Elastic-Plastic Models for Stable Crack Growth†

by

James R. Rice*

March 1973

Published in:
Mechanics and Mechanisms of Crack Growth, Proc. of April 1973 Conf.*, British Steel Corp. Physical Metallurgy Centre Report (edited by M. J. May), 1975, pp. 14-39.

★ † For presentation at conference on "Mechanics and Mechanisms of Crack Growth", Cambridge, England, 4 - 6 April 1973; to be published in Conference Proceedings.

* Professor of Engineering, Brown University, Providence, R. I. 02912 U.S.A.

Introduction

Typically, cracks do not abruptly begin to propagate in ductile solids. Instead it is usual that the initiation of crack extension is followed by stable growth under a continuous increase of the applied load, or at least of the load-point displacement. Ultimately the required increase for continuing quasi-static crack advance falls to zero, and unstable propagation ensues. Sometimes the stable growth phase is so imperceptible that initiation and propagation are essentially coincident. This is often the case for 'plane strain' fractures under small scale yielding conditions. Nevertheless, it is evident that a realistic theoretical framework for fracture must include not only models for the initiation of crack extension, but also models for subsequent stable crack growth and, especially, for its terminal loss of stability.

An example cited by Drucker and Rice (1970), illustrated in fig. 1, shows the manner in which some degree of stable growth is inherent to materials which do not fully recover their strain upon unloading. The idealized non-linear elastic (left) and rigid-plastic (right) materials behave in an essentially similar fashion when subjected to monotonic load, without crack growth. But if we imagine that the specimen boundaries are held fixed in their displaced positions while the cracks are saw-cut ahead in the two specimens, it is seen that very different behavior results. The non-linear elastic material retains no memory of the initial crack size, and readjusts its deformation field so that a new site of strain concentration is formed at the cut-ahead tip. By contrast, there is no strain concentration created at the cut-ahead tip in the rigid-plastic material and the deformation field accumulated during the initial loading remains. Indeed, a strain concentration can be maintained in this case only if the specimen boundaries are continuously displaced as the crack advances.

The example illustrates by a limiting case the factor which would seem to be the primary cause of stable crack growth in elastic-plastic materials: The crack is advancing into plastically deformed material which has already accommodated, in part, the deformations imposed by displacement of the specimen boundaries or of the elastic surroundings of an embedded plastic zone. Hence a lessened crack tip strain concentration results, in comparison to that when all the deformation is imposed at one crack length. Similar consideration of the relative importance of recoverable or elastic, versus permanent or plastic, deformation effects are often important to the interpretation of stable growth. The former effects are de-stabilizing, the latter stabilizing.

There are, of course, other contributory factors to stable growth. For example, a crack tip stress concentration of sign opposite to that caused by tensile loadings is created by the residual stress field due to the wake of plastically strained material trailing behind an advancing tip. In addition, the near tip stress distribution in lightly hardening materials is to a certain extent determined solely by equilibrium considerations and the yield criterion. As this stress distribution moves through the material, along with the tip, some incompatible elastic strain increments result and these have the effect of inducing further plastic straining at the tip (e.g., McClintock and Irwin, 1965; Rice, 1968a). Indeed, this latter effect of the material's recoverable deformations results in the greatest contribution to the crack tip strain concentration under steady state conditions; without it, stable growth would be much more extensive.

Some Unresolved Questions on Stable Crack Growth

Most experiments on stable crack growth are dominated by three dimensional effects. For example, in thin plate specimens the crack ultimately extends as a single or double slant fracture and, in addition, there are more complicated 3-D features in the growth from an initially flat starting crack to the final slant geometry. In moderately thick plate specimens the stable growth is most often seen to occur by a tunnelling ahead of the crack front in the center of the fracture surface, which leaves trailing uncracked ligaments that presumably experience little triaxiality and fail in ductile shear. The net result is a flat central portion of the fracture with bordering slanted shear lips, and it is likely that most of the observed resistance to growth is contributed by the shear lip sections (Bluhm, 1962).

Remarkably little study seems to have been done on the thickness dependence of stable growth and, in particular, on determining material behavior in the limiting case of idealized plane strain, for which the crack front remains straight as it extends and results in a (macroscopically) flat fracture surface. Some central questions are: (i) How much plane-strain stable crack growth occurs, (ii) how does it vary over the range of near tip yield states from well contained plasticity to fully plastic yielding of the uncracked ligament, and (iii) how do the stress triaxiality and flow field variations among different fully plastic specimen geometries influence growth in the latter case?

I place this emphasis on plane strain because many applications are of this type. If, for example, a large flaw in a thick-walled pressure vessel is considered, it is possible that crack advance could occur under plane-strain-like conditions with a contained plastic zone at the tip, and it is important to know if the stable growth observed in a small, fully plastic laboratory specimen will be duplicated to nearly the same extent in the application. Also, the specimen stable growth behavior is likely to be completely irrelevant, even if the application involves, say, fully plastic conditions over the uncracked ligament ahead of a long part-through surface flaw in a plate or shell, if the specimen behavior is dominated by shear ligaments trailing behind a tunnelling crack front.

The theory of initiation of ductile plane-strain crack extension (Rice and Johnson, 1970) is based on the fact that large strains, much in excess of elastic values, result ahead of an initially sharp crack tip only over a highly localized region of linear extent comparable in size to the crack tip opening displacement, δ_t . Fig. 2, for example, shows the variation of logarithmic or 'true' strain with distance from a stationary crack tip, for small scale yielding conditions (in which case $\delta_t \approx K^2/2E\sigma_o$, where σ_o is the tensile yield stress). Distance is given in terms of X , as measured off before deformation, and the analysis is based on a full account of finite changes in geometry; in fact, an analysis based on the conventional 'small geometry change' assumptions results in a prediction of no intense straining whatsoever directly ahead of a sharp plane-strain crack tip.

Since δ_t sets the size scale of the region of intense straining, it is to be expected that extensive void coalescence at the crack tip (i.e., the initiation of crack extension) will begin when δ_t attains a size which is comparable to the spacing of voids that have been nucleated by the decohesion or cracking of second phase particles, in the triaxially elevated stress field experienced by material points before their envelopment by the large strain region.

Of course the exact ratio of δ_t to this spacing must depend on particle volume fraction, on the ease of nucleating additional voids from, say, smaller precipitate particles, on the material's strain hardening, etc., and the precise influence of these factors is not clear. Indeed, a precise characterization of when crack extension should be said to 'initiate' does not seem to be possible. Nevertheless, the general concept that initiation corresponds to $\delta_t \approx$ spacing of void sites seems well attested to experimentally in data cited, e.g., by Rice and Johnson, Smith and Knott (1971), and Rice (1973).

A part of this data refers to measured K_{Ic} values. If these are taken to denote the onset of propagation, as is certainly valid in some cases, it would seem correct to infer that initiation and propagation were then essentially coincident, and that any stable crack growth was in those cases limited to an amount of order δ_t involved in the progressive blunting of the tip. On the other hand, the fully-plastic plane-strain bend specimens of Smith and Knott, while verifying the above initiation concept, definitely showed measurable stable crack growth. These results are in general accord with our previous consideration of growth effects as corelating with the relative dominance of recoverable vs. permanent deformation effects. For example K_{Ic} specimens have only minute plastic zones and are generally prepared for the higher strength materials which undergo proportionally larger elastic strains at yield. The Smith and Knott specimens were fully plastic and were of a low strength material. Still, the amount of data on any one material in plane strain crack advance seems too limited to generalize on the above results. Also, there are certainly cases, e.g. maraging steels (J. Srawley, private communication), in which stable growth occurs under K_{Ic}-like test conditions.

Plasticity Analysis for Growing Cracks

There are few available continuum plasticity solutions for growing cracks. Those which are available are due mainly to McClintock and coworkers. All treat the material as perfectly plastic, without alteration of the yield surface by hardening or by Bauschinger effects, local vertices, etc. These omissions should be kept in mind because the mathematical predictions rely on a certainly oversimplified representation of material response to the strongly 'non-radial' stressing experienced near the tip.

Most of the progress has been on the anti-plane strain elastic-plastic model (McClintock, 1958; McClintock and Irwin, 1965; Rice, 1968a; Chitaley and McClintock, 1971). McClintock (1968) has also obtained solutions for plane strain crack advance in rigid-plastic materials by methods of slip line theory, but the only work on the elastic-plastic case is Rice's (1968a) analysis of the nature of the crack tip strain singularity under conditions of steady state crack advance.

Nevertheless, there seems to be enough background for the following general discussion of elastic-plastic deformation near a growing crack. The accompanying analysis is certainly not precise, but is suggestive instead, and it does seem to embody the essential features of the problem as revealed by the limited progress just noted toward precise solutions in particular cases. Further, the context is wide enough to include any case for which the deformation field can be modelled as two-dimenaional: plane strain, either exactly or locally near the tip of a three-dimensional crack; anti-plane strain, possibly in combination with plane strain for an approximate model of slant fracturing; or plane stress, within the usual approximations of the 2-D theory and assuming that the plastic zone is diffuse rather than localized in the Dugdale sense. Later some of these separate cases are discussed more precisely on their own. Also, the present considerations do not include cases for which crack advance is accompanied by an extended zone of partially degraded material at the tip. This would be the case, for example, in the 2-D plane stress analysis of a moderately thick plate, when the crack has tunnelled far ahead in the center so as to leave extensive ligaments which are not yet completely sheared (Bluhm, 1962). It would also be the case if McClintock's (1969) postulated unstable, decohering zone extended over a size scale comparable to an overall dimension of the plastic region.

