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LIMITATIONS TO THE SMALL SCALE YIELDING APPROXIMATION FOR CRACK TIP PLASTICITY

By J. R. Rice
Division of Engineering, Brown University, Providence, R.I., U.S.A.

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Summary

Recent finite-element results by S. G. Larsson and A. J. Carlsson suggest a limited range of validity to the 'small scale yielding approximation', whereby small crack tip plastic zones are correlated in terms of the elastic stress intensity factor. It is shown with the help of a model for plane strain yielding that their results may be explained by considering the non-singular stress, acting parallel to the crack at its tip, which accompanies the inverse square-root elastic singularity. Further implications of the non-singular stress term for crack tip deformations and fracturing are examined. It is suggested that its effect on crack tip parameters, such as the opening displacement and J-integral, is less pronounced than its effect on the yield zone size.

1. Introduction

Recently, Larsson and Carlsson (1973) have shown that the range of validity of the 'small scale yielding' approximation for crack tip plastic zones is substantially more limited than previous analyses had suggested (see, for example, Rice [1967a, 1968a]). To describe the approximation, let r, θ be polar coordinates centred at the tip of a crack in a body under plane strain deformations. The small-displacement-gradient linear elastic solution results in stresses of the form

$$\sigma_{ij} = Kr^{-1/2}f_{ij}(\theta) + \text{non-singular terms} \tag{1.1}$$

near the crack tip, where K is the stress intensity factor and where the set of universal functions f_{ij} is normalized so that the singular part of the stress acting ahead of the tip, normal to the plane of the crack, is $K(2\pi r)^{-1/2}$. The small scale yielding approximation then incorporates the notion that, even though (1.1) is inaccurate within and near a small crack tip yield zone, its dominant singular term should in some sense still govern the deformation state within that zone. Hence, the actual elastic–plastic problem is replaced by a problem formulated in boundary layer style, whereby a semi-infinite crack in an infinite body is considered and the actual conditions of boundary loading are replaced by the asymptotic boundary conditions that

$$\sigma_{ij} \to Kr^{-1/2}f_{ij}(\theta) \qquad \text{as } r \to \infty. \tag{1.2}$$

Hence, as is often said, the small yield zone is 'surrounded' by the dominant elastic singularity, and the applied loadings and geometric shape of the body influence conditions within the plastic region only insofar as they enter the formula for K, as computed elastically.

A consequence of this formulation is that the plastic zone dimension r_p and the crack tip opening displacement δ_t, when definable, are given by formulae of the type

$$r_p = \alpha K^2/\sigma_0^2, \qquad \delta_t = \beta K^2/E\sigma_0, \tag{1.3}$$

where E is elastic tensile modulus, σ_0 is yield strength, and α and β are dimensionless factors which may, for example, depend on Poisson's ratio, strain-hardening exponent, etc., but are independent of the applied load and specimen geometry. Now, by comparing equations such as (1.3), generated by the boundary layer formulation, to available complete elastic–plastic solutions, RICE (1967a, 1968a) found that the approximation was valid up to substantial fractions of the loads corresponding to general yielding. Of course, in the limit of very small load levels, the solutions coincide exactly. It turns out to be important that such complete solutions were, however, available only for the anti-plane strain case and for the Barrenblatt–Dugdale–BCS (Bilby–Cottrell–Swinden) yield model.

By contrast, LARSSON and CARLSSON (1973) performed plane strain elastic–plastic calculations, by the finite element method, for a variety of specimen geometries, and found significant discrepancies with the boundary layer formulation, even within the rather small range of yield zone sizes allowed by the ASTM limits for fracture test correlation in terms of K-values. For example, by fitting their numerical results to (1.1) they found that at loads corresponding to the ASTM limit, α would have to differ by a factor of two between the compact tension and center cracked specimens.

