J. R. Rice,[1] *P. C. Paris,*[2] *and J. G. Merkle*[3]

Some Further Results of J-Integral Analysis and Estimates

Published in "Progress in Flaw Growth and Fracture Toughness Testing", Special Tech. Publication 536, ASTM, Philadelphia, 1973, pp. 231-245

REFERENCE: Rice, J. R., Paris, P. C., and Merkle, J. G., **"Some Further Results of J-Integral Analysis and Estimates,"** *Progress in Flaw Growth and Fracture Toughness Testing, ASTM STP 536,* American Society for Testing and Materials, 1973, pp. 231–245.

ABSTRACT: It is shown that the J-integral can be directly evaluated from single load-displacement records for a series of crack toughness specimens having the common feature that their only significant length dimension is that of the uncracked ligament. For the special case of bending loads on the ligament of a deeply cracked bar, J is shown to be twice the work of deformation divided by the ligament area. This and like results are employed to discuss Charpy and "equivalent energy" toughness measures and also to evolve yet simpler estimating procedures for the J-integral.

KEY WORDS: fracture properties, mechanical properties, cracks, crack propagation, plastic deformation, plasticity theory, toughness

In recent work the J-integral [*1,2*][4] has been adopted as a failure criterion [*3-5*] and analytical estimation procedures [*5,6*] have been developed for its use. This discussion will show that for certain geometric configurations of interest in crack toughness testing and structural applications, the J-integral can be evaluated from a single load-displacement record. These configurations have the feature that there is only one geometric dimension of interest, namely the uncracked ligament dimension. Some additional, related simplifications in analysis procedures and estimates for J are discussed as well.

In review, within the framework of a total strain formulation of elastic-plastic

[1] Professor of Engineering, Brown University, Providence, R. I. 02912.
[2] Chairman of the Board, Del Research Corp., Hellertown, Pa. 18055.
[3] H.S.S.T. Program Staff, Oak Ridge National Laboratory, Oak Ridge, Tenn. 37830.
[4] The italic numbers in brackets refer to the list of references appended to this paper.

deformation, a path independent integral may be defined for two-dimensional problems:

$$J = \int_{\Gamma} \left(W\,dy - T \cdot \frac{\partial u}{\partial x} ds\right) \tag{1}$$

Here, x and y are rectangular coordinates normal to the crack front, y being perpendicular to the crack surface; ds is an increment of arc length along any contour, Γ, beginning along the bottom surface of the crack and ending along the top surface; $\vec{T}$ is the stress vector exerted on the material within the contour; $\vec{u}$ is the displacement; and W is the integral of stress working density (or strain energy density). An alternate and equivalent definition of J is given by

$$J = \int_0^{\delta} \left(-\frac{\partial P}{\partial a}\right)_{\delta} d\delta, \text{ or } J = \int_0^{P} \left(\frac{\partial \delta}{\partial a}\right)_{P} dP \tag{2}$$

These relate J to the rate of change with respect to crack size, a, of the area under the load versus load-point-displacement, P versus δ, curves. Here P is the force per unit length of crack front, for example, the force per unit thickness in the two-dimensional case, and the curves are considered to be generated for different crack sizes, a, in virgin specimens subjected to monotonic loading.

The latter definition was used by Begley and Landes [3,4] in their experimental evaluation of the J versus δ relation. Indeed, this same definition is readily extended to three dimensions to define J at each point along the crack front, even though the line integral definition could then be applied only in the limit of a vanishingly small circuit Γ surrounding the crack tip at that point. For example, in the case of a symmetrically loaded, cracked axi-symmetric bar, Eq 2 would apply if P was interpreted as the force per unit of circumference around the crack front. More generally, if a planar crack has a front denoted by the contour, L, and if we consider an infinitessimal advance, Δa, where Δa is an arbitrary function of position along L, then local values of J along L are defined by writing

$$\int_L (J\,\Delta a)\,dL = \int_0^{\delta} (-\Delta_{\text{force}})_{\delta}\, d\delta \tag{3}$$

to first order in Δa. Here $(\Delta_{\text{force}})_{\delta}$ is the change in load point force, corresponding to a given δ, between the crack configuration advanced by Δa and the initial configuration.

It is of interest also simply to note that for bodies loaded in the linear-elastic range,

$$J_{\text{elastic}} = \mathcal{G} = \frac{K^2}{E} \tag{4}$$

or J_{elastic} is the well known Griffith energy release rate, but only for this special case.