Let r, θ be polar coordinates centered at the possibly moving crack tip, fig. 3. Now, solutions for stationary cracks in elastic-plastic or rigid-plastic non-hardening materials, subjected to monotonic loading, result in asymptotic, singular near-tip strain states which may be expressed in the form

$$\epsilon_{ij} = \frac{\delta_t}{r} f_{ij}(\theta) \qquad i,j = x,y,z \tag{1}$$

Here δ_t is the crack tip opening displacement (or sliding displacement, in anti-plane strain); the set of functions f_{ij} are dimensionless functions of θ , which are non-zero in fan-like regions of characteristics near the tip and are typically of order unity, although sometimes in fully plastic solutions they can be singular at some value of θ , so as to correspond to a slipping discontinuity emanating from the crack tip at that angle. This equation and the subsequent discussion, except where qualified, neglects the large local geometry changes illustrated in fig. 2, treating the strains as if they were infinitesimal. Indeed, an important unsolved problem is the generalization of fig. 2 to the advancing crack case. For rigid-plastic materials, δ_t is linearly related to the imposed displacements of loaded boundaries; for well-contained plastic yielding, it is related to the square of the elastic stress intensity factor.

In fact, (1) comes from solutions of <u>incremental</u> type, and it is more proper to write

$$d\varepsilon_{ij} = \frac{d\delta_t}{r} f_{ij}(\theta) . \tag{2}$$

Now consider the following sequence of operations, first in a <u>rigid</u>-plastic material: Boundary displacements are imposed at a crack length ℓ_o, resulting in strains given by integrating (2). Then the crack is cut ahead by $(\Delta\ell)$, and further boundary displacements are imposed, with the strains again given by integrating (2), but with r,θ now referring to polar coordinates emanating from the new crack tip. The same procedure is adopted after a further cut $(\Delta\ell)_2$, followed by additional boundary displacements, and so on. The net result is a crack surface of staircase appearance, with the step heights marking the δ_t's due to the further boundary displacements imposed after each cutting operation. Hence, in the limit of a continuous increase of crack length with boundary displacement, the strains are still given by integration of (2), with the r,θ values for a particular material point changing continuously with crack length. It is no longer correct to think of a crack tip opening displacement; $d\delta_t$ is the opening which <u>would have</u> resulted had the crack not advanced. But instead, since it has advanced by, say, $d\ell$ while $d\delta_t$ is imposed, there is a <u>crack tip opening angle</u> instead, and if $2\phi_t$ denotes the total angle of opening, then

$$\tan \phi_t = \frac{1}{2}\frac{d\delta_t}{d\ell} . \tag{3}$$

Specific expressions for $d\delta_t$ have the form, for <u>rigid</u>-plastic specimens,

$$d\delta_t = \beta \, dq \tag{4}$$

where q is the load-point displacement and where in general β may depend on, say, the ratio of crack length to some specimen dimension such as the uncracked ligament size. This ratio β of $d\delta_t$ to dq is determined, for example, from the flow fields of rigid-plastic slip-line solutions for plane strain specimens.

The same analysis is approximately applicable to growing cracks in elastic-plastic materials, with one important omission having to do with the induced incompatible elastic straining that will be discussed subsequently (and which contributes essentially additive strain increments to eq. 2), and with an interpretation of $d\delta_t$ that is guided by the following considerations, which are given separately for fully plastic specimens and for nominally elastic specimens with well-contained yielding (it is not yet clear as to how intermediate cases may be treated).

Consider, then, an elastic-plastic specimen deformed such that the uncracked ligament is fully plastic. Let Q be the applied load per unit thickness into the plane of the paper in fig. 3 . The load point displacement may be decomposed into elastic and plastic parts as

$$q = q^e + q^p , \tag{5}$$

where the elastic displacement is defined as that which would be recovered on unloading if the material were somehow frozen against reverse plastic flow. It is given by

$$q^e = C(\ell)\, Q , \tag{6}$$

where $C(\ell)$ is the elastic compliance. The part of the compliance due to the presence of the crack can be expressed in terms of Irwin's relation between compliance and elastic stress intensity factors. Thus if the stress intensity factor is written as

$$K = k(\ell)\, Q , \tag{7}$$

then for plane strain conditions (e.g., Rice, 1968a)

$$\begin{aligned} \frac{1}{2}\frac{dC(\ell)}{d\ell} &= \frac{1-\nu^2}{E}[k(\ell)]^2 , \quad \text{or} \\ C(\ell) &= C(0) + \frac{2(1-\nu^2)}{E}\int_0^{\ell}[k(\ell')]^2 d\ell' , \end{aligned} \tag{8}$$

where $C(0)$ is the elastic compliance of the specimen (or, with another interpretation of Q and q , of the specimen plus loading apparatus) when no crack is present. Now, suppose that the crack advances by $d\ell$ while some increment dq of the load point displacement is imposed. We can accomplish this in two steps: First, advance the crack by $d\ell$, but diminish q however appropriate to ensure that no further plastic deformation occurs during the advance. This can never be accomplished precisely, especially in view of the effects of the induced incompatible elastic straining, but we shall suppose that the first step corresponds to $dq^p = 0$, so that only q^e has been diminished. Then after the crack advance, the load point is displaced by dq plus an amount which annuls the diminishing of q^e. Hence the effective increment of load point displacement in the second step is

$$dq - dq^e = dq^p \quad , \tag{9}$$

where the essentially negative dq^e is

$$dq^e = d[C(\ell)\, Q(\ell)] \quad , \tag{10}$$

it being noted that for the non-hardening idealization Q is some definite function of ℓ, decreasing with increasing ℓ, that gives the plastic limit load for a crack of length ℓ. Thus, applying (4), the increment $d\delta_t$ to be employed in (2) is

$$d\delta_t = \beta\, dq^p = \beta\, \{dq + d[-C(\ell)\, Q(\ell)]\} \quad , \tag{11}$$

where the (positive) differential $d[-CQ]$ gives the additional contribution to $d\delta_t$ which occurs because of the material's elasticity.

For the opposite extreme of well-contained plastic yielding, the following viewpoint seems to be in approximate accord with experience gained by rigorous analyses of the anti-plane strain case. Let ω be the distance by which the elastic-plastic boundary leads the crack, in the direction of its growth. This is defined in fig. 4 a and b for plastic regions typical respectively of plane strain and of anti-plane strain or 2-D plane stress with diffuse yielding. For a stationary crack loaded monotonically in the small scale yielding range, in a previously undeformed material, ω is a pure number times K^2/σ_o^2 and

$$\delta_t = \alpha \frac{\sigma_o}{E} \omega \tag{12}$$

where α is a pure number equal to about 3 for plane stress or anti-plane strain and, depending on the precise interpretation of ω as discussed subsequently, between 5 and 10 for plane strain. In the case of the growing crack we will assume that ω bears approximately the same relation to K^2/σ_o^2 as for the stationary crack, and that when the material is further loaded to make ω increase <u>without</u> change of crack length,

$$d\delta_t = \alpha \frac{\sigma_o}{E} d\omega \tag{13}$$

is the incremental increase in opening displacement.

Now consider an advance $d\ell$ in crack length which takes place under an increment of applied load which, together with the crack advance, has the net effect of increasing ω by $d\omega$. As before, we accomplish this in two steps. First we let the crack extend but simultaneously decrease the applied load by an amount which is just sufficient to ensure that the plastically active region remains fixed relative to the material (i.e., ω <u>diminishes</u> by $d\ell$) . Again, it is not possible to precisely satisfy this and, additionally, the increment of crack advance induces the incompatible elastic deformation which is deferred for subsequent consideration. After this first step the prescribed load increment <u>plus</u> an increment of load to correct for the load decrease in the first step are applied, so that the net increase of plastic region relative to the material is $d\omega + d\ell$. Hence from (13) the increment $d\delta_t$ which is to be employed in (2) is

$$d\delta_t = \alpha \frac{\sigma_o}{E} (d\omega + d\ell) = \delta_t \left(\frac{d\omega}{\omega} + \frac{d\ell}{\omega}\right) . \tag{14}$$

Here the use of δ_t in the second version of the equation is purely notational. The term is defined by (12) and represents the opening displacement that would be present if all the load were applied at the current crack length; it is not intended to imply that a non-zero opening displacement can be defined for a <u>growing</u> crack, based on the present 'small geometry change' analysis, or, if one is defined from a more exact analysis, that it will be related to δ_t as given by (12). In this connection, Chitaley and McClintock (1971) have given a formula for a non-zero opening displacement accompanying a growing anti-plane strain crack. Their result is in error, however, and comes about because of some approximations, made for the sake of numerical simplicity, in their evaluation of the displacement. The correct value which follows from their analysis is zero, and this can be seen by direct determination of the displacement field by integration of their formulae for the near tip strain field, and by examining the (vanishing) limit of the resulting integral as the crack tip is approached.