Larsson and Carlsson were able to explain their results in terms of a suggestion by the present writer that differences from specimen to specimen in the 'non-singular terms' of (1.1) could be responsible for the discrepancies. Indeed, from the analyses of WILLIAMS (1957) and IRWIN (1960), a more detailed form than (1.1) for the in-plane stress components is

$$\begin{bmatrix}\sigma_{xx} & \sigma_{xy}\\ \sigma_{yx} & \sigma_{yy}\end{bmatrix} = \frac{K}{\sqrt{r}}\begin{bmatrix} f_{xx}(\theta) & f_{xy}(\theta)\\ f_{yx}(\theta) & f_{yy}(\theta)\end{bmatrix} + \begin{bmatrix} T & 0\\ 0 & 0\end{bmatrix} + \text{terms which vanish at crack tip.} \tag{1.4}$$

Here, (x, y) is the plane of straining and the crack coincides with the x-axis, so it is seen that the portion of the non-singular stress field which does not vanish at the tip amounts to a uniform stress $\sigma_{xx} = T$ acting parallel to the crack plane. Thus, by first determining T in terms of the applied load for each of their specimens, LARSSON and CARLSSON (1973) were able to verify that a two-parameter boundary layer formulation, in which (1.2) is replaced by the requirement of an asymptotic approach to the field given by the two leading terms of (1.4), could closely match their results for the different specimens.

The aim in the present paper is to study further this T-effect, and to clarify the manner in which it results in deviations from (1.3) at such substantially lower levels of applied load than had been expected from earlier studies. Much of this discussion is given in terms of a simple model for plane strain yielding, consisting of two slip bands emanating symmetrically from the crack tip. It is also shown that there is no similarly strong T-effect on formulae for the value of the J-Integral. This and related implications of the T-effect for fracture are discussed.

2. A MODEL FOR PLANE STRAIN YIELDING

The model for plane strain yielding is illustrated in Fig. 1. Plastic relaxation occurs by sliding on two bands at angles $\pm\phi$ with the crack plane. These bands sustain a

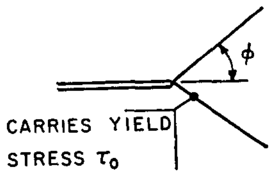

FIG. 1. Crack tip yield model.

yield stress τ_0 in shear, and their length r_p is determined by the following approximate argument (RICE, 1967a).† Consider first a mode II shear crack under stress intensity factor $K^{(s)}$, so that the elastic field analogous to (1.1) results in

$$\sigma_{yx}^{(s)} = \frac{K}{(2\pi r)^{1/2}} + \dots \tag{2.1}$$

for the shear stress exerted directly ahead of the crack, in its own plane. If this is relaxed through sliding in the crack plane under a yield stress τ_0, the small scale yielding estimates of the extent of the plastic zone and the crack tip sliding displacement are

$$r_p^{(s)} = (\pi/8)[K^{(s)}/\tau_0]^2, \qquad \delta_t^{(s)} = (1-\nu^2)[K^{(s)}]^2/E\tau_0. \tag{2.2}$$

Now, for the mode I tensile case, the elastic field (1.1) results in a shear stress

$$\sigma_{\phi r} = \frac{\sin\phi\cos(\frac{1}{2}\phi)K}{2(2\pi r)^{1/2}} + \dots \tag{2.3}$$

along the planes at angles $\pm\phi$ where sliding is presumed to take place. By comparing this to (2.1), we can identify $K^{(s)}$ as

$$K^{(s)} = \tfrac{1}{2}\sin\phi\cos(\tfrac{1}{2}\phi)K \tag{2.4}$$

and, as an approximation, estimate the extent of the plastically relaxed zones and crack tip sliding displacement in each from (2.2). Hence, for small scale yielding,

$$\left.\begin{aligned} r_p &\approx r_p^{(s)} = (\pi/64)\sin^2\phi\,(1+\cos\phi)K^2/\tau_0^2, \\ \delta_t &\approx 2\delta_t^{(s)}\sin\phi = \tfrac{1}{4}(1-\nu^2)\sin^3\phi\,(1+\cos\phi)\,K^2/E\tau_0 \end{aligned}\right\} \tag{2.5}$$

where a trigometric identity is used and where δ_t is the total opening at the tip between upper and lower crack surfaces. In fact, this expression for δ_t differs from that given originally (RICE, 1967a) in that the sin ϕ multiplying $\delta^{(s)}$, and giving its projection onto the y-direction, had been omitted.

