over shear modulus*. In short, the simple formula (20) merits continued expansion only when the critical hardening modulus as predicted is so small as to be comparable to a representative stress level.

5. CONCLUDING DISCUSSION

Although the constitutive laws have been developed from the point-of-view of describing brittle rock under compressive principal stresses, the analysis raises some issues that seem fundamental to plasticity theory at large. The importance of yield surface vertices in altering incremental deformation predictions as based on a smooth yield surface has long been recognized, and its relevance to structural bifurcation analyses has been clear to at least some workers (see HUTCHINSON (1974) for discussion). In the present paper, a physically motivated constitutive law has been formulated that suggests the presence of a vertex and approximates incremental response from it. By performing the localization calculation with this law, it has been demonstrated that substantial alterations in the predictions of h_{cr} result from vertex-like behavior. Furthermore, while there is a long standing association in plasticity literature between normality and uniqueness in hardening materials, we have specifically investigated the effect of deviations from normality. In particular, it has been shown that non-normality permits non-uniqueness with positive hardening. This is of obvious importance in the deformation of brittle rock and other geological materials in which the plasticity arises from frictional sliding and local tensile fissuring, and normality cannot reasonably be expected. That localization can occur for positive, although small, values of the tangent modulus indicates a tendency for brittle rock to deform by localizing deformation. This is particularly significant in that the instability we have discussed is a limiting one (e.g. HILL (1962)), in the sense that local non-uniformities, for instance in pore pressure or crack content, may cause localization before the onset of the instability described here; it may also be preceded by geometrical instabilities analogous to neck formation in the extension test.

Indeed, when applied to stressed regions on a tectonic scale, the analysis discussed here may well be relevant to the inception of localized shear in the form of unstable earth faulting. The problem is greatly complicated by heterogeneities in rock properties and pre-rupture stress distributions. But the analysis suggests that, depending on the nature of the prevailing stress-state (characterized here by N) and on the constitutive parameters μ, β, and h_1, localization may set in either while the rock mass is continually hardening ($h_{cr} > 0$) or when it is past the peak strength and progressively softening ($h_{cr} < 0$). Premonitory events in the region which ultimately faults are detectable by alterations in seismic and electrical properties (e.g. NUR (1972), SCHOLZ, SYKES and AGGARWAL (1973), and ANDERSON and WHITCOMB (1975)), and these should be explainable in terms of the extent to which the rock has deformed into the inelastic range prior to the localization instability. It is possible that distinctly different patterns of premonitory signals will result when localization occurs *before* vs. *after* the peak stress is attained; also, for fluid-saturated rock such as that adjacent to faults in the upper crust, the instability analysis must be carried out with due attention to dilatant hardening effects (RICE, 1975). Perhaps the least understood constitutive feature, and that most influential on predictions of localization, is the structure of incremental stress–strain relations at a yield vertex. There seems at present to be an absence of relevant experimental studies on this in the rock mechanics literature.

More generally, the results of this analysis lend support to the hypothesis that the inception of rupture can be modelled as a constitutive instability. As remarked in the Introduction, investigation of this point-of-view has been limited, and we would argue that it merits greater attention than that given thus far. At the same time, it must be recognized that, generally, the predictions will be strongly dependent on subtleties of the incremental constitutive description which are themselves not well understood.

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REFERENCES

ANDERSON, D. L. and WHITCOMB, J. H. 1975 *J. Geophys. Res.* **80**, 1497.

BATDORF, S. B. and BUDIANSKY, B. 1949 N.A.C.A. Tech. Note 1871.

BERG, C. A. 1970 *Inelastic Behavior of Solids* (edited by KANNINEN, M. F., ADLER, W. F., ROSENFIELD, A. R. and JAFFEE, R. I.), p. 171. McGraw-Hill, New York.

BIENIAWSKI, Z. T. 1967 *Int. J. Rock Mech. Min. Sci.* **4**, 407.

BRACE, W. F. 1964 *State of Stress in the Earth's Crust* (edited by JUDD, W. R.), p. 111. Elsevier, New York.

BRACE, W. F., PAULDING, B. W., Jr. and SCHOLZ, C. 1966 *J. Geophys. Res.* **71**, 3939.

HILL, R. 1952 *J. Mech. Phys. Solids* **1**, 19.

1962 *Ibid.* **10**, 1.

1967 *Ibid.* **15**, 79.