Note that in integrating (2), the r, θ values associated with a given material point change as the crack advances. In fact, writing

$$x = \ell + r\cos\theta \quad , \quad y = r\sin\theta$$

(fig. 3) one may show that the change in θ at a fixed material point (i.e., fixed x, y coordinates) is

$$d\theta = \frac{d\ell}{r}\sin\theta \quad . \tag{15}$$

The missing ingredient is the induced incompatible elastic straining. The origin of the effect is this: The yield condition of a non-hardening material and the stress equilibrium equations are alone sufficient to determine much of the form of the near tip stress field, in particular the variation of σ_{ij} with θ as $r \to 0$, apart from a limited non-uniqueness related to the angular extents of the regions which are at yield very near the tip, which are unknown, in general, *a priori*. Rice and Tracey (1973) have given a rigorous analysis of the plane strain case as $r \to 0$, and show, for example, that the near tip field has a centered fan structure in any angular sector at the tip sustaining singular strain rates. Earlier analyses have been given by Rice (1968 a,b), by Hutchinson (1968) for 2-D plane stress, and by McClintock (1958) for anti-plane strain.

Hence we can assume that there is a near tip stress distribution $\sigma_{ij} = \sigma_{ij}(\theta)$ which moves through the material with the advancing crack tip. Let $\varepsilon^e_{ij} = \varepsilon^e_{ij}(\theta)$ be the associated *elastic* strains, related to this stress field by the appropriate linear Hookian formulae. Then the *elastic* strain increment induced at a given material point, having polar coordinates r, θ, by an increment $d\ell$ of crack advance is

$$d\varepsilon^e_{ij} = \varepsilon^{e\prime}_{ij}(\theta)\frac{d\theta}{d\ell}d\ell = \varepsilon^{e\prime}_{ij}(\theta)\frac{\sin\theta}{r}d\ell \quad , \tag{16}$$

where the prime means derivative with respect to θ and (15) is employed. In general, this elastic strain increment field is *not* derivable from a set of displacement increments; i.e., it is incompatible. Thus a plastic strain increment is induced as well and, for a Mises material, this satisfies

$$d\varepsilon_{ij}^{p} = d\varepsilon_{ij} - d\varepsilon_{ij}^{e} = d\Lambda\ s_{ij}(\theta) \quad , \tag{17}$$

where s_{ij} is the deviatoric part of the stress state, where $d\Lambda$ is a non-negative scalar, and where it is the total strain increment $d\varepsilon_{ij}$ which is derivable from a field of displacement increments.

By eliminating $d\Lambda$ in (17), one arrives at a set of linear combinations of the $d\varepsilon_{ij}$'s which are set equal to corresponding linear combinations of the $d\varepsilon_{ij}^{e}$'s . Since the latter have a $1/r$ term by (16), this set of equations for the $d\varepsilon_{ij}$'s has solutions for displacement increments in the form

$$du_i = [\alpha_i(\theta) \log \frac{1}{r} + \beta_i(\theta)]d\ell \quad , \tag{18}$$

where the set of functions $\alpha_i(\theta)$ are determined in terms of $\varepsilon_{ij}^{e}(\theta)$ and arise because of the incompatibility. Further, since the elastic strains are of order σ_o/E , the α_i's are of form σ_o/E times terms of order unity. The strain increment field resulting from (18) thus takes the form

$$d\varepsilon_{ij} = \frac{\sigma_o}{E}\frac{1}{r}[g_{ij}(\theta) \log \frac{1}{r} + h_{ij}(\theta)]d\ell + d\varepsilon_{ij}^{e} \quad , \tag{19}$$

where the first set of terms, containing the bracket, is the plastic portion of the strain increment. The set of functions g_{ij} , which result from the α_i's are of order unity within fan-like sectors at the crack tip and these, as well as the h_{ij}'s , share the same directions of zero deformation rates as do the f_{ij}'s in (1,2), since all are distributed in proportion to s_{ij} .

Now, earlier in this section, crack advance in elastic-plastic materials was subdivided into two steps, the first of which was then assumed to induce no overall plastic deformation dq^{p} or no increase of extent of the yield zone. We shall now take the point of view that eq. (2), with (11) or (14) for $d\delta_t$, properly accounts for the strain increments induced in the second step, but that the first step also induces plastic straining due to the incompatibility effect. This induced plastic straining should, however, vanish at the boundaries of the plastically active zone. For this reason we make the approximate identification

$$h_{ij}(\theta) \approx g_{ij}(\theta) \log R(\theta)$$

in (19) so that the strain induced in the <u>first</u> step is now written

$$d\varepsilon_{ij} = \frac{\sigma_o}{E}\frac{d\ell}{r} g_{ij}(\theta) \log \frac{R(\theta)}{r} + d\varepsilon_{ij}^{e} . \qquad (20)$$

Evidently, $R(\theta)$ is the outer cut-off radius at angle θ for this effect. In problems of contained plasticity, we can approximately identify $R(\theta)$ as the distance to the elastic-plastic boundary or, in accord with the use of the same symbol by Rice and coworkers, as the distance from the crack tip at which the representation (1) for ε_{ij} would predict strains corresponding to initial yield. For fully plastic plane strain specimens with deep double edge cracks, in which Prandtl fans result, R might be chosen as the radius of the fan regions; however, it is not evident as to how it should be chosen for bend or single edge cracked tension specimens in which discrete slip surfaces emanate from the tip. Indeed, in the latter case $d\varepsilon_{ij}^{e}$ vanishes anyway so that the g_{ij}'s vanish.

The final equation for an increment of strain therefore amounts to summing (20) for the first step and (2) for the second. Hence the total plastic strain increment is

$$d\varepsilon_{ij}^{p} = \frac{\sigma_o}{E}\frac{d\ell}{r} g_{ij}(\theta) \log \frac{R(\theta)}{r} + \frac{d\delta_t}{r} f_{ij}(\theta) , \qquad (21)$$

where:

$$d\delta_t = \alpha \frac{\sigma_o}{E} (d\omega + d\ell) , \quad \text{small scale yielding;} \qquad (14')$$

$$d\delta_t = \beta \, dq + \beta \, d[-C(\ell)Q(\ell)] , \quad \text{fully plastic .} \qquad (11')$$

Strain Accumulation at Growing Crack Tips

The functions f_{ij} and g_{ij} of (21) are non-zero only within angular sectors at the crack tip sustaining centered fan singularities. For contained yielding under plane stress or anti-plane strain conditions, these sectors exist directly ahead of the tip (fig. 4b) so that the crack advances into its own strain singularity. By contrast, for contained plane strain yielding, the singular sectors are above and below the tip (fig. 4a), and begin only at an angle of 45°, so that at least on average the crack grows between its singularities. The regions are also directly above and below the tip for fully plastic plane strain specimens, although in some cases the fans degenerate to concentrated slip bands (look ahead to figs. 5 a and b).

Consider first the plane stress and anti-plane strain type of singular zone. If for contained yielding we identify $R(0)$ as the frontal plastic zone extent ω of fig. 4b, then along the line directly ahead of the growing fracture (i.e., $\theta = 0$) eq. (21) reduces to

$$d\varepsilon_{ij}^{p} = \frac{\sigma_o}{E}\frac{d\ell}{r} g_{ij}(0) \log\frac{\omega}{r} + \frac{d\delta_t}{r} f_{ij}(0) . \quad (22)$$

Obviously, a $1/r$ strain field as in (1) is the result of integrating this expression for a stationary crack. To see the opposite extreme, consider steady state conditions of crack advance for which $d\delta_t/d\ell$ and ω are imagined to remain constant as the crack grows. Then, since $d\ell = -dr$, (22) may be immediately integrated to give the strain distribution

$$\varepsilon_{ij}^{p} = \int_{\omega}^{r} \frac{d\varepsilon_{ij}^{p}}{d\ell}(-dr)$$

$$= \frac{1}{2}\frac{\sigma_o}{E} g_{ij}(0) \left(\log\frac{\omega}{r}\right)^2 + \frac{d\delta_t}{d\ell} f_{ij}(0) \log\frac{\omega}{r} . \quad (23)$$

This is markedly different from (1), and we see that the $1/r$ strain singularity has now changed to a logarithmic singularity. The latter term of (23) contains the effect of crack growth into already deformed material; the former results from the induced incompatible elastic straining which accompanies growth.

In fact, under steady state conditions $d\delta_t/d\ell$ is of the order σ_o/E from (14) and, unless $f_{ij}(0)$ is much greater than $g_{ij}(0)$, the term arising from elastic incompatibility is the dominant term very near the tip. This is the case for anti-plane strain (Rice, 1968a). On the other hand, if during the early phases of stable growth the plastic zone size must be greatly increased to maintain continuing crack extension, as seems to be the case, the term in (22) containing $d\delta_t$ is dominant instead. Indeed, this is the only term which contributes during the initial loading, before growth begins.