If we choose the value of ϕ as that which maximizes the extent of the yielded zone, then $\cos\phi = \frac{1}{3}$, so that $\phi = 70{\cdot}6°$, and (2·5) become

$$\left.\begin{aligned} r_p &= \frac{\pi}{18}\left(\frac{K}{\sqrt{3}\tau_0}\right)^2 \approx 0{\cdot}17\left(\frac{K}{\sqrt{3}\tau_0}\right)^2, \\ \delta_t &= \frac{16}{27}\sqrt{\frac{2}{3}}\,\frac{(1-\nu^2)K^2}{E(\sqrt{3}\tau_0)} \approx 0{\cdot}44\,\frac{K^2}{E(\sqrt{3}\tau_0)} \quad (\text{for } \nu = 0.3), \end{aligned}\right\} \tag{2.6}$$

† A numerical solution of this model for $\phi = 45°$ has been reported by BILBY and SWINDEN (1965), who modelled the crack and yield bands by a finite set of discrete dislocations, having fixed positions but variable Burgers vectors.

where the results are given in terms of the equivalent tensile strength $\sigma_0 = \sqrt{3}\,\tau_0$ for purposes of comparison with (1.3) and with numerical finite-element solutions to the full elastic–plastic equations for a non-hardening von Mises material. The most accurate of such solutions for the small scale yielding formulation is probably that of RICE and TRACEY (1973), employing singular elements. They reported a maximum plastic zone extent at 71° with numerical coefficients of 0·152 for r_p and 0·493 for δ_t. Similar results were obtained by LARSSON and CARLSSON (1973) and also by LEVY, MARCAL, OSTERGREN and RICE (1971), in an earlier implementation of singular elements, except that the latter obtained a numerical coefficient about 14 per cent lower for δ_t. Thus, the simple model seems to be in fair agreement with more accurate solutions. Indeed, if we thought of the yield zone as not being confined to discrete bands, but rather as a diffuse zone, and used (2.5_1) to predict the distance to the elastic–plastic boundary, then a yield zone shape in good agreement with that of the numerical solutions results over the 'centred fan' range of ϕ from 45° to 135°.

Now let us consider the effect of the T-term on this model. Evidently, a uniform stress field $\sigma_{xx} = T$ creates a uniform shear stress

$$\sigma_{\phi r} = -T \sin\phi \cos\phi \tag{2.7}$$

along a plane at angle ϕ with the crack plane. Since it is uniform, a solution to the yield model for the case of $T = 0$ also provides the solution when $T \neq 0$ if we make the replacement

$$\tau_0 \to \tau_0 + T \sin\phi \cos\phi. \tag{2.8}$$

Hence, the solution for the modified boundary layer formulation, in which T is accounted for as discussed earlier, is given directly from (2.5) as

$$\left.\begin{aligned} r_p &= (\pi/64)\sin^2\phi\,(1+\cos\phi)\,K^2/(\tau_0 + T\sin\phi\cos\phi)^2,\\ \delta_t &= \tfrac{1}{4}(1-\nu^2)\sin^3\phi\,(1+\cos\phi)\,K^2/[E(\tau_0+T\sin\phi\cos\phi)]. \end{aligned}\right\} \tag{2.9}$$

To see the real significance of these results, let us keep in mind that K and T are directly proportional to the applied loadings. For example,

$$K = \sigma_{yy}^{\infty}(\pi a)^{1/2}, \qquad T = \sigma_{xx}^{\infty} - \sigma_{yy}^{\infty} \tag{2.10}$$

for the Inglis–Kolosov configuration of a crack of length 2a under remotely uniform biaxial stressing (Fig. 2). Now, if (2.9) is expanded in a series about $T = 0$, using the value of $\phi = 70{\cdot}6°$ which maximizes r_p in that case, then

$$\left.\begin{aligned} r_p &= \frac{\pi}{18}\left(\frac{K}{\sqrt{3}\tau_0}\right)^2\left[1-\tfrac{4}{3}\sqrt{\frac{2}{3}}\left(\frac{T}{\sqrt{3}\tau_0}\right)+\ldots\right],\\ \delta_t &= \frac{16}{27}\sqrt{\frac{2}{3}}\frac{(1-\nu^2)K^2}{E(\sqrt{3}\tau_0)}\left[1-\tfrac{2}{3}\sqrt{\frac{2}{3}}\left(\frac{T}{\sqrt{3}\tau_0}\right)+\ldots\right], \end{aligned}\right\} \tag{2.11}$$