The Double Edge Notched Plate in Tension

Consider a wide plate of width, *W*, subjected to a tension force, *P*, per unit thickness which is transmitted through a narrow neck of width, *b*, between coplanar cracks from each edge. The displacement between the load points, δ_{total}, may be regarded as the sum of the displacement without cracks, $\delta_{\text{no crack}}$, plus the displacement due to introducing the cracks, δ_{crack}, or,

$$\delta_{\text{total}} = \delta_{\text{no crack}} + \delta_{\text{crack}} \tag{5}$$

or alternately as the elastic displacement, δ_{elastic}, plus the displacement due to plasticity, δ_{plastic}, or,

$$\delta_{\text{total}} = \delta_{\text{elastic}} + \delta_{\text{plastic}} \tag{6}$$

These forms for displacement will also be useful in conjunction with other crack configurations.

For the current configuration of double edge notched tension it is useful to consider Eq 6. The plastic part of the displacement must have the form,[5]

$$\delta_{\text{plastic}} \doteq bh\left(\frac{P}{b}\right) \tag{7}$$

where the cracks are deep enough that plasticity is confined to the neck between the cracks.

Making use of Eq 2 as the definition of J along with Eq 6, it becomes

$$J = \int_0^P \left(-\frac{\partial \delta_{\text{total}}}{\partial b}\, dP\right)$$

or

$$J = J_{\text{elastic}} + \int_0^P \left(-\frac{\partial \delta_{\text{plastic}}}{\partial b}\right) dP \tag{8}$$

[5] This form may be developed from a variety of special plastic or elastic-plastic analyses, but it is sufficient to argue that it results from dimensional analysis when it is noted that the material parameters E, σ_{yp}, and n or equivalents are the only other parameters involved in the function $h\left(\frac{P}{b}\right)$.

but from the form of Eq 7, it is noted that

$$-\frac{\partial \delta_{\text{plastic}}}{\partial b} = \frac{P}{b} h'\left(\frac{P}{b}\right) - h\left(\frac{P}{b}\right) = \frac{1}{b}\left[P\left(\frac{\partial \delta_{\text{plastic}}}{\partial P}\right)_b - \delta_{\text{plastic}}\right] \qquad (9)$$

Substituting Eqs 4 and 9 into Eq 8 and integrating by parts leads to:

$$J = \mathcal{G} + \frac{1}{b}\left[2\int_0^{\delta_{\text{plastic}}} P d\delta_{\text{plastic}} - P\delta_{\text{plastic}}\right] \qquad (10)$$

Note that the bracket in Eq 10 is directly interpretable as a particular part of the area under load versus displacement curve. This is of importance since J can be evaluated at any point by having a single load displacement record up to that point (recall that the Begley-Landes procedures required at least two load displacement records for differing crack sizes [*3,4,6*]).

Note also that for no plastic deformation, $\delta_{\text{plastic}} = 0$, Eq 10 reduces to:

$$J = \mathcal{G} \qquad (11)$$

or on the other hand, if the material is rigid-perfectly-plastic (or typically for very large plastic versus elastic deformations) Eq 10 reduces to:

$$J = \frac{P_{\text{limit}}\,\delta}{b} = \sigma_{\text{limit}}\,\delta \qquad (12)$$

All of these forms, Eq 10, 11, and 12 are useful in evaluating J for various load displacement records of double edge notched plates.

The Internally Notched Plate in Tension

Consider a rectangular plate with an internal crack of length $2a$, which is centrally located so that narrow uncracked ligaments of width b exist between the crack ends and the specimen edges. If we write δ for the load point displacement and $2P$ for the applied force per unit thickness, supposing this to be centrally applied so that P acts through each ligament, then Eq 2 again gives J. The analysis is identical in form to that of the last section: An equation of the type 7 applies, and the final result for J is just as in Eq 10 except that now, of course, the internally notched load-deflection curve is intended.

Likewise, the special versions Eqs 11 and 12 apply. The plane strain limit stress σ_{limit} is known to be higher by a factor $1 + \pi/2 \approx 2.6$ for the double edge cracked specimen as compared to the internally cracked specimen. Hence, within the rigid-plastic approximation, identical δ values correspond to J values differing by 2.6 for specimens with the same ligament size b.