If the material point under consideration does not lie in the plane of the crack, (21) must be integrated by considering θ as well as r to be variable as the crack grows. It is simplest to replace r with $y/\sin\theta$, since the y coordinate remains fixed, and to write $d\ell = r\,d\theta/\sin\theta$ from (15). Then, under steady state conditions the strain is given by

$$\varepsilon^p_{ij} = \frac{\sigma_o}{E}\int_{\theta_*}^{\theta} g_{ij}(\theta)\log\left[\frac{R(\theta)\sin\theta}{y}\right]\frac{d\theta}{\sin\theta} + \frac{d\delta_t}{d\ell}\int_{\theta_*}^{\theta} f_{ij}(\theta)\frac{d\theta}{\sin\theta}\ , \qquad (24)$$

where θ_* is the angle of the material point when it first enters the region near the tip for which (21) applies. For $\theta_* \to 0$, this reduces to (23) upon proper computation of the limit.

Eq. (24) is also applicable to contained plane strain yielding. Now, however, for material points which approach the plane of fracture, the angle θ_* will not approach zero but instead will equal $45°$, corresponding to the centered fan boundary, fig. 4a. Hence, even for points very near the crack tip, the term in (24) containing $d\delta_t/d\ell$ will contribute non-singular strain only, whereas the first term, reflecting the induced elastic incompatibility, contributes a part of the strain which is logarithmically singular. This contrasts with the plane stress and anti-plane strain cases, for which stronger singularities are induced with crack growth.

Nevertheless, it is difficult to conclude much about the plane strain case from (24), because at least on average, the crack does not advance into the fan region. Further, it is known that deformations near a stationary crack can be correctly interpreted only when the actual finite geometry changes accompanying crack tip blunting are included (Rice and Johnson, 1970, and discussion in connection with fig. 2), and these are presumably as important for a growing crack. While rigid plastic considerations suggest that the crack advances with no opening displacement at its tip and with a finite opening angle, given by (3), inclusion of the elastic incompatibility effect results in (18) for displacement increments, and these suggest that $du_i/d\ell \approx -\partial u_i/\partial x$ is logarithmically infinite at the tip. Hence, while the opening displacement at the tip is zero, the displacement profile is predicted to exhibit a vertical tangent there (again, within an analysis neglecting large geometry changes), and this argues for the presence of a somewhat opened tip as the crack grows. This is an important area for future clarification.

It should finally be emphasized that the features of actual combined stress plastic behavior which are ignored in the perfectly plastic model (in particular, the development or 'vertices' or sharply rounded 'noses' on combined stress yield surfaces, Bauschinger effects, etc.) would largely tend to make the material behave a little more like the non-linear elastic material in fig. 1, and a little less like the rigid-plastic material of the figure or the elastic-non-hardening-plastic material of the preceding calculations.

Plane Strain Crack Growth

The general analysis can be carried through in further detail for plane strain crack growth, at least for contained yielding. In that case the functions g_{ij} can be inferred from Rice's (1968a) analysis of the steady state, and $R(\theta)$ and the f_{ij}'s can be taken approximately from the same source or from the Rice and Tracey (1973) numerical solution. But this is not likely to be productive because, as was remarked upon for the stationary crack case (fig. 2 and Rice and Johnson, 1970), a realistic description of plane strain fracturing must necessarily involve an analysis of the large local geometry changes near the blunted tip. Indeed, the material points which enter this region come from the angular sector directly ahead of the crack, where the small geometry change solution predicts that there is no intense prior straining at all. It is likely that this is why there is typically so little stable growth in plane strain as compared to plane stress or to anti-plane strain predictions (e.g., McClintock and Irwin, 1965). There is as yet no suitable analysis of the large local geometry changes for growing cracks and the problem is likely to be further complicated by a zig-zag mechanism of growth on a microstructural scale (McClintock, 1969b).

It may, however, be fruitful to discuss plane strain crack growth and instability on a comparative basis, for different fully plastic specimens and for well-contained yielding, utilizing concepts from the general formulation.

Fig. 5 shows three fully yielded perfectly-plastic specimens. Slip line patterns are as illustrated and limit loads are given in terms of the shear yield stress τ_o, here taken equal to $\sigma_o/\sqrt{3}$. Also, below each specimen the corresponding formula for $d\delta_t$ is given in the form of (11), relating it to the plastic increment of overall deformation. Here θ is the net rotation of one end of the bend specimen relative to the other and q is the net extension for the tensile specimens. Specimen widths are W and uncracked ligament dimensions are b . Approximating the elastic stress intensity factor for the tensile specimens (b) and (c) by the value for an infinite periodic array of colinear cracks (Paris and Sih, 1965), one derives through the theory of eqs. (6-8) that

$$q^e = C\,Q \approx \left[\frac{L}{W} + \frac{4}{\pi}\log\left(\frac{1}{\sin\lambda}\right)\right]\frac{(1-\nu^2)Q}{E} \qquad \text{where} \quad \lambda = \pi b/2W \,. \tag{25}$$

Here L is the overall length of the tensile specimens, and appears because of the $C(0)$ term in (8). In fact, if we wish to think of displacement as being controlled not at the specimen load point, but rather at some point of the loading system, then we shall understand that L is the 'effective' length and thus contains an additive term accounting for the compliance of the loading system. Similarly, from Wilson's (1970) formula $K \approx 4\,M\,b^{-3/2}$ for a deeply cracked bend specimen,

$$\theta^e = C\,M \approx \left(\frac{12\,L}{W^3} + \frac{16}{b^2}\right)\frac{(1-\nu^2)M}{E} \,, \tag{26}$$

where again L is the effective specimen length.

If we now compute $d\delta_t$ from (11) for the three specimens, using the above compliance formulae and the limit load and $d\delta_t$ formulae as given in the figure, then:

For a deeply cracked bend specimen, since $db = -d\ell$,

$$\begin{aligned} d\delta_t &= .37\,b\left[d\theta + \frac{24\,(.63)(1-\nu^2)\sigma_o\,L\,b}{\sqrt{3}\;E\;W^3}\,d\ell\right] \\ &\approx .37\,b\,d\theta + 2.7\,\frac{\sigma_o}{E}\,\frac{L\,b^2}{W^3}\,d\ell \,. \end{aligned} \tag{27}$$

The second term containing $d\ell$ is of interest because it includes the effect of the material's elasticity on further deforming the crack tip region, even if the crack advances under conditions for which $d\theta = 0$.

For the single edge crack tensile specimen (fig. 5b), under a force acting centrally on the ligament so as to cause extension without rotation of the specimen ends, since $db = -d\ell$,

$$d\delta_t = dq + \frac{2\,(1-\nu^2)\sigma_o}{\sqrt{3}\,E}\left[\frac{L}{W} + \frac{4}{\pi}\{\log\left(\frac{1}{\sin\lambda}\right) - \frac{\lambda\cos\lambda}{\sin\lambda}\}\right] d\ell$$

$$\approx dq + \frac{\sigma_o}{E}\left[\frac{L}{W} + \ldots\right] d\ell \ , \qquad (28)$$

where the dots, depending on λ , are negligible if L is several times W and λ is not too small.

For the deeply double edge cracked specimen (fig. 5c), since $db = -2d\ell$,

$$d\delta_t = 2\left\{dq + \frac{(2+\pi)(1-\nu^2)\sigma_o}{\sqrt{3}\,E}\left[\frac{L}{W} + \frac{4}{\pi}\{\log\ \frac{1}{\sin\lambda} - \frac{\lambda\cos\lambda}{\sin\lambda}\}\right](2d\ell)\right\} \qquad (29)$$

$$\approx 2\,dq + 10.3\ \frac{\sigma_o}{E}\left[\frac{L}{W} + \ldots\right] d\ell \ .$$

These formulae all have the form that the first term accounts for further imposed displacement of the load point, whereas the latter term containing $d\ell$ incorporates the effect of the specimen's elasticity. It is a destablizing term in that it tends to promote a running fracture, although the specimens cannot be unambiguously compared on the basis of it because of the elastic incompatibility effect in (21), which is likely to be different among the different specimen designs, being least prominent in fig. 5b. The fracture criterion may also be affected by differences in overall flow pattern and stress triaxiality. Nevertheless, if we compare the terms when L/W is chosen as 4 , to account for specimen and loading system compliance, and if we take $1/2$ as a representative value of the terms represented by the dots in (28,29), then when load point displacements are held fixed,

$$\begin{aligned} d\delta_t &\approx 2.7\ (\sigma_o/E)d\ell \ , && \text{bend specimen, } b = W/2 \\ &\approx 0.7\ (\sigma_o/E)d\ell \ , && \text{bend specimen, } b = W/4 \\ &\approx 4.5\ (\sigma_o/E)d\ell \ , && \text{single edge crack} \\ &\approx 46.3\ (\sigma_o/E)d\ell \ , && \text{double edge cracks} \end{aligned} \qquad (30)$$

Thus as judged from these terms alone, the deeply cracked bend specimen would seem most conducive to stable extension of the crack. It is also likely to overestimate the amount of stable crack growth that will be observed in other specimens. For example, the ratio $d\delta_t/d\ell$ is so large for the deeply double edge cracked specimen that a running fracture might commence immediately upon initiation of crack extension, for materials which show substantial stable growth in other specimen geometries. Of course, an increase by, say, two in the chosen value for L would increase the numbers in (30) accordingly to 5.4, 1.4, 8.5, and 87.5 as we go down the list.