where by comparison with (2.6), the bracketed terms represent the deviation from the small scale yielding approximation. Thus,

$$r_p = \frac{\pi^2}{18}\,a\left(\frac{\sigma_{yy}^{\infty}}{\sqrt{3}\tau_0}\right)^2\left[1+\tfrac{4}{3}\sqrt{\frac{2}{3}}\left(\frac{\sigma_{yy}^{\infty}-\sigma_{xx}^{\infty}}{\sqrt{3}\tau_0}\right)+\ldots\right] \tag{2.12}$$

for the configuration of Fig. 2.

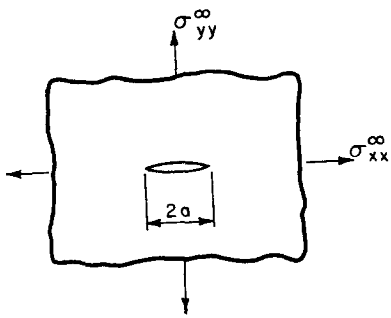

FIG. 2. Crack under biaxial tension.

There is a fundamental difference between these expansions and similar expansions as carried out from the available complete elastic–plastic solutions for anti-plane strain or for the Barrenblatt–Dugdale–BCS model. Namely, for every such solution the corresponding series for r_p and δ_t are of the form

$$r_p = \alpha(K/\sigma_0)^2[1+\lambda(\sigma_{appl}/\sigma_0)^2+\ldots], \tag{2.13}$$

where α and λ are constants and where σ_{appl} is some nominal applied stress. A typical example for the latter type of model, wherein yield is supposed to be confined to plastic zones sustaining the tensile yield strength σ_0 and lying in the plane of the crack, is

$$r_p = a\{[\cos(\pi\sigma^{\infty}_{yy}/2\sigma_0)]^{-1}-1\} = (\pi^2/8)\,a(\sigma^{\infty}_{yy}/\sigma_0)^2[1+(5\pi^2/48)(\sigma^{\infty}_{yy}/\sigma_0)^2+\ldots] \tag{2.14}$$

for the configuration of Fig. 2. The feature of interest for all such solutions is that the deviation from the small scale yielding approximation is *quadratic* in the applied load, whereas for the present inclined shear band model (and, by implication, for the exact elastic–plastic plane strain solution) the deviation is *linear* in the applied load. Indeed, this difference would seem to be at the root of the Larsson–Carlsson observation of a substantially more limited range of validity to the small scale yielding approximation than had been evident from the earlier solutions.

In retrospect, it is easy to see how this distinction comes about: the non-vanishing but non-singular T-terms are the source of the linear deviation in (2.11) and (2.12). This term is completely without effect on the Barrenblatt–Dugdale–BCS model. For example, changing T by changing σ^{∞}_{xx} in Fig. 2, or by alterations of boundary conditions in other cases which would induce a uniform σ_{xx} if the response were elastic, has no effect on the solutions for this model. Of course, the same is not true for the plane strain model of Fig. 1, as (2.7) to (2.9) show. In anti-plane strain, there is a similar possibility of a non-vanishing but non-singular term amounting to a uniform shear stress σ_{xz}, and this would presumably result also in a linear deviation. But this term exists only when loadings are unsymmetrical relative to the crack line, and its effect has been undetected simply because solutions have been done only for symmetrical loadings.

It is interesting to examine further the predictions of the simple model as given by (2.9). The *apparent* α-value, in the notation of (1.3), will be called R_0 and is given by

$$R_0 = r_p/(K/\sqrt{3}\tau_0)^2 = (3\pi/64)\sin^2\phi\,(1+\cos\phi)/[1+(T/\tau_0)\sin\phi\cos\phi]^2. \tag{2.15}$$

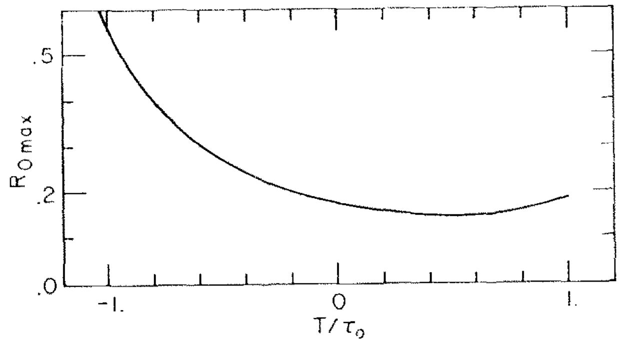

FIG. 3. Effect of T on yield zone size for a given K.