The Notched Round Bar in Tension

For a deeply externally notched round bar, leaving a circular neck of radius r, subject to a centered tension force P, the displacement between the load points due to introducing the crack (or notch) is of the form (see footnote 5):

$$\delta_{\text{crack}} = rg\left(\frac{P}{r^2}\right) \tag{13}$$

Introducing Eq 5 into Eq 2, it is noted that J is always "due to the crack," or

$$J = \frac{1}{2\pi r}\int_0^P \left(-\frac{\partial \delta_{\text{crack}}}{\partial r}\right)_P dP \tag{14}$$

but from the form of Eq 13, it is noted that

$$\left(-\frac{\partial \delta_{\text{crack}}}{\partial r}\right)_P = \frac{2P}{r^2}\,g'\left(\frac{P}{r^2}\right) - g\left(\frac{P}{r^2}\right) = \frac{1}{r}\left[2P\left(\frac{\partial \delta_{\text{crack}}}{\partial P}\right)_r - \delta_{\text{crack}}\right] \tag{15}$$

Substituting Eq 15 into Eq 14 and integrating by parts leads to:

$$J = \frac{1}{2\pi r^2}\left[3\int_0^{\delta_{\text{crack}}} P d\delta_{\text{crack}} - P\delta_{\text{crack}}\right] \tag{16}$$

Again J can be evaluated on a single record of load versus load-point-displacement where provision is made to eliminate the displacement component present with the crack absent. Also for the rigid-plastic case, it is again noted that

$$J = \frac{P_{\text{limit}}}{\pi r^2}\,\delta = \sigma_{\text{limit}}\,\delta \tag{17}$$

The Remaining Uncracked Ligament Subject to Bending[6]

Consider a plate which is deeply cracked from one edge, where the crack is approaching the other edge, leaving a narrow neck of width b subject to remotely applied in-plane bending of moment M per unit thickness.

For such a case, Eq 2 can simply be modified by substituting moment M for force P, and the angle change between point of moment application θ, for displacement δ, or

$$J = \int_0^M \left(-\frac{\partial \theta_{\text{total}}}{\partial b}\right)_M dM \tag{18}$$

[6] Dr. J. Srawley of NASA-Lewis Research Center has independently developed an analysis leading to a result similar to Eq 22 of this section.

where, as with Eq 5,

$$\theta_{\text{total}} = \theta_{\text{no crack}} + \theta_{\text{crack}} \tag{19}$$

The form (see footnote 5) of the displacement due to introduction of the crack in this case is:

$$\theta_{\text{crack}} = f\left(\frac{M}{b^2}\right) \tag{20}$$

From Eq 20, it is found that

$$\left(-\frac{\partial\theta_{\text{total}}}{\partial b}\right)_M = \left(-\frac{\partial\theta_{\text{crack}}}{\partial b}\right)_M = \frac{2M}{b^3} f'\left(\frac{M}{b^2}\right) = \frac{2M}{b}\left(\frac{\partial\theta_{\text{crack}}}{\partial M}\right)_b \tag{21}$$

Substituting Eq 21 into Eq 18 and integrating:

$$J = \frac{2}{b}\int_0^{\theta_{\text{crack}}} M d\theta_{\text{crack}} \tag{22}$$

This result, Eq 22, leads to an even simpler interpretation than for previous results, Eqs 10 and 16, since the integral is simply the area under the M versus θ_{crack} curve. This is simply the work done in loading, with the deformations with no crack present eliminated from the calculations. Again, it is implied that a single test is sufficient to evaluate J.

Also, if a remaining ligament is subject principally to bending but the load is applied by a force(s) P, then Eq 22 reverts to:

$$J = \frac{2}{b}\int_0^{\delta_{\text{crack}}} P\, d\delta_{\text{crack}} \tag{23}$$

This result, Eq 23, is, for example, applicable to the deeply notched 3-point bend specimen or compact tension specimen. Notch depths must at least be sufficient so that plasticity encountered is confined to the uncracked ligament region ahead of the crack.

In the case of the 3-point bend specimen, elastic displacements with no crack, $\delta_{\text{no crack}}$, may be appreciable (compared to displacements, elastic and plastic, due to the crack) and would have to be eliminated in evaluating J from Eq 23. However, the displacements of the load points in a deeply notched compact tension specimen with no crack present would be negligible compared to displacements due to the crack. Therefore, the raw load versus load-point-displacement record could be analysed using Eq 23 with good results.

Moreover for cases where the plastic displacement becomes very large compared to elastic contributions, Eq 23 can be used on the raw load-displacement record with little error.