A comparison with the well-contained yielding case can be made by assigning a numerical value to α in eqs. (12-14). This rests on the certainly arbitrary choice of the frontal plastic zone extent ω as the basis for assessing the contribution of growth to $d\delta_t$ in (14). From the numerical solution of Rice and Tracey (1973) for monotonic loading of a stationary crack, if ω is taken to be the actual frontal extent of yielded material, then

$$\omega \approx \frac{1}{10}\,(K^2/\sigma_o^2) \quad , \quad \text{and} \quad \alpha \approx 5 \, . \tag{31}$$

On the other hand, if we define an effective plastic zone boundary as the contour $r = R(\theta)$, where $R(\theta)$ is the distance from the tip at which the singular strain expression (1), if assumed to hold at distances from the tip comparable to overall plastic zone size, would predict strains corresponding to first yield, then from the Rice and Tracey plot of $R(\theta)$,

$$\omega \approx \frac{1}{20}\,(K^2/\sigma_o^2) \quad , \quad \text{and} \quad \alpha \approx 10 \, . \tag{32}$$

It may, in fact, be more correct to base the estimate on the singularity amplitude $R(\theta)$, but the arbitrariness of the entire procedure should be kept in mind. With this latter estimate we obtain, from (14) that

$$d\delta_t \approx 10\,(\sigma_o/E)d\ell \tag{33}$$

for crack advance under a constant value of K, and hence of ω. The latter version of (14) makes it evident that essentially the same result would be obtained for conditions of constant load or of load-point displacement during advance, since the plastic zone dimensions are here assumed to be small in comparison to crack length and ligament size. If this factor of 10 in (33) is accurate, then by comparison with (30), different fully plastic specimens may either strongly underestimate or overestimate the destabilizing tendency present for well-contained plasticity. Again, these remarks exclude consideration of the uncertain elastic incompatibility effect in the different cases, and the crucial factor of large local geometry changes is not in the model.

Of course, in the above estimates of growth induced $d\delta_t$'s, as well as in the elastic incompatibility term of (21), what finally matters in comparing growth effects among different materials, when overall plastic zone dimensions are similar among the cases examined, is the ratio σ_o/E. When this ratio is high, the destabilizing terms are important and less stable growth, if any at all, is to be expected in comparison to cases for which the ratio is low. Generally speaking, it has been possible to determine the nominally elastic plane strain fracture toughness, K_{Ic}, in specimens of reasonably small size only for the higher strength materials, with σ_o/E greater than about .006 or so (i.e., $\sigma_o \gtrsim 60$ ksi in aluminum alloys, and $\gtrsim 180$ ksi in steels). Hence, while it is often assumed that such fractures are accompanied by, at most, very small amounts of stable growth (at least during the plane strain portion, prior to formation of a thumbnail crack in a plate specimen, after which substantial stable growth of a 3-D kind may occur), this may not hold true in general. In particular, it is plausible that if lower strength materials were tested in specimens of sufficient overall size to maintain K_{Ic} - like conditions of well contained yielding, that quite substantial amounts of stable crack growth would be observed.

In such an event, a plane strain 'resistance curve', of the type employed by Krafft et al. (1961) would presumably be applicable. Amounts of stable crack growth and, in particular, instability points would then be determined by the well known condition of tangential contact between the 'R curve' and that of G vs. ℓ for different values of applied load or load-point displacement. However, the difficulty is that the specimen size for such evaluations would be excessive. A more practical range of specimen sizes, and perhaps also of thick-walled structural sizes, would then typically involve large scale plastic yielding during plane strain crack advance. The proper generalization of the parameters upon which the R-curve procedure is founded and, indeed, the validity of the concept itself remain uncertain under these conditions.

It is also worthy of note that an interpretation of crack growth behavior which is too strongly tied to non-hardening slip line fields, as in fig. 5, may be misleading. These fields are of value in obtaining overall flow fields and applied forces, but it is possible, at least for a stationary crack, that the material's strain hardening serves to create deformation fields very near the tip which are more unique than the variety pictured in fig. 5 would indicate. This was argued by Begley and Landes (1972) in their use of the J integral as a fracture initiation criterion; they proposed that one-parameter near tip singular strain fields of the Hutchinson-Rice-Rosengren type would exist at the tip and that J could be chosen as the parameter. Alternatively, to the extent that a crack tip opening displacement is well-defined outside the context of a non-hardening model, so also could the <u>actual</u> δ_t be chosen. Indeed, the actual δ_t would then have to be uniquely related to J, in the same form for all specimen geometries and extent of plastic yielding.

But the actual δ_t or J value is <u>not</u> uniquely related to the δ_t as inferred from the non-hardening slip line solutions, and as given by the formulae in fig. 5. Suppose, for example, that the specimens are loaded well into the plastic range so that contributions to J from the plastic range overwhelm those accumulated in the elastic-plastic regime. Then from the method for J estimation of Bucci et al. (1972),

$$\begin{aligned} J &= 1.26\,\tau_o b\theta \quad \text{, bend specimen} \\ &= 2\,\tau_o q \quad \text{, single edge cracked specimen} \\ &= (2+\pi)\tau_o q \quad \text{, double edge cracked specimen ,} \end{aligned} \tag{34}$$

where now $\tau_o = \sigma_o/\sqrt{3}$ is to be chosen by identifying σ_o as, say, the ultimate tensile strength to closely match limit load predictions to actual fully-plastic load levels. Now, the δ_t's <u>as predicted from the slip line solutions</u> are $.37\,b\theta$, q, and $2q$ for the three specimens and thus one computes

$$\begin{aligned} \delta_t &= .51\,J/\sigma_o \quad \text{, bend specimen} \\ &= .87\,J/\sigma_o \quad \text{, single edge crack specimen} \\ &= .67\,J/\sigma_o \quad \text{, double edge crack specimen} \end{aligned} \tag{35}$$

Thus, if there is a unique one-parameter near tip field, which means that the actual δ_t is uniquely related to J, then the inference is that slip-line field predictions cannot be relied upon very near the tip. The presence or absence of such a one-parameter field over a microstructurally significant size scale must depend both on the amount of strain hardening and on absolute specimen size, as discussed by McClintock (1971) and by Begley and Landes (1972).

For comparison with (35), the non-strain hardening, incremental, plane strain, small scale yielding solution of Rice and Tracey (1973), which is presumably the most accurate available solution, gives

$$\delta_t = .49\ K^2/E\sigma_o = .54\ J/\sigma_o \ , \tag{36}$$

taking $J = (1-\nu^2)K^2/E$ and setting $\nu = .3$. However, Rice and Tracey note that the value J_{tip} of J on a contour immediately about the tip is approximately .8 of this value, according to their incremental numerical results, which would then suggest a numerical factor of .68 when (36) is based on J_{tip}. For power law strain hardening materials, with $\tau \propto \gamma^N$ in the plastic range under pure shear, numerical results by Tracey (1973) closely substantiate a formula by Rice (1973), based on analogy with Mode III, that

$$\delta_t = .49\ \frac{K^2}{E\sigma_o}\left[\frac{2\ (1+\nu)\sigma_o(1+N)}{\sqrt{3}\ E\ N}\right]^N , \tag{37}$$

where the opening is defined from the intersection of a 45° line drawn back from the tip with the deformed crack surfaces as predicted from the small geometry change solution. This is equivalent to replacing σ_o in (36) by the equivalent tensile flow stress corresponding to a strain $\gamma = N/(1+N)$ in pure shear. When re-expressed in terms of J_{tip}, (37) may well constitute the unique relation of the actual δ_t to J as referred to above. By comparison, the Dugdale-BCS formula $\delta_t = J/\sigma_o$ is not accurate, although the Well's procedure of using $2\sigma_o$ to account for triaxial constraint agrees closely with the more exact estimates.

While Begley and Landes achieve remarkable success in correlating the initiation of plane strain crack extension with a critical J criterion, there is as yet no way by which the approach can be generalized to growing cracks. Perhaps it will be possible to identify an effective dJ in a manner parallel to the deduction of an effective $d\delta_t$ in (11) and (14), but this remains to be seen.

2-D Models for Crack Growth in Plate Specimens

Here we consider crack growth in plate specimens under conditions which are nominally referred to as "plane stress". It has been noted that such cases are strongly influenced, and sometimes dominated, by 3-D features such as crack tunnelling and transition from flat to shear mode separation. However, we shall examine the problem from the standpoint of two-dimensional models here, assuming that the plastic zone is large compared to plate thickness and that any 3-D features can be imbedded within the 2-D model in some average sense.

From observations, it is known that the plane stress plastic zone can be either diffuse as pictured in fig. 4b (e.g., Gerberich, 1964) or localized in the Dugdale-BCS sense to a thin zone ahead of the tip of height comparable to plate thickness (Hahn and Rosenfield, 1965). The exact reasons for occurrence of one type of zone instead of the other are unknown; it is likely that yield instability leads to the localized zone, but other possibilities exist (Drucker and Rice, 1970). Also, the plastic zone may be diffuse far from the tip but localized nearby, due either to necking or to reduction of the thickness average yield strength by crack tunnelling.