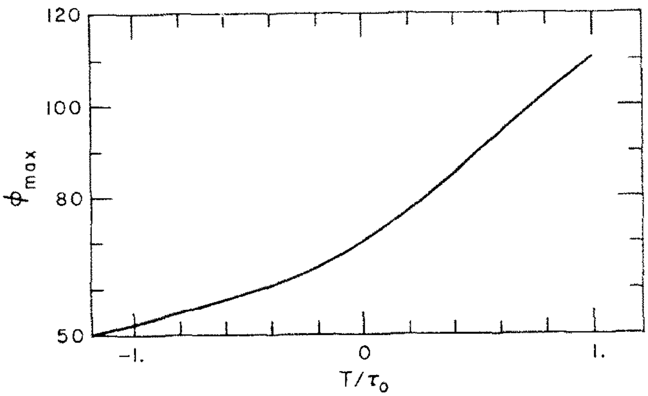

FIG. 4. Angle ϕ at which yield zone size is maximum.

Figures 3 and 4 show the variation, with T/τ_0, of the value $R_{0,\max}$ resulting when this is at a maximum, and the corresponding values, $\phi_{\max}$, of ϕ. Now consider the results summarized in Table 1 for four specimens analyzed by LARSSON and CARLSSON (1973). The first column shows T in ratio to K for each specimen, with a being crack depth. The data are taken from solutions for loadings at the ASTM limit, $K = 0{\cdot}6\,\sigma_0\sqrt{a}$, and corresponding ratios of T/τ_0 are listed. Next is shown $R_{0,\max}$ from the numerical solution and from (2.15). Finally $\phi_{\max}$ is shown from the same sources. It is given as a

TABLE 1. *T-effect on yield zones for fracture specimens.*

Specimen	$T\sqrt{a}/K$	T/τ_0	$R_{0,\max}$ LARSSON and CARLSSON (1973)	$R_{0,\max}$ Equations (2.9) and (2.15)	$\phi_{\max}$ (deg.) LARSSON and CARLSSON (1973)	$\phi_{\max}$ (deg.) Equations (2.9) and (2.15)
Center cracked	−0·59	−0·57	0·23	0·29	58–63–69	58
Doubled edge cracked	−0·14	−0·13	0·20	0·19	64–67–73	67
Bend	0·03	0·03	0·15	0·17	63–71–78	71
Compact tension	0·29	0·28	0·11	0·15	68–83–105	81

range for the numerical solution because of the typically broad and not precisely defined maximum for R_0, and the angles such as 58°–63°–69° for the center cracked specimen correspond to those for which the yield zone extent is $0{\cdot}98\ R_{0,\max}$–$R_{0,\max}$–$0{\cdot}98\ R_{0,\max}$, respectively. The agreement is remarkably good. Also, as remarked earlier with $T = 0$, a reasonable indication of the shape of the plastic zone is given by regarding (2.9) as being the distance to the elastic–plastic boundary at any angle ϕ. The writer is pleased to acknowledge the assistance of Professor A. J. Carlsson in providing the numerical evaluation of the formula (2.9) and the comparison with his finite-element data as discussed here.

It is also worthy of note that the variation of crack tip opening displacement between specimens at the same K is considerably less than that in plastic zone size. For example, if the values of T/τ_0 and $\phi_{\max}$ at the ASTM limits are taken from the Table, then (2.9) with $\nu = 0{\cdot}3$ gives *apparent* β-values, in the notation of (1.3), which range from 0·49 for the center cracked to 0·41 for the compact tension specimen. By comparison, 0·44 is the actual β-value, from (2.6) for $T = 0$. The numerical solutions of Larsson and Carlsson (1973) showed a similarly small variation in apparent β-values.