Hutchinson, J. W. 1970 *Proc. Roy. Soc. Lond. A.* **319,** 247.
1974 *Advances in Applied Mechanics* (edited by Yih, C.-S.) **14,** 67. Academic Press, New York.
Jaeger, J. C. and Cook, N. G. W. 1969 *Fundamentals of Rock Mechanics.* Methuen, London.
Nur, A. 1972 *Bull. Seismol. Soc. Amer.* **62,** 1217.
Prager, W. 1961 *Introduction to the Mechanics of Continua.* Ginn, Boston.
Rice, J. R. 1973 *Plasticity and Soil Mechanics* (Proceedings of the Symposium on the Role of Plasticity in Soil Mechanics. Cambridge, England. 13th–15th September 1973), (Edited by Palmer, A. C.), p. 263. Department of Engineering, University of Cambridge, England.
1975 *J. Geophys. Res.* **80,** 1531.
Scholz, C. H., Sykes, L. R. and Aggarwal, Y. P. 1973 *Science* **181,** 803.
Sewell, M. J. 1974 *J. Mech. Phys. Solids* **22,** 469.
Thomas, T. Y. 1961 *Plastic Flow and Fracture in Solids.* Academic Press, New York.
Tokuoka, T. 1972 *Arch. Rational Mech. Analysis* **46,** 114.
Walsh, J. B. and Brace, W. F. 1973 *Rock Mechanics Symposium* (AMD—Vol. 3) (edited by Sikarskie, D. L.), p. 1. American Society of Mechanical Engineers, New York.
Wawersik, W. R. and Brace, W. F. 1971 *Rock Mech.* **3,** 61.
Wawersik, W. R. and Fairhurst, C. 1970 *Int. J. Rock Mech. Min. Sci.* **7,** 561.

Appendix I

Localization calculation for isotropic-hardening case

The plane of localization has the orientation for which the hardening modulus h is a maximum. After using (17) in (16) this orientation can be determined by seeking the maximum in h with respect to the n_K (K = I, II, III), the components of the unit normal to the plane in the direction of the principal axes. Since the n_K are subject to the constraint

$$n_{\mathrm{I}}^2+n_{\mathrm{II}}^2+n_{\mathrm{III}}^2 = 1, \qquad \text{(AI.1)}$$

the method of Lagrange multipliers is used. The necessary conditions for a maximum are then

$$n_K(\sigma'_K/\bar{\tau})[(\sigma'_K/\bar{\tau})+\chi] = n_K\lambda, \qquad K = \mathrm{I, II, III}, \qquad \text{(AI.2)}$$

where λ is the undetermined multiplier, $\chi = \frac{1}{3}(\beta+\mu)(1+\nu)/(1-\nu)-\sigma'_{22}/\bar{\tau}(1-\nu)$, and $\sigma'_{\mathrm{I}} \geq \sigma'_{\mathrm{II}} \geq \sigma'_{\mathrm{III}}$ are the principal deviatoric stresses. Three possibilities can be identified, as follows.

(i) *None of the* n_K *is zero.* Since each equation may be divided through by the respective component of the unit normal, (AI.2) reduces to two independent equations:

$$(\sigma'_{\mathrm{I}}/\bar{\tau})[(\sigma'_{\mathrm{I}}/\bar{\tau})+\chi] = (\sigma'_{\mathrm{II}}/\bar{\tau})[(\sigma'_{\mathrm{II}}/\bar{\tau})+\chi] = (\sigma'_{\mathrm{III}}/\bar{\tau})[(\sigma'_{\mathrm{III}}/\bar{\tau})+\chi].$$

It is not possible to choose χ such that these equations are satisfied. Therefore at least one of the n_K is zero.

(ii) *Two of the n_K are zero.* Although a solution is possible, computing the corresponding value of h_{cr} for this solution reveals that it is less than that given by (iii) for the range of β and μ investigated. Physically, this is expected since this configuration is one for which there is no shear stress on the plane of localization.

(iii) *One of the n_K is zero.* In this case, it is easiest to proceed by guessing which of the n_K is zero, obtaining an expression for h_{cr}, and then checking the guess. Since it is anticipated that the plane of localization contains the maximum shear stress, we take $n_{II} = 0$. Equations (AI.2) now reduce to the single equation

$$(\sigma'_I/\bar{\tau})[\chi + (\sigma'_I/\bar{\tau})] = (\sigma'_{III}/\bar{\tau})[\chi + (\sigma'_{III}/\bar{\tau})]. \tag{AI.3}$$

The solution is easily found to be

$$\chi = -[(\sigma'_I/\bar{\tau}) + (\sigma'_{III}/\bar{\tau})] = \sigma'_{II}/\bar{\tau}.$$

Writing $\sigma'_{22}/\bar{\tau} = n_I^2(\sigma'_I/\bar{\tau}) + n_{III}^2(\sigma'_{III}/\bar{\tau})$ in χ and using (AI.1) gives two linear equations for n_I^2 and n_{III}^2. Solving and forming the ratio n_I/n_{III} gives (19).