It is often remarked that the Dugdale-BCS line plastic zone leads to no predictions of stable growth, because the distribution of opening displacement $\delta(x)$ within the plastic zone is independent of the detailed history of prior load and crack length increments. However, a physically appealing re-interpretation of the model by Wnuk (1972), in terms of a 'final stretch' fracture criterion, does lead to growth effects. He argues that only the crack opening accumulated while a material point is within some distance Δ from the tip should enter the criterion. Here Δ is supposed to be much smaller than the plastic zone extent ω, and we may for example take it to correspond to the distance from the tip at which strongly localized deformation sets in because of tunnelling or necking. The Dugdale-BCS model leads to the opening displacement expression (e.g., Rice, 1968a)

$$\delta = \frac{8\,\sigma_o\,\omega}{\pi\,E}\left[\xi - \frac{r}{2\omega}\log\frac{1+\xi}{1-\xi}\right], \qquad \xi = \sqrt{1 - r/\omega} \quad , \tag{38}$$

where r is distance from the tip and where the formula appropriate to small scale yielding is used, in which case $\omega = (\pi/8)K^2/\sigma_o^2$. Thus, if the crack has grown enough so that ω changes smoothly with crack length during subsequent growth, if δ_{fs} is the critical final stretch, and if $\Delta << \omega$, then the criterion reads:

$$\delta_{fs} = \left\{\frac{8\ \sigma_o}{\pi\ E}\left(\omega + \frac{d\omega}{d\ell}\Delta\right)\right\} - \left\{\frac{8\ \sigma_o}{\pi\ E}\,\omega\left[1 - \frac{\Delta}{2\omega}\left(1 + \log\frac{4\omega}{\Delta}\right)\right]\right\} \tag{39}$$

This determines the required increase of plastic zone size (or K) with crack length to continually meet the fracture criterion. Rearranged,

$$\frac{d\omega}{d\ell} = \frac{\pi\ E}{8\ \sigma_o\Delta}\,\delta_{fs} - \frac{1}{2}\left(1 + \log\frac{4\omega}{\Delta}\right) . \tag{40}$$

The equation, when integrated, gives the analog of a 'resistance curve', now phrased in terms of requisite plastic zone size. Since ω is also known as a function of load and crack length, the load point instability ensues at the load and crack length for which there is tangential contact between this resistance curve and plots of ω vs. ℓ at fixed loads. This concept is illustrated in fig. 6 for a failure criterion phrased in the terms

$$\mathcal{G}(Q,\ell) = \mathcal{R}(\ell - \ell_o) , \tag{41}$$

where ℓ_o is the initial crack length. However, it should be noted that the resistance curve interpretation of instability for Wnuk's model would not be rigorously valid beyond the small scale yielding assumptions.

To see the extent of growth effects predicted by (40), note that the plastic zone size which must be achieved to maintain steady-state crack growth, with $d\omega/d\ell = 0$, is

$$\omega_{st-st} = \frac{\Delta}{4}\exp\left[\frac{\pi\ E\ \delta_{fs}}{4\ \sigma_o\Delta} - 1\right] \tag{42}$$

On the other hand, if the same final stretch is identified as the δ_t at which crack extension initiates, then the requird ω to initiate crack extension is, from (38),

$$\omega_{init} = \frac{\pi\ E}{8\ \sigma_o}\,\delta_{fs} . \tag{43}$$

Because of the exponential term in (42), a very much larger plastic zone is typically required for steady state conditions than for initiation. The former corresponds to the horizontal plateau of the R-curve in fig. 6; the latter to the initial bending over of the R-curve from the vertical. Further, for all other parameters the same, the disparity between the two, and hence the amount of growth before instability, lessens the greater is σ_o/E .

The actual extents of stable growth in thin sheets can be substantial. Broek (1968) shows examples of approximately 50% increase of crack length, amounting to 20 to 30 times the plate thickness, in tensile tests of 2024-T3 aluminum sheets. He also shows that even if parameters entering the resistance curve description are phrased in terms of elastic fracture mechanics, the final locus of stress levels and cracks lengths at instability need not correspond to a critical K value.

When the plane stress plastic zone is viewed as being diffuse, as in fig. 4b, strain distributions result in front of the crack of a kind given in eqs. (22,23). Because all of the details of the general formulation have not yet been worked out for this case, and because of its qualitative similarity with McClintock's anti-plane shear case, we shall consider the latter here. Then the equation for increments of plastic shear strain γ_{yz}^p on the line directly ahead of the crack is (e.g., Rice, 1968a)

$$d\gamma_{yz}^p = \frac{\tau_o}{G}\frac{d\ell}{r}\log\frac{\omega}{r} + \frac{\tau_o}{G}\frac{d\omega + d\ell}{r} \quad , \tag{44}$$

where G is the shear modulus. This equation is seen to be a special form of (22), with $d\delta_t$ in the form of (14). Two special integrals for the crack tip strain distribution, corresponding to the extremes of a stationary crack and of a crack growing in steady state are

$$\gamma_{yz}^p = \frac{\tau_o}{G}\left(\frac{\omega}{r} - 1\right), \quad \gamma_{yz}^p = \frac{\tau_o}{G}\left\{\frac{1}{2}\left[\log\frac{\omega}{r}\right]^2 + \log\frac{\omega}{r}\right\} , \tag{45}$$

respectively, the latter being a special case of (23).

McClintock and Irwin (1965) have illustrated the stable growth effects predicted from (44,45) according to the fracture criterion that a critical plastic strain $\gamma_{yz}^p = \gamma_f$ be continually maintained at some small distance ρ

ahead of the tip for crack growth. In that event (45) can be solved for the plastic zone sizes required for the initiation of growth and for steady-state growth as

$$\omega_{init} = \rho\left(\frac{\gamma_f}{\tau_o/G} + 1\right) \quad ; \quad \omega_{st\text{-}st} = \rho \exp\left[\sqrt{\frac{2\gamma_f}{\tau_o/G} + 1} - 1\right] \qquad (46)$$

The ratio of the latter to the former can be taken as the index of growth effects. One finds

$$\begin{aligned} \omega_{st\text{-}st}/\omega_{init} &= 1.04 \quad \text{when} \quad \gamma_f/(\tau_o/G) = 1 \\ " &= 3 \qquad\quad " \qquad\qquad = 10 \\ " &= 18 \qquad\quad " \qquad\qquad = 25 \\ " &= 169 \qquad\quad " \qquad\qquad = 50 \\ " &= 5100 \qquad\quad " \qquad\qquad = 100 \; . \end{aligned} \qquad (47)$$

For a fixed value of γ_f the ratio is seen to decrease very rapidly with increasing yield strength level, at least in the range for which γ_f exceeds τ_o/G by 25 or more.

McClintock and Irwin (1965) have employed the above fracture criterion, together with a numerical integration of (44) to predict growth and instability for an edge crack in a half-space. Rice (1968a) has further shown that this criterion leads to a universal resistance curve, depending only on $\ell - \ell_o$, when phrased in terms of the required plastic zone extent ω. This R-curve has $\omega_{st\text{-}st}$ for its horizontal plateau, and initially deviates from the vertical at ω_{init}. The reader is referred to these works for further details. For example, Rice shows by a generalization of the McClintock-Irwin work that the locus of loads and crack lengths at which load maximum instability occurs is given approximately by

$$\omega(Q,\ell) = \rho \exp\left\{\sqrt{\frac{2\gamma_f}{\tau_o/G} + \left[1 + \frac{\partial\omega(Q,\ell)}{\partial\ell}\right]^2} - \left[1 + \frac{\partial\omega(Q,\ell)}{\partial\ell}\right]\right\} \qquad (48)$$

where it is assumed that the frontal extent ω of the plastic zone is given as a function of load Q and crack length ℓ.

A separate approach to stable growth has been developed by Andersson (1973), with specific reference to a finite element model. His fracture criterion is the attainment of a critical angle of opening between the elements at the crack tip, and would seem to be self-consistent so long as it is understood that the critical angle cannot be considered to be independent of the size chosen for the near tip elements. Indeed, if only a critical angle entered the growth criterion, and not some inherent length scale, there would be no size effect to fracturing. His calculation of growth proceeds by advancing the crack length by one element when the criterion is met, and by then adding further load increments as necessary until it is met at the new crack tip.

In summary, it would seem that the main features of stable growth under plane stress conditions are understood, at least qualitatively, but it is not yet clear as to how parameters of the models are to be chosen in correspondence to the 3-D features which dominate in the fracturing region.

Acknowledgement

I wish to acknowledge support for this study by the National Aeronautics and Space Administration, under grant NGL-40-002-080 to Brown University.

dwa

References

H. Andersson, 1973, Ph.D. Thesis, Lund Technical University.

J. A. Begley and J. D. Landes, 1972, In Fracture Toughness, Proceedings of the 1971 National Symposium on Fracture Mechanics, Part II, ASTM-STP-514, Philadelphia, pp. 1 and 24.