3. Absence of T-Effect on the J-Integral

Recently Broberg (1971) and Begley and Landes (1972) have suggested that failure criteria could be based on the J-Integral, in the sense of its use by Rice (1968a, b) as a measure of the intensity of crack tip deformations in non-linear materials. By its compliance interpretation, it is known that

$$J = (1-\nu^2)\frac{K^2}{E} \tag{3.1}$$

for small scale yielding. It is of interest to know if there is a linear deviation of J from this formula, due to a T-effect, as is the case with plastic zone size (equations (2.11) and (2.12)). In fact, it is shown here that there is no T-effect on J, so that

$$J = (1-\nu^2)\frac{K^2}{E}\left[1+\mu\left(\frac{\sigma_{\text{appl}}}{\sigma_0}\right)^2+\dots\right], \tag{3.2}$$

i.e. the deviation is *quadratic* in applied load.

To see this, consider the modified boundary layer formulation for a semi-infinite crack in an infinite body, with boundary conditions of asymptotic approach to the first two terms of (1.4). There will be a yield zone at the crack tip, but we focus on the nature of the solution in elastically deformed material outside of a circle of radius greater than the greatest extent of the non-linear zone. By doing a 'Williams (1957) expansion' of the stress field for this *outer* region we arrive at the representation

$$\sigma_{ij} = T\delta_{ix}\delta_{jx}+Kr^{-1/2}f_{ij}(\theta)+r^{-1}A_{ij}(\theta)+r^{-3/2}B_{ij}(\theta)+r^{-2}C_{ij}(\theta)+\dots. \tag{3.3}$$

Of course, the terms of exponent more negative than $-\frac{1}{2}$ are customarily deleted because if this were written for the *inner* region, they would result in unbounded energy.

By seeking the most general r^{-1} stress field corresponding to symmetrical loadings about the x-axis, one easily shows from the equilibrium equations and compatibility equation $\nabla^2(\sigma_{rr}+\sigma_{\theta\theta}) = 0$, for an isotropic material, that

$$A_{rr} = \xi \cos\theta, \qquad A_{\theta\theta} = \eta \cos\theta, \qquad A_{r\theta} = \eta \sin\theta, \tag{3.4}$$

where ξ and η are constants and where $\theta = \pm\pi$ are the crack surfaces. In fact, these expressions when multiplied by r^{-1} correspond to the stress field due to an edge dislocation with Burgers vector in the y-direction plus a concentrated line force pointing in the x-direction. If the A_{ij}-terms are to satisfy traction-free boundary conditions on the crack surfaces, $\eta = 0$, and if there is to be no net force transmitted across a circuit surrounding the crack tip, $\xi = 0$. Hence, there is no r^{-1} term in (3.3):

$$A_{ij} = 0. \tag{3.5}$$

Choosing the path Γ in the definition of J to be a large circle of radius r surrounding the tip,

$$\begin{aligned} J &= \int_\Gamma [W\,dy - T_i(\partial u_i/\partial x)\,ds] \\ &= r\int_{-\pi}^{+\pi} [\tfrac{1}{2}\sigma_{ij}\varepsilon_{ij}\cos\theta - \sigma_{ix}(\partial u_i/\partial x)\cos\theta - \sigma_{iy}(\partial u_i/\partial x)\sin\theta]\,d\theta. \end{aligned} \tag{3.6}$$

The displacement derivatives and strains will have expansions in powers of r identical to those for the stresses. Further, since J is path-independent, r may be chosen as large as we wish. By considering the different powers of r which remain in (3.3) when the r^{-1} term is deleted, one sees that in the limit $r \to \infty$, only the first two terms of (3.3) will contribute to J. But this means that J takes on the same value which it would have if there were no plastic zone and the material responded elastically everywhere, and this value is well-known to be that given by (3.1) independently of T. Hence there is no T-effect on the J-integral.