Substituting the expressions for n_I^2 and n_{III}^2 in (17) and the resulting expressions into (16) yields, after rearrangement, equation (20). Now, we recognize that the expression for h resulting from choosing n_I or n_{III} equal to zero may be obtained merely by substituting $\sigma'_I/\bar{\tau}$ or $\sigma'_{III}/\bar{\tau}$, respectively, for $N = \sigma'_{II}/\bar{\tau}$. By comparing these expressions, the inequality (18) may be derived. Using, in addition, the relation

$$(\sigma'_I/\bar{\tau})^2 + (\sigma'_{II}/\bar{\tau})^2 + (\sigma'_{III}/\bar{\tau})^2 = 2,$$

it is easily shown that for $n_I = 0$ to yield the maximum h, it is necessary that $(\beta + \mu)$ be negative. This is not possible for frictional materials showing positive dilatancy, but may be of interest for loose granular materials which compact during shear.

APPENDIX II

Localization calculation for the modified constitutive law (13)

In this Appendix, the localization analysis, presented in the text and Appendix I for the isotropic-hardening constitutive law, is outlined for the modified law (13) incorporating vertex effects. An expression, equivalent to (16), for the hardening modulus at localization can be obtained from (16) by making the identifications listed after (13). This gives

$$\frac{h}{G} = \frac{h_1}{G}\,\frac{\psi - \beta\mu\sigma_{22}'^2 - \frac{4}{3}\beta\mu(\sigma_{21}^2 + \sigma_{23}^2) - \sigma'_{22}\bar{\tau}(\mu+\beta)(1 + h_1/G) - \bar{\tau}^2(a - 4\beta\mu h_1/3G)}{\psi + \bar{\tau}^2 a h_1/G}, \tag{AII.1}$$

where $a = 1 + 3(1-\nu)h_1/G(1+\nu)$ and $\psi = (\sigma'_{22})^2(h_1/K) + a(\sigma_{23}^2 + \sigma_{21}^2)$. The necessary conditions for a maximum in h are given by exactly equations (AI.2) if χ is now used to denote

$$\chi = \frac{-2\sigma'_{22}(a - h_1/K)(\bar{\tau}^2 a h_1/G - \Phi) + (\psi + \bar{\tau}^2 a h_1/G)[\frac{4}{3}\beta\mu\sigma'_{22} + \bar{\tau}(\mu+\beta)(1 + h_1/G)]}{-a(\Phi - \bar{\tau}^2 a h_1/G) - \frac{4}{3}\beta\mu(\psi + \bar{\tau}^2 a h_1/G)},$$

where

$$\Phi = -\beta\mu\sigma_{22}'^2 - \frac{4}{3}\beta\mu(\sigma_{21}^2 + \sigma_{23}^2) + \sigma'_{22}\bar{\tau}(\mu+\beta)(1 + h_1/G) - \bar{\tau}^2(a + 4h_1\beta\mu/3G).$$

Note that in the limit $h_1/G \to 0$, then $h/G \to 0$; and in the limit $h_1/G \to \infty$, then (AII.1) and χ reduce to the corresponding expressions of the isotropic-hardening case.

By the same process as in Appendix I, it is determined that $\chi = \sigma'_{II}/\bar{\tau}$ is the appropriate solution. Now, however, the equations for n_I^2 and n_{III}^2 are *quadratic*:

$$An_I^4 + 2Bn_I^2 + C = 0, \tag{AII.2}$$

where

$$\left.\begin{aligned} A &= (4-3N^2)[\beta\mu N + (\mu+\beta)], \\ B &= -(4-3N^2)^{\frac{1}{2}}(1-\beta(\sigma'_{III}/\bar{\tau}))(1-\mu(\sigma'_{III}/\bar{\tau})), \\ C &= -B - (\sigma'_{III}/\bar{\tau})^2(\mu+\beta)(h_1/K) + (\mu+\beta)h_1/K - 2(\sigma'_{II}/\bar{\tau})[\beta\mu h_1/G - h_1/K]. \end{aligned}\right\}$$

In writing the expressions for the coefficients, terms involving h_1/G (or h_1/K) have been neglected when added to unity. The normalizing condition $n_I^2 + n_{III}^2 = 1$ can be used to determine n_{III}^2 and the remainder follows as in Appendix I.

Equation (AII.2) yields a suitable solution (i.e. $0 \leq n_I^2 \leq 1$) for the range of β and μ investigated, except at very small values of h_1/G. As h_1/G becomes small, equation (AII.2) yields no solution for *triaxial extension* for large values of μ. In this case, the solution giving the maximum h_{cr} is such that the normal to the plane of localization is in the direction of the algebraically greatest principal stress ($n_I = 1$, $\sigma'_{22} = \sigma'_I$, $\sigma_{23} = \sigma_{21} = 0$).