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PLANE STRAIN SLIP LINE THEORY FOR ANISOTROPIC RIGID/PLASTIC MATERIALS

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Summary

A slip line theory governing states of incipient plane flow at the yield point load is developed for *anisotropic* rigid/plastic materials which exhibit a reduced yield criterion, governing states of plane flow, that depends only on the deviatoric parts of the in-plane stress tensor. It is shown that every homogeneous, incompressible material which complies with the principle of maximum plastic work, but which is of otherwise arbitrary anisotropy, is of this class. The stress equilibrium requirements are seen to take a remarkably simple form expressing the constancy of the quantities ***mean in-plane normal stress plus or minus arc length around the governing yield contour in a Mohr stress plane*** along members of the two slip line families. Further, this generalization of the Hencky equations is valid for *every* material of the considered class. Some special features of yield contours containing corners and flat segments are discussed, and velocity equations are given for materials complying with the maximum work inequality. The theory is applied to obtain the solution for indentation of an arbitrarily anisotropic half-space with a flat-ended, rigid, frictionless punch. A simple, universal formula, involving only geometrical dimensions of the governing yield contour, is derived for the yield point indentation pressure.

1. Introduction

The plane strain slip line theory for isotropic rigid/plastic materials is well known, and Hill (1950) has extended it to include anisotropy in the form of an ellipsoidal yield condition. Here, the theory is further generalized to treat incipient plane flow at the yield point load† for incompressible, homogeneous rigid/plastic materials, which comply with the principle of maximum plastic work, but which are otherwise of *arbitrary* anisotropy. The results take a remarkably simple form. They allow, for example, a completely general solution to the Prandtl punch problem, valid for any such material.

2. Theoretical Development

2.1 *Yield condition and flow rule*

The plane of deformation is x, y and σ_x, σ_y, τ_{xy} are the associated in-plane normal stresses and shear stress. Our formulation, to follow, of stress equilibrium equations in regions of incipient flow will be valid for the class of all homogeneous rigid/plastic

† It is well known that the 'yield point' load of a rigid/plastic solid coincides with the 'limit load' of the corresponding elastic/perfectly plastic solid when the latter is analyzed according to the conventional assumptions of the small displacement-gradient formulation.

materials having a *reduced* yield criterion (i.e. a criterion appropriate to plane flow) which is expressed solely in terms of the stress variables

$$(\sigma_x - \sigma_y)/2 \quad \text{and} \quad \tau_{xy}. \tag{2.1}$$

That is, the criterion is expressible as a unique contour, generally closed, in the Mohr stress plane of Fig. 1, having (2.1) as axes. This specifically excludes an *explicit* dependence of the reduced yield criterion on the mean in-plane stress

$$\sigma = (\sigma_x + \sigma_y)/2, \tag{2.2}$$

or on any of the stresses σ_z, τ_{zx}, τ_{zy} necessary to enforce the plane flow.

We shall now see that any incompressible material, of arbitrary anisotropy, which complies with the principle of maximum plastic work is of this class. Let

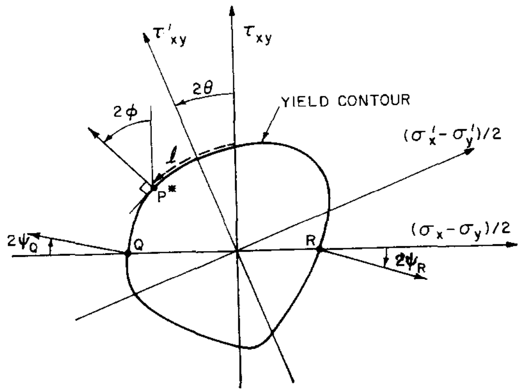

FIG. 1. Yield contour in Mohr plane.

ε_x, ε_y, γ_{xy} denote a set of incipient plane deformation rates so that the principle becomes

$$(\sigma_x - \sigma_x^\circ)\varepsilon_x + (\sigma_y - \sigma_y^\circ)\varepsilon_y + (\tau_{xy} - \tau_{xy}^\circ)\gamma_{xy} \geqslant 0, \tag{2.3}$$

where σ_x°, σ_y°, τ_{xy}° are the in-plane components of *any* other stress state, whether appropriate to a state of plane flow or not, which does not violate the yield criterion. But incompressibility with plane flow implies $\varepsilon_x + \varepsilon_y = 0$, so that (2.3) may be written

$$[(\sigma_x - \sigma_y)/2 - (\sigma_x^\circ - \sigma_y^\circ)/2](\varepsilon_x - \varepsilon_y) + [\tau_{xy} - \tau_{xy}^\circ]\gamma_{xy} \geqslant 0. \tag{2.4}$$