J. I. Bluhm, 1962, In Fracture of Structural Metals, Watertown Arsenal Laboratories Monograph Series -48, U. S. Army Materials Research Agency, Watertown, Mass.

D. Broek, 1968, Int. J. Fracture Mech., v. 4, p. 19.

R. J. Bucci, P. C. Paris, J. D. Landes, and J. R. Rice, 1972, In Fracture Toughness, Proceedings of the 1971 National Symposium on Fracture Mechanics, Part II, ASTM-STP-514, Philadelphia, p. 40.

A. D. Chitaley and F. A. McClintock, 1971, J. Mech. Phys. Solids, v. 19, p. 147.

D. C. Drucker and J. R. Rice, 1970, Engr. Fracture Mech., v. 1, p. 577.

W. W. Gerberich, 1964, Exp. Mech., v. 4, p. 335.

G. T. Hahn and A. R. Rosenfield, 1965, Acta Met., v. 13.

J. M. Krafft, A. M. Sullivan, and R. W. Boyle, 1961, In Proceedings of the Symposium on Crack Propagation, Cranfield College of Aeronautics, p. 8.

F. A. McClintock, 1958, J. Appl. Mech., v. 25, p. 582.

F. A. McClintock and G. R. Irwin, 1965, In Fracture Toughness Testing and Its Applications, ASTM-STP-381, Philadelphia, p. 84.

F. A. McClintock, 1969a, Int. J. Fracture Mech., v. 4, p. 101.

F. A. McClintock, 1969b, In Physics of Strength and Plasticity, edited by A. S. Argon, M.I.T. Press, Cambridge, Mass., p. 307.

F. A. McClintock, 1971, In Fracture: An Advanced Treatise, vol. 3, edited by H. Liebowitz, Academic Press, N. Y., p. 47.

P. C. Paris and G. C. Sih, 1965, In Fracture Toughness Testing and Its Applications, ASTM-STP-381, p. 30.

J. R. Rice, 1968a, In Fracture: An Advanced Treatise, vol. 2, edited by H. Liebowitz, Academic Press, N. Y., p. 191.

J. R. Rice, 1968b, J. Appl. Mech., v. 35, p. 379.

J. R. Rice and M. A. Johnson, 1970, In Inelastic Behavior of Solids, edited by M. F. Kanninen et al., McGraw-Hill, N. Y., p. 641.

J. R. Rice, 1973, In Preprint Volume of Third International Congress on Fracture, Munich.

R. F. Smith and J. F. Knott, 1971, In Practical Applications of Fracture Mechanics to Pressure Vessel Technology, Inst. Mech. Engrs., p. 65.

W. K. Wilson, 1970, Engr. Fracture Mech., v. 2, p. 169.

M. P. Wnuk, 1972, In Proceedings of the International Conference on Dynamic Crack Propagation, Lehigh University, in press.

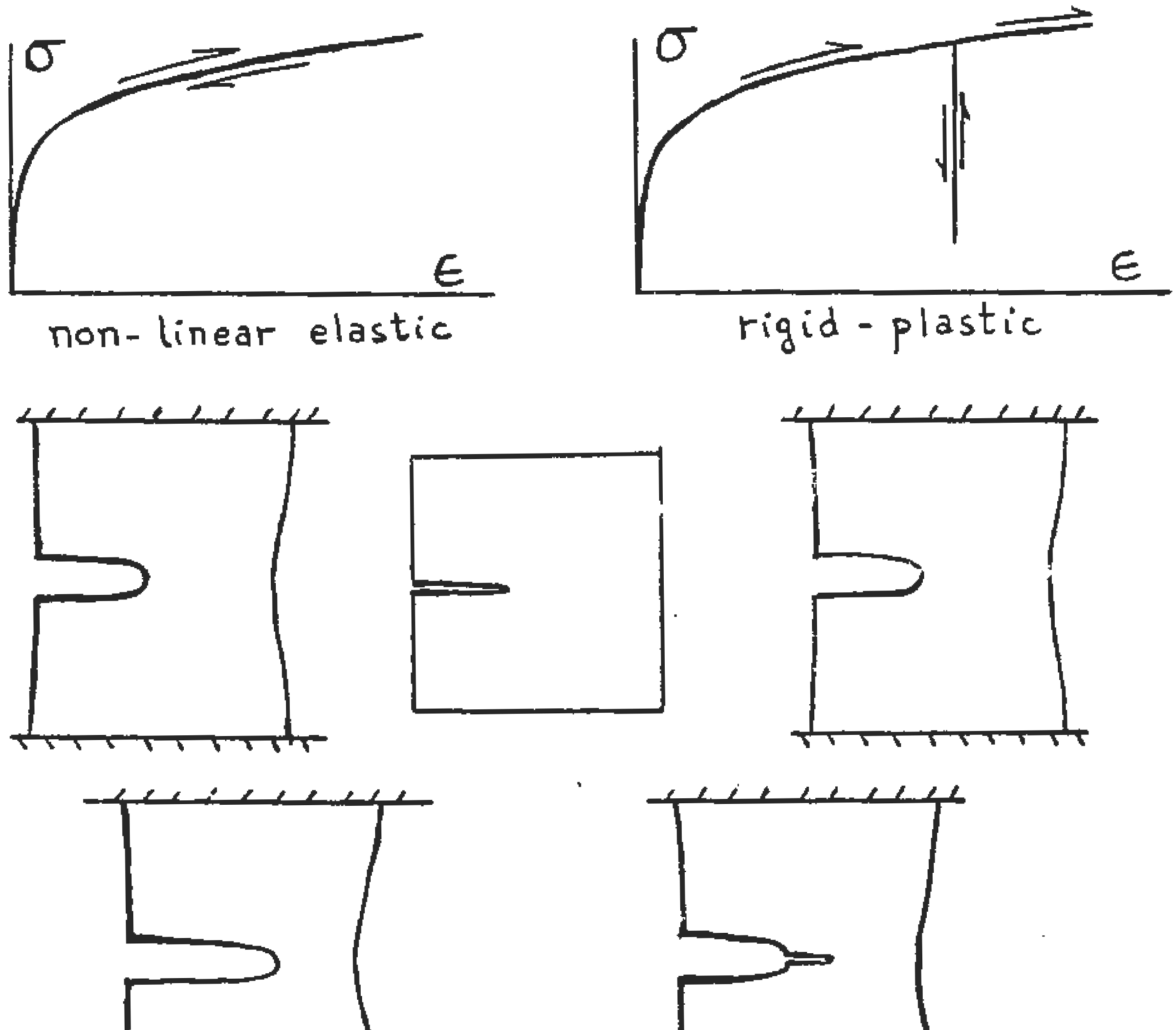


fig. 1 Crack advance in idealized materials.