Charpy and "Equivalent Energy" Toughness Measures

In the case of Charpy tests on the upper shelf (above the transition temperature) for constructional steels, large plastic displacements normally precede crack extension. In this case J should be approximately proportional to the work $\int P\,d\delta$ done up to any point. Hence, in the spirit of the Begley-Landes [3,4] use of J as a failure criterion,

$$J_{Ic} = \frac{{K_{Ic}}^2}{E} = \frac{2}{b} \int_0^{\delta_{cr}} P\,d\delta_{crack} \tag{24}$$

The Charpy energy (CVN) includes mainly the energy dissipated up to extension of the crack

$$\int_0^{\delta_{cr}} P\,d\delta_{crack}$$

including the elastic energy stored, which for constant modulus and proportions would be proportional to σ_{yp}^2. From Eq 24 and these considerations, a suggested form would be:

$$K_{Ic}^2 = A(\text{CVN}) - B\sigma_{yp}^2 \tag{25}$$

where A and B are constants with appropriate dimensions. The similiarity between Eq 25 and the Rolfe-Novak-Barsom [7] correlation, that is, for their choice of units,

$$K_{Ic}^2 = 5\sigma_{yp}\,(\text{CVN}) - \frac{\sigma_{yp}^2}{4} \tag{26}$$

is thus indeed of interest here. Or consider the Sailors-Corten [7] suggestion,

$$K_{Ic} = 15.5\,(\text{CVN})^{1/2} \tag{27}$$

The results herein may at least be a partial explanation of why large brittle-behaving tests for K_{Ic} (such as by ASTM Test for Plane-Strain Fracture Toughness of Metallic Materials (E 399-72)) can be correlated with small ductile-behaving upper shelf Charpy energies (CVN). However, it is not the purpose here to go into such matters in any detail and many discrepancies, such as Charpy energy dissipated after the beginning of crack extension and also the initial crack tip sharpness, would have to be considered.

Witt [8] has proposed an "equivalent energy" fracture criterion, whereby, for plane strain response as a special case, the energy per unit volume absorbed up to the fracture for a family of geometrically similar specimens would be considered to be inversely proportional to a linear dimension of the specimen. We see from Eq 22 that this result is to be expected on the basis of a critical J-criterion for compact tension or deeply cracked bend specimens. On the other hand, we see from Eqs 10 and 16 that the same result is not expected for the other specimens under tensile rather than bending loadings. However, within the rigid-plastic approximation, all these formulae reduce to a pure number times the absorbed energy divided by the uncracked ligament area. Hence, the J-criterion (and also one based on crack tip opening displacement, for that matter) is approximately consistent with the equivalent energy concept in the limited context of correlating a series of geometrically similar specimens which fail well into the plastic range. The pure number is different for bending and for tensile loads (2 and 1, respectively) so that a critical J-criterion applying among specimens of such different type would require that the absorbed energy for a given ligament area differ by about a factor of 2 between different fully plastic specimens.

Estimates of *J* From Single Points on Load Displacement Records

In the preceding discussion, it was shown that in some crack configurations analysis of a single load displacement record is sufficient to evaluate J at points on that record. This is considerably simpler than the Begley-Landes procedure [3,4], but, of course, their method is not restricted to a few specific configurations. Nevertheless, the restrictions do allow using compact tension and bend test configurations which are convenient and normally used for fracture toughness testing (for example, ASTM E 399-72). However, the single load-displacement record analyses suggested herein require measurements of areas on those records. It is therefore appropriate to inquire whether an estimate of J can be made from a single point on a load displacement record, such as the critical point of crack extension, as is possible for linear elastic fracture toughness tests (ASTM E 399-72). This possibility would make J-type testing and analysis even more practical for toughness evaluation.

In a previous paper [6], estimating procedures using plastic-zone-corrected linear-elastic analysis in combination with limit analysis provided some simplifications. The same approach will be used here. Basically, the procedure involves predicting the shape of the load-displacement record by deriving expressions for displacement based on linear-elastic "compliance" analysis and inserting an effective crack size, a_{eff}, equal to the actual crack size plus the usual plastic zone, r_y, correction, or,

$$a_{eff} = a + r_y \qquad (28)$$

wherein

$$r_y = \beta \frac{K^2}{\sigma_{yp}^2} \tag{29}$$

and where β varies between $1/2\pi$ and $1/6\pi$ depending on whether plane stress or plane strain conditions, respectively, tend to dominate.