Assume tentatively that one or more of the stresses σ, σ_z, τ_{zx}, τ_{zy} enters the reduced yield condition, so that a family of distinct yield contours can be created in the Mohr plane by assignment of distinct values to the influential stress(es). Hence there will be a family of stress points in the Mohr plane, one for each contour, corresponding to any given set of incipient plane strain rates. If any one of these is identified as $\sigma_x, \ldots$, and any other as $\sigma_x^\circ, \ldots$, then the left side of (2.4) will either be negative, positive, or zero. If it is negative we have a contradiction. If it is positive we can create a contradiction by reversing the arbitrary assignments of $\sigma_x, \ldots$ and $\sigma_x^\circ, \ldots$. If it is zero we can make it non-zero by choosing a slightly different strain rate direction, although this is only obvious after realizing that by a standard argument based on (2.4) each separate contour must be such that the strain rate vector

$[(\varepsilon_x - \varepsilon_y), \gamma_{xy}]$ is in the direction of the outward normal to it at a smooth point, or within the fan set by limiting normals at a corner, and that each contour is convex. Thus, the assumption that any of σ, σ_z, τ_{zx}, τ_{zy} enter the reduced yield criterion leads to a means of violating the maximum plastic work inequality, and hence we conclude that the reduced yield criterion can depend only on the stress variables (2.1).

The normality property just cited must, of course, be valid for the *unique* yield contour shown to result in the Mohr plane. The basis of this, namely the plastic work inequality, is known to hold for any polycrystal flowing according to critical shear stress laws for the separate slip systems within each crystalline grain or, more generally, for any composite solid in which each constituent element separately complies with the inequality (BISHOP and HILL, 1951). When the angle of the outward normal vector to the yield contour is 2ϕ, as measured anticlockwise from the τ_{xy} axis of Fig. 1, we require that $[(\varepsilon_x - \varepsilon_y), \gamma_{xy}]$ be co-directional with $[-\sin 2\phi, \cos 2\phi]$ and hence that

$$(\varepsilon_x - \varepsilon_y) \cos (2\phi) + \gamma_{xy} \sin (2\phi) = 0. \tag{2.5}$$

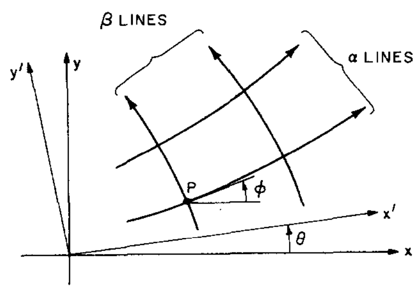

FIG. 2. Plane of deformation and slip lines.

The nature of the anisotropy implied by a given yield curve is best understood by considering an x', y' coordinate system, rotated by angle θ anti-clockwise from the base x, y system as in Fig. 2. According to the laws of stress transformation in a plane, the yield curve referred to the associated stress measures $(\sigma'_x - \sigma'_y)/2$, τ'_{xy} is given by a rotation of axes by 2θ from the base system in the Mohr plane, as illustrated in Fig. 1. Thus, if θ is chosen so that the τ'_{xy} axis passes through the point P* on the yield curve, it is seen that the distance from the origin of the Mohr plane to P* is just the shear strength for an in-plane stress state of pure shear relative to the primed axes (this being accompanied by whatever reaction stresses are needed to maintain plane flow). Similarly, from the laws of strain transformation in a plane, (2.5) becomes

$$(\varepsilon'_x - \varepsilon'_y) \cos [2(\phi - \theta)] + \gamma'_{xy} \sin [2(\phi - \theta)] = 0. \tag{2.6}$$

Thus, if the maximum plastic work inequality and hence the normality rule applies, we may choose the primed system so that $\theta = \phi$ at the point P* on the yield curve, thereby having a primed system according to which the incipient deformation is a pure shear. Then, the primed Mohr coordinates of P* are the in-plane stresses necessary to enforce this shear and, in particular, τ'_{xy} is the plastic dissipation per unit shear. These remarks also suggest lower and upper bound means of calculating the reduced yield criterion for an anisotropic crystal, polycrystal or composite.

2.2 Equilibrium equations in the plastic region

To derive the consequences of stress equilibrium equations in the plastic region we shall make no use of the maximum work inequality and flow rule. Instead we assume only that limiting states of stress at incipient flow are given by a unique yield contour in the Mohr stress plane.