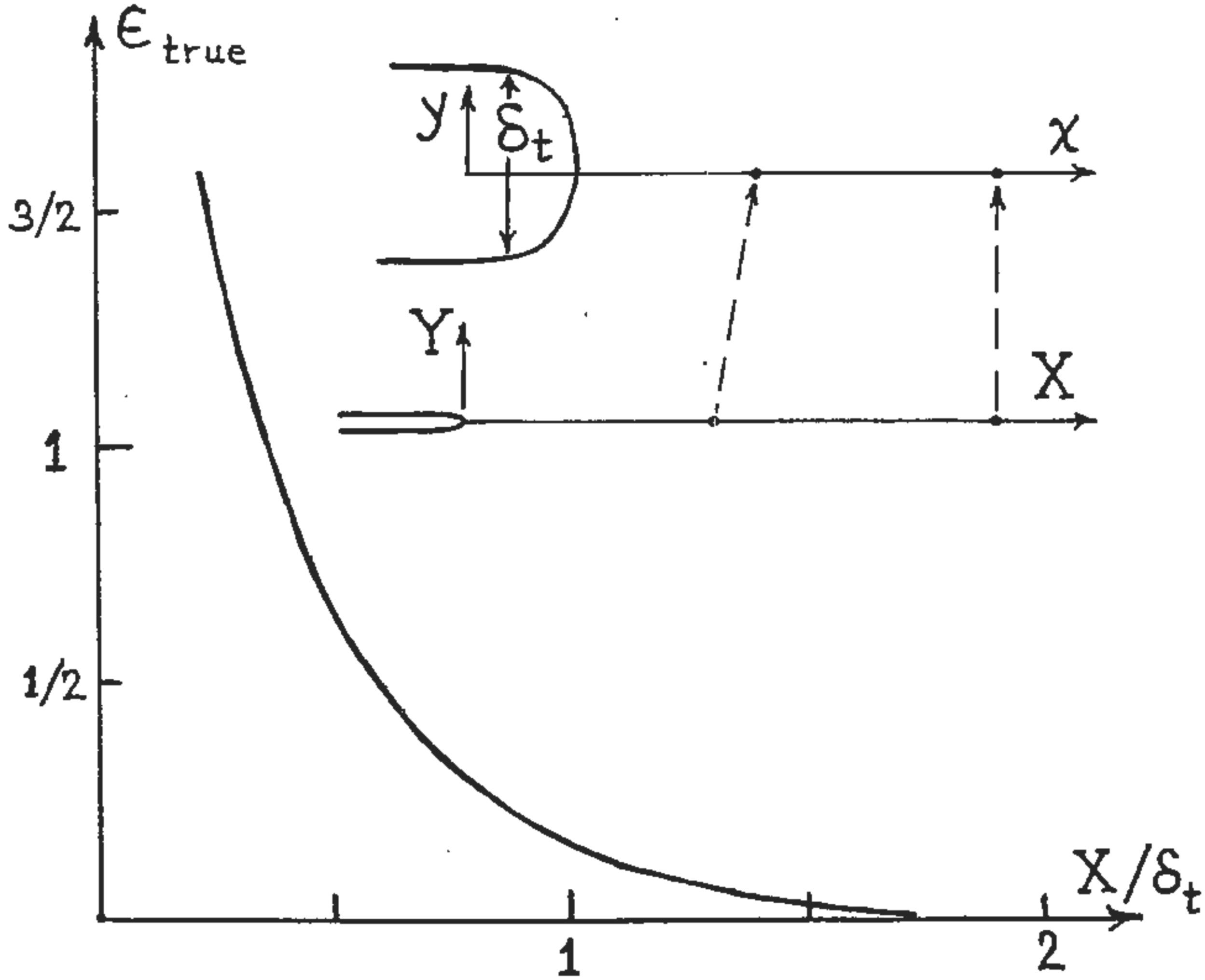


fig. 2 Blunted shape of crack and distribution of strain $\epsilon_{true} = \log(\partial y/\partial Y)$ directly ahead of tip, from 'small scale yielding' results of Rice and Johnson (1970), for stationary crack.

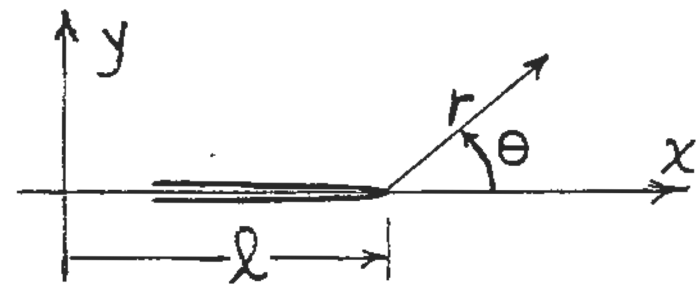


fig. 3 Coordinates for growing crack

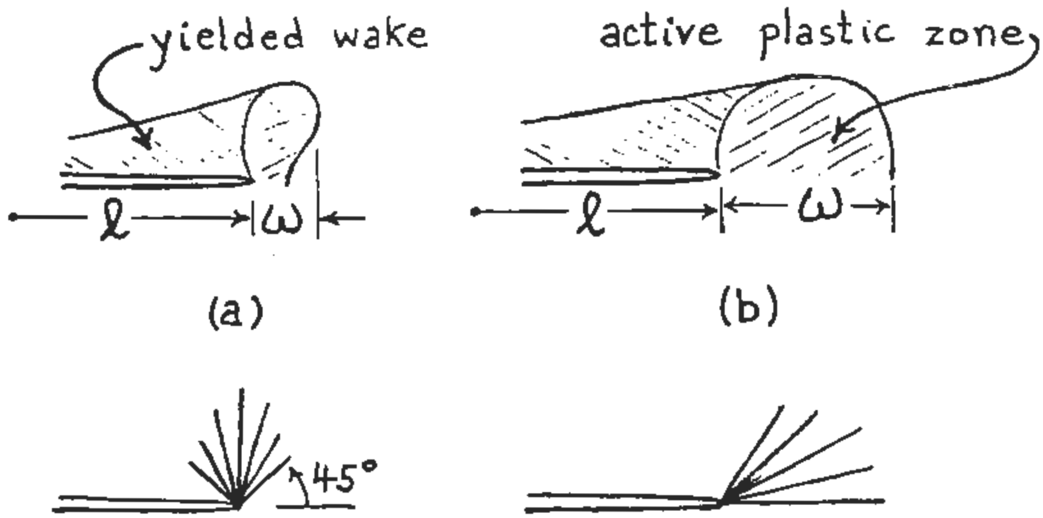
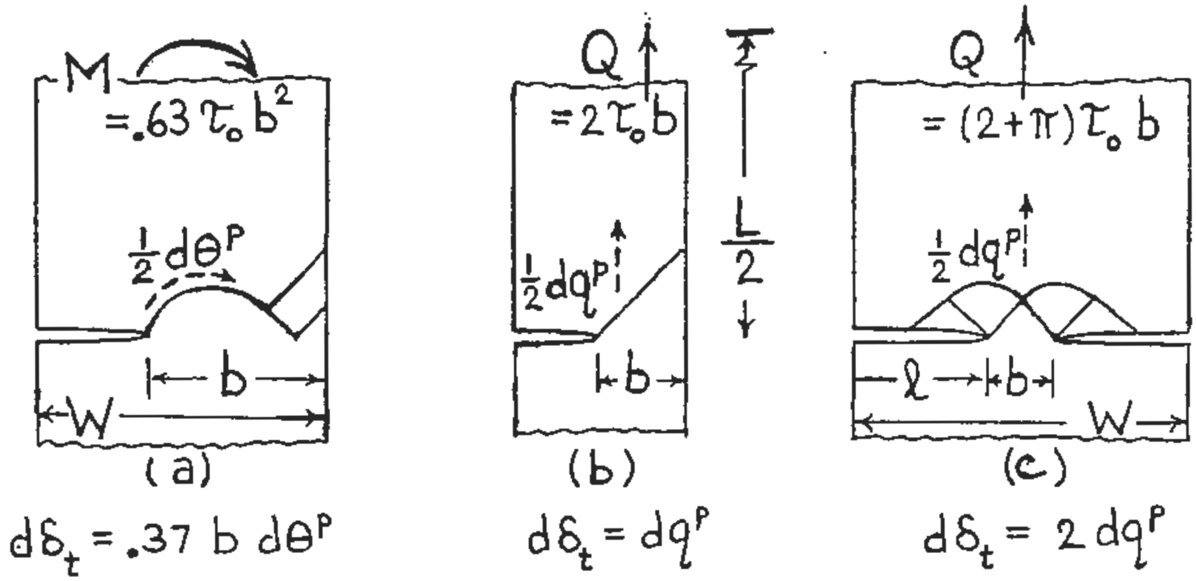
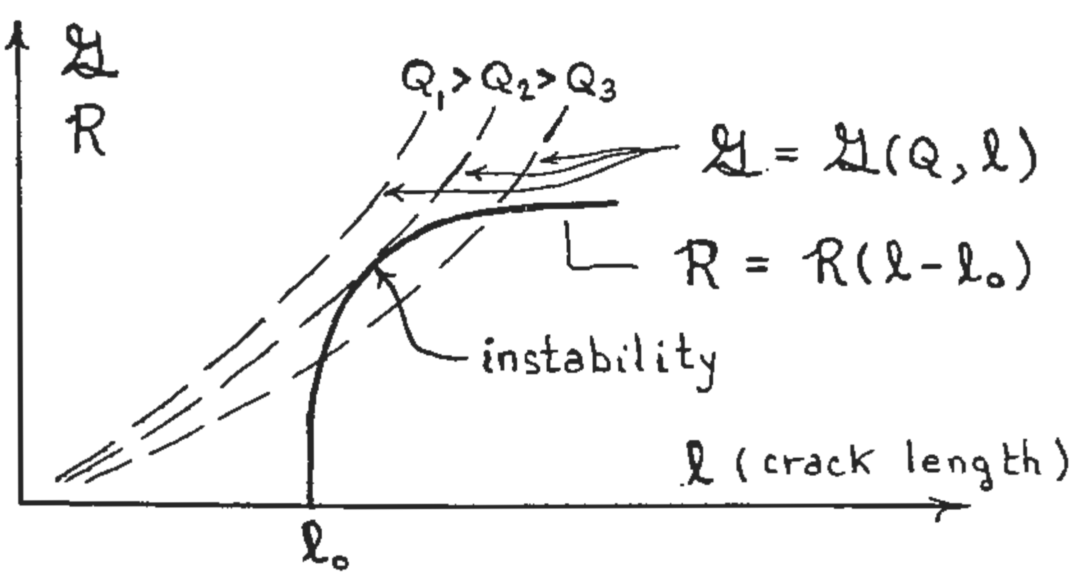


fig. 4 Contained yielding at crack tips. (a) Plane strain type. (b) Anti-plane strain or 2-D plane stress.



$d\delta_t = .37\, b\, d\theta^p$ $\quad$ $d\delta_t = dq^p$ $\quad$ $d\delta_t = 2\, dq^p$

fig. 5 Fully yielded plane strain specimens.



→ 0 at instability

$$\mathcal{G}(Q,\ell) = R(\ell-\ell_0) \quad ; \quad \frac{\partial \mathcal{G}}{\partial Q}\frac{dQ}{d\ell} + \frac{\partial \mathcal{G}}{\partial \ell} = \frac{dR}{d\ell}$$

fig. 6 'Resistance Curve' analysis of instability.