In the preceding paper [6], numerical calculations were made but herein analytical expressions are desired for displacements. In order to derive elastic displacements, a method can be developed [9] relating energy principles to usual methods of linear-elastic fracture mechanics analysis. Start by considering the strain-energy, U, of a cracked configuration as that with no crack present, $U_{\text{no crack}}$, plus that due to introducing the crack, U_{crack}. Then:

$$U_{\text{total}} = U_{\text{no crack}} + U_{\text{crack}} \tag{30}$$

By Castigliano's theorem the displacement of the load point is

$$\delta_{\text{total}} = \frac{\partial U_{\text{total}}}{\partial P} = \frac{\partial U_{\text{no crack}}}{\partial P} + \frac{\partial U_{\text{crack}}}{\partial P} \tag{31}$$

or

$$= \delta_{\text{no crack}} + \delta_{\text{crack}}$$

Therefore, it is seen that

$$\delta_{\text{crack}} = \frac{\partial U_{\text{crack}}}{\partial P} \tag{32}$$

But, by definition

$$\mathcal{G} = \frac{\partial U_{\text{crack}}}{\partial a} = \frac{K^2}{E} \tag{33}$$

Combining Eqs 32 and 33 gives

$$\delta_{\text{crack}} = \frac{\partial}{\partial P} \int_0^a \left(\frac{\partial U_{\text{crack}}}{\partial a}\right) da$$

or

$$= \frac{1}{E} \int_0^a \frac{\partial (K^2)}{\partial P}\, da \tag{34}$$

Therefore, if an analytical expression for the crack tip stress intensity parameter, K, is known, via Eq 34 an expression for elastic displacement of the load point may be derived.

For example, for the deeply notched plate subject to bending, as discussed earlier, Wilson [10] gives

$$K = \frac{4M}{b^{3/2}} \tag{35}$$

(actually, the coefficient is about 3.98). Substituting Eq 35 into 34 with appropriate modification leads to

$$\theta_{\text{crack}} = \frac{16M}{Eb^2} \tag{36}$$

Also, for the deeply notched round bar subject to tension

$$K = \frac{P}{2\sqrt{\pi}r^{3/2}} \tag{37}$$

which with Eq 34 gives

$$\delta_{\text{crack}} = \frac{P}{Er} \tag{38}$$

For the double edge cracked strip subject to tension with a strip width, W, a crack depth on each side, c, and remaining neck of width, b, (namely, $W - 2c = b$), an explicit form for K is not known. However, the following inequalities (from other solutions of similar problems) apply.

$$\frac{P}{\sqrt{W}}\sqrt{\tan\frac{\pi c}{W}} \leqslant K \leqslant \frac{\sqrt{2}P}{\sqrt{\pi b}} \tag{39}$$

Again utilizing Eq 34 gives

$$\frac{4P}{\pi E}\ln\left(\frac{2W}{\pi b}\right) \leqslant \delta_{\text{crack}} \leqslant \frac{4\sqrt{2}P}{\pi E}\ln\left(\frac{W}{b}\right) \tag{40}$$

Note that the result is not bounded as $W \rightarrow \infty$; therefore, the "deeply notched" strip cannot be treated herein in a direct fashion. Nevertheless, the inequality of Eq 40 itself is interesting with extremes differing by the $\sqrt{2}$ and $\ln \dfrac{2}{\pi}$. However, it will not be pursued further here.

For the notch round bar in tension Eqs 28, 29, and 38 may be combined to give δ_{crack} which is a plastic-zone corrected nonlinear approximation for the displacement for use up to limit load [6]. The result is

$$\delta_{crack} = \frac{r}{E}\left(\frac{P}{r^2}\right) \frac{1}{\left(1 - \frac{\beta}{4\pi\sigma_{yp}^2}\left[\frac{P}{r^2}\right]^2\right)} \tag{41}$$

It is noticed that the form conforms with Eq 13 as might be expected.