Referred to the rotated x', y' system of Fig. 2, the equilibrium equations are

$$\left.\begin{aligned}\frac{\partial\sigma}{\partial x'}+\frac{\partial}{\partial x'}\left(\frac{\sigma_x'-\sigma_y'}{2}\right)+\frac{\partial\tau_{xy}'}{\partial y'}&=0,\\ \frac{\partial\sigma}{\partial y'}-\frac{\partial}{\partial y'}\left(\frac{\sigma_x'-\sigma_y'}{2}\right)+\frac{\partial\tau_{xy}'}{\partial x'}&=0.\end{aligned}\right\} \tag{2.7}$$

Consider these in a plastic region about some point P of the x, y plane. This point corresponds to a point P* on the yield contour, and we shall assume for the time being that the contour contains no corner in its neighborhood (see Section 2.4). Let l be the arc length around the yield contour to P*, increasing in the anti-clockwise sense of traversal as in Fig. 1. Now, points in the immediate neighborhood of P in the x, y plane correspond to points along the yield contour separated, say, by dl from P*. If we project such an increment of arc length onto the $(\sigma_x'-\sigma_y')/2$ and τ_{xy}' axes there results

$$d\left(\frac{\sigma_x'-\sigma_y'}{2}\right)=-dl\cos[2(\phi-\theta)],\qquad d\tau_{xy}'=-dl\sin[2(\phi-\theta)], \tag{2.8}$$

where 2ϕ is the angle of the outward normal to the yield contour at P*. Thus, the equilibrium equations (2.7) may be written in terms of derivatives of σ and l only in a plastic region:

$$\left.\begin{aligned}\frac{\partial\sigma}{\partial x'}-\frac{\partial l}{\partial x'}\cos[2(\phi-\theta)]-\frac{\partial l}{\partial y'}\sin[2(\phi-\theta)]&=0,\\ \frac{\partial\sigma}{\partial y'}+\frac{\partial l}{\partial y'}\cos[2(\phi-\theta)]-\frac{\partial l}{\partial x'}\sin[2(\phi-\theta)]&=0.\end{aligned}\right\} \tag{2.9}$$

If we now choose the rotation angle θ of the x', y' system to equal one-half of the angle 2ϕ of the outer normal to the yield contour at P*, these become

$$\frac{\partial}{\partial x'}(\sigma-l)=\frac{\partial}{\partial y'}(\sigma+l)=0\quad\text{at P.} \tag{2.10}$$

We therefore shall cover the region at yield in the x, y plane with two orthogonal families of curves, labelled α and β as in Fig. 2, and called 'slip lines'. In particular, the mesh of slip lines is defined such that the anti-clockwise angle between the α line through a generic point P and the x direction is ϕ, where 2ϕ is the angle of the outward normal to the yield contour at the corresponding stress state P*. For this mesh the derivatives on x', y' in (2.10) are the same as directional derivatives on arc length in the α, β directions through P. Hence, the integrated equations of stress equilibrium in a plastic region are

$$\left.\begin{aligned}\sigma-l&=\text{constant along an }\alpha\text{ line,}\\ \sigma+l&=\text{constant along a }\beta\text{ line.}\end{aligned}\right\} \tag{2.11}$$

This is a simple generalization of the Hencky equations for an isotropic material. In that case the yield contour is a circle of radius equal to the yield stress k in shear, and the arc length around it to a point having the normal angle 2ϕ is $l = 2k\phi$. Hence, (2.11) expresses the well-known constancy of $\sigma \pm 2k\phi$ along slip lines. Also, HILL's (1950, pp. 337–338) results for an ellipsoidal yield criterion take the form of (2.11) with a term analogous to l which is given in terms of an elliptic integral. It becomes clear after some study that his term is nothing more than an expression for the arc length around the elliptical yield curve in the Mohr plane. SHOEMAKER (1963) considered a yield curve in the form of a square in the Mohr plane, as an anisotropic but simple approximation to the isotropic case, and likewise derived stress equations coincident with (2.11).

HILL (1950) noted that Hencky's first theorem on the geometry of slip lines was readily extended to the case of an elliptical yield curve, whereas the second theorem appeared to have only a very complicated extension which was not of obvious utility. The same is true in the general case. The extension of Hencky's first theorem is that the change in l between two lines of one family, at their points of intersection with a line of the other, is the same for each intersecting line of the second family. Since ϕ is a function of l (apart from the inessential non-uniqueness at a corner, to be considered later) this restricts the admissible geometries of orthogonal curves purporting to be α, β families. It is easy to see that constant state regions and centered fans are admissible fields for every material. They will arise in later applications.