The significance of this is made evident by the work of CHEREPANOV (1967), HUTCHINSON (1968), and RICE and ROSENGREN (1968) on crack tip singularities in 'power-law' strain hardening materials. The strength of their leading singular term, which dominates the deformation field near the crack tip, well within the plastic zone, is expressed solely in terms of J. From this we conclude that there is no T-effect on the dominant singularity, although there will of course be an effect on the overall shape of the plastic zone. But this needs two qualifications. First, as remarked by RICE and ROSENGREN (1968) and MCCLINTOCK (1971), and as is also evident from an earlier anti-plane strain analysis by RICE (1967b), the question as to whether the 'dominant' singularity really governs over physically significant size scales for fracturing depends on how strongly the material strain-hardens. Indeed, with the non-hardening idealization there is no such one-parameter characterization and different specimens may have different near tip fields, at the same J-value, when load magnitudes are beyond the range of validity of the unmodified boundary layer formulation (1.2). For example, the T-effect leads to slight differences in a crack tip parameter such as the opening displacement in non-hardening calculations, as remarked earlier. The second point is that the dominant singularity, when present, is parameterized in terms of the value, J_{tip}, of J on a contour of vanishing radius about the tip. This will equal the value (3.1) as computed on contours in the elastic region only to the extent that a 'total strain' formulation of plasticity if appropriate. It is likely that the development of pointed vertices on small-offset yield surfaces makes this a good approximation to actual behavior for cases without substantially non-radial loading. However, some approximation is involved and J_{tip} may therefore differ from (3.1) by an amount which depends on conditions in plastically deformed regions away from the tip, where there *is* a T-effect. For example, the incremental, small scale yielding, non-hardening solutions by LEVY *et al.* (1971) and RICE and TRACEY (1973), which take no account of vertex formation, result in a J_{tip} value about 20 per cent less than that of (3.1). This percentage reduction could be affected linearly by T, although the net effect on J_{tip} would seem small.

4. DISCUSSION AND SUMMARY

The model discussed here shows deviations from the small scale yielding solution at relatively low levels of applied load, in agreement with the results of LARSSON and CARLSSON (1973). Their cause is evidently due to the non-singular stress term T acting parallel to the crack plane. The effect on the plastic zone size is quite pronounced at load levels corresponding to the ASTM limit, although there seems to be less effect on near-tip parameters such as the crack tip opening displacement and J-integral. As regards the near-tip stress state, recall that the Prandtl field, which is thought to give the stress state as $r \rightarrow 0$ for a non-hardening material (RICE, 1967a, 1968a, b), involves a positive σ_{xx} both ahead and behind the tip. This suggests that specimens with a negative value of T would tend to show a more rapid fall-off, with increasing r, from the hydrostatically elevated stresses of the Prandtl field than would be the case for those with positive T. Indeed, this agrees with the numerical results of LARSSON and CARLSSON (1973) who find, for example, that the center-cracked specimen exhibits a considerably more rapid stress fall-off than do the other specimens listed in Table 1.

This latter kind of T-effect is likely to be important for stress induced fracture mechanisms, such as cleavage micro-cracking, whereas the T-effect on crack tip deformation parameters would seem more relevant to cases of ductile void-growth. Effects of both kinds could be involved when high stress levels are important to void nucleation by the cracking or de-cohesion of second-phase particles (RICE and JOHNSON, 1970). As for experimental studies which might reveal a T-effect on critical K-values for fracture, HALL (1971) has compared four crack test specimen designs for 2219-T87 aluminum and 5Al–2·5Sn ELI titanium alloys. Two of his specimens, namely the bend and compact tension, coincide with those of Table 1. With aluminum, Hall finds a critical K-value for the compact tension specimen which is typically about 25 per cent lower than that for the bend specimen, when comparison is made at the maximum load allowed in the ASTM procedures. This, incidentally, corresponds to a plastic zone extent which is only 6 to 7 per cent of the crack depth a. On the other hand, with titanium Hall finds a less definitive effect, and the bend specimen results instead in the lower critical K-value, by about 10 per cent.

The inclusion of T as a second crack tip parameter was shown to characterize suitably small *plane strain* yield zones, when K alone becomes inadequate. More generally, for actual three-dimensional tensile mode crack tip stress states, it would seem necessary to supplement K with two parameters, say S and T. Here, T is the non-singular σ_{xx} introduced earlier whereas S represents a similar non-singular σ_{zz}, acting perpendicular to the principal plane of deformation. For plane strain, $S = \nu T$, but this will not be so in general. Just as T influences yielding in the plane, S would seem to influence the transition to a non-plane-strain yielding mode involving throughthickness deformation as observed, for example, in thin notched sheets with 'plane stress' yielding.

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