Modifying Eq 16 by using integration over the complementary energy area of the P versus δ_{crack} diagram (or integrating by parts), it can be written

$$J = \frac{1}{2\pi r^2}\left(2P\,\delta_{crack} - 3\int_0^P \delta_{crack}\,dP\right) \tag{42}$$

and substituting Eq 41 into 42 and integrating gives

$$J = \frac{P\delta_{crack}}{\pi r^2}\left[1 + \frac{3\pi\sigma_{yp}^2 r^3}{E\beta P\delta_{crack}} \ln\left(1 - \frac{\beta}{4\pi\sigma_{yp}^2}\left[\frac{P}{r^2}\right]^2\right)\right] \tag{43}$$

Noting that the limit load must have the form

$$P_{limit} = D_1 \pi r^2 \sigma_{yp} \tag{44}$$

and noting that the integral term in the bracket of Eq 42 applies only up to limit load and is to be dropped beyond, then Eq 43 may be rewritten:

up to limit load

$$J = \frac{P\delta_{crack}}{\pi r^2}\left(\frac{1}{4} + \frac{3\pi\beta D_1^2}{16}\left[\frac{P}{P_{limit}}\right]^2\right) \tag{45}$$

beyond limit load

$$J = \frac{P\delta_{crack}}{\pi r^2}\left(1 - \frac{3}{4}\frac{(\delta_{crack})_{limit}}{\delta_{crack}}\left[1 - \frac{\pi\beta D_1^2}{4}\right]\right) \tag{46}$$

Equations 45 and 46 are estimating formulas for J for the notch round tension bar which require only the point on the P versus δ_{crack} curve at which the estimate is to be made. Of course, P_{limit} and δ_{limit} appear in the equations, but they can be approximated from Eqs 44 and 41, and it is noted that small changes in their values have little effect on J as estimated by Eqs 45 and 46. This is true also of the values chosen for β and D_1, so these equations do seem to be a practical method of estimating J. The only impracticality is that δ_{crack} is to be evaluated, eliminating the stretch of the bar without a crack present. However, since plasticity is confined to the neck, in other words to δ_{crack}, the $\delta_{no\ crack}$ to be subtracted from the total displacement (measured) can be computed from the simple formula for elastic elongation of a prismatic tensile bar.

For the deeply notched plate subject to bending (such as the 3-point bend or compact tension test specimens), Eqs 28 and 29 may be put into Eq 36 to obtain a plastic zone corrected nonlinear load-displacement relationship, again applicable up to limit load [6]

$$\theta_{crack} = \frac{16}{E}\frac{M}{b^2}\frac{1}{\left(1 - \frac{16\beta}{\sigma_{yp}^2}\left[\frac{M}{b^2}\right]^2\right)^2} \tag{47}$$

The agreement with the form of Eq 20 is noted. Equation 22 may be modified by integrating over the complementary energy of the M versus θ_{crack} diagram (or integrating by parts) and becomes

$$J = \frac{2}{b}\left(M\theta_{crack} - \int_0^M \theta_{crack}\, dM\right) \tag{48}$$

Substituting Eq 47 into 48, the integration is performed and leads to

$$J = \frac{2M\theta_{crack}}{b}\left(1 - \frac{1}{2}\left[1 - \frac{16\beta}{\sigma_{yp}^2}\left(\frac{M}{b^2}\right)^2\right]\right) \tag{49}$$

The limit moment has the form

$$M_{limit} = D_2 b^2 \sigma_{yp} \tag{50}$$

Moreover, the induced moment, M, is often applied by a remote loading force, P, in which case conversions may be made from

$$M\theta_{crack} = P\delta_{crack}$$

and (51)

$$\frac{M}{M_{limit}} = \frac{P}{P_{limit}}$$

Making use of Eqs 50 and 51 in Eq 49 and noting that the integral term in the bracket in Eq 48 applies only up to the limit load,

up to limit load

$$J = \frac{P\delta_{\text{crack}}}{b}\left(1 + 16\beta D_2^2\left[\frac{P}{P_{\text{limit}}}\right]^2\right) \tag{52}$$

beyond limit load

$$J = \frac{P\delta_{\text{crack}}}{b}\left(2 - \frac{(\delta_{\text{crack}})_{\text{limit}}}{\delta_{\text{crack}}}\left[1 - 16\beta D_2^2\right]\right) \tag{53}$$

Equations 52 and 53 are the single point J estimating formulas for deep notched bending or compact tension tests or similar configurations imposing bending on a narrow remaining rectangular ligament. For practical purposes the notches need only be deep enough to confine plasticity to the uncracked ligament region (and away from top or bottom of the compact tension).

From Green and Hundy's analysis and other experiments [2,3,4], it is known that D_2 is about 0.36. Therefore, in the case of plane stress where β is about $1/2\pi$, the coefficients in Eqs 52 and 53 are

$$16\beta D_2^2 \approx 0.35$$

and, for a tendency toward plane strain, it might become up to 3 times smaller. Noting this value it can be seen that Eqs 52 and 53 are insensitive to this change in stress state and also insensitive to any inaccuracy in approximating P_{limit} or δ_{limit}. Thus, they are indeed estimating formulas for J which are practical in form.

Again, for application of Eqs 52 and 53 to bend tests, the δ_{crack} values to be used should eliminate the component of load point displacement which would be present without a crack, $\delta_{\text{no crack}}$. However, in applying these equations to compact tension tests, the (unnotched) $\delta_{\text{no crack}}$ would be negligible compared to δ_{crack} so that total load point displacement, as measured, may be used without large errors. Moreover, the load point displacement could be sufficiently measured by the usual clip-in gage measuring notch opening at the load-line of the compact tension specimen (which is easy to arrange with a machined-in clip gage seat at the load line).

Summary

This discussion has attempted to point out analysis advantages in certain configurations with a single dominating characteristic (crack plane) dimension. These analyses result in methods of evaluating J from single load versus load point displacement relationships or tests results, namely, Eqs 10, 16, and 22. This makes single test evaluation of fracture toughness, as evaluated by J_{Ic}, feasible for several commonly used test configurations.

Moreover, these special analytical results, for example, Eq 22, lead to better possibilities for understanding upper shelf Charpy versus fracture toughness, K_{Ic}, correlations and the like. Equation 22 also implies that the "equivalent-energy" approach [8] is definitely applicable to bend and compact tension type tests.

Finally, some estimating formulas were developed for approximate calculations of J from single load-displacement points for notched round, bending, and compact tension configurations. It was suggested that these estimating formulas for J, which should be noted to be just as simple to use as procedures or formulas for K (for example, ASTM E 399-72), are put forth as practical quick-estimate possibilities.

And though J-integral methods may be as yet relatively unexplored compared to linear-elastic fracture mechanics, it is hoped that the simplicities and usefulness of these methods are illustrated herein.

Acknowledgments

The sponsorship of work leading to and encouraging many of the results of this paper by the Fracture Mechanics Section of Westinghouse Research and Development is gratefully acknowledged. One of the authors also wishes to acknowledge support by NASA under Grant NGL-40-002-080 at Brown University. Another author wishes to acknowledge support by the Heavy Section Steel Technology Program, sponsored by A.E.C. at Oak Ridge National Laboratory. The assistance of Dr. R. J. Bucci in checking many of the derivations in this paper is also gratefully acknowledged.

References

[1] Rice, J. R., *Journal of Applied Mechanics, Transactions,* American Society of Mechanical Engineers, Vol. 35, 1968, pp. 379-386.
[2] Rice, J. R. in *Fracture,* Vol. 2, H. Liebowitz, Ed., Academic Press, 1968, pp. 191-311.
[3] Begley, J. A. and Landes, J. D. in *Fracture Toughness, ASTM STP 514,* American Society for Testing and Materials, 1972, pp. 1-23.
[4] Landes, J. D. and Begley, J. A. in *Fracture Toughness, ASTM STP 514,* American Society for Testing and Materials, 1972, pp. 24-39.
[5] Broberg, K. B., *Journal of the Mechanics and Physics of Solids,* Vol. 19, No. 6, 1971, pp. 407-418.
[6] Bucci, R. J., Paris, P. C., Landes, J. D., and Rice, J. R. in *Fracture Toughness, ASTM STP 514,* American Society for Testing and Materials, 1972, pp. 40-69.
[7] Sailors, R. H. and Corten, H. T. in *Fracture Toughness, ASTM STP 514,* American Society for Testing and Materials, 1972, pp. 164-191.

[*8*] Witt, F. J. "Fracture Behavior of Reactor Pressure Vessel Steel in the Frangible, Transitional, and Tough Regimes," *Nuclear Engineering and Design,* Vol. 20, 1972.

[*9*] Paris, P. C., "The Mechanics of Fracture Propagation and Solutions to Fracture Arrestor Problems," Document D2-2195, The Boeing Company, 1957.

[*10*] Wilson, W. K., "Stress Intensity Factors for Deep Cracks in Bending and Compact Tension Specimens," *Journal of Engineering Fracture Mechanics,* Vol. 2, No. 2, Nov. 1970.