2.3 *Velocity equations*

We will consider the velocity equations only in cases for which the maximum work principle is adopted, and hence for which the normality rule holds. Then (2.6) is valid. If in particular we choose the rotation angle θ of the x', y' system equal to the value of ϕ at the representative point P of Fig. 2, and recall also that $\varepsilon'_x + \varepsilon'_y = 0$, then

$$\varepsilon'_x = \varepsilon'_y = 0 \quad \text{at P.} \tag{2.12}$$

From this it is clear that the α, β directions are of zero extension rate, and hence the relevant velocity equations are always the Geiringer equations

$$\left.\begin{aligned} dv_\alpha - v_\beta\, d\phi &= 0 \quad \text{on an } \alpha \text{ line,} \\ dv_\beta + v_\alpha\, d\phi &= 0 \quad \text{on a } \beta \text{ line.} \end{aligned}\right\} \tag{2.13}$$

Here, v_α, v_β are velocities in the α, β directions, chosen positive in the direction of the arrows in Fig. 2. Given that the strain rate is to be in the direction of the *outward* normal to the yield curve, the velocity field is subject to the further constraint $\gamma_{\alpha\beta} > 0$.

Of course, the results of this section and Section 2.2 may equally well be obtained by examining the stress equilibrium and normality flow rule equations for the existence of characteristics. The α, β lines are then found to be the common characteristics of both and (2.11, 13) arise as consistency requirements. This approach was adopted by HILL (1950) in his study of the ellipsoidal criterion, and he has also noted the general coincidence of stress and flow characteristics which follows from the normality rule.

2.4 *Corners and flats*

There is no need to alter any of the preceding results when there is a corner on the yield contour. They need only a proper interpretation.

First, the procedure outlined for obtaining ϕ from the stress state at a generic point of the x, y plane breaks down when that state corresponds to the corner. This causes no difficulty when the corner state maps into an arc in the x, y plane. Then, the α, β lines are well-defined on the two sides of the arc and join across it with a discontinuous change in ϕ corresponding to half the angular range of the fan of normals to the yield contour at the corner. Since (2.11) applies on either side of the arc and l and σ are continuous across it, (2.11) continues to apply along the entire length of the kinked α, β lines traversing the arc.

As for the velocity equations, v_x and v_y are continuous across the arc but v_α and v_β must be discontinuous because of the discontinuity in α and β directions. Denote the sides of the arc by superscripts $-$ and $+$, letting ϕ^- and ϕ^+ be the corresponding limiting values of ϕ at the corner. Then, the kink angle is $\phi^+ - \phi^-$ and velocity continuity on fixed reference axes requires that

$$\left.\begin{aligned} v_\alpha^+ &= v_\alpha^- \cos(\phi^+ - \phi^-) + v_\beta^- \sin(\phi^+ - \phi^-), \\ v_\beta^+ &= -v_\alpha^- \sin(\phi^+ - \phi^-) + v_\beta^- \cos(\phi^+ - \phi^-). \end{aligned}\right\} \tag{2.14}$$

Now, it is a remarkable fact that these same jump equations are obtained if we divide both members of (2.13) by $d\phi$ and view the results as a set of coupled differential equations for the variation of v_α, v_β with ϕ at any fixed point along the arc. Specifically, (2.14) is easily verified to be the solution of the equations at $\phi = \phi^+$ subject to initial conditions $v_\alpha, v_\beta = v_\alpha^-, v_\beta^-$ at $\phi = \phi^-$, and in this sense (2.13) remains valid as well along the kinked slip lines traversing the arc.

The other possibility is that the stress state of the corner maps into a finite region of the x, y plane. Then, ϕ and hence the α, β lines cannot be uniquely defined from stress considerations alone. In fact, (2.11) properly determines the stress field in such a region no matter how we choose to draw the α, β lines within it (except, of course, that we will always have ϕ within the range set by limiting normals at the corner). This is because l is constant by hypothesis within the region, so that the entire content of (2.11) is that σ is constant there as well. That this is correct may be seen directly from (2.7) since the Mohr stress variables are fixed to their values at the corner. The only way of choosing ϕ within the region is from flow considerations, and if we choose α, β directions to comply with zero stretching rate directions of the incipient flow field, then it is immediately obvious that (2.13) applies. If the region is non-deforming, there is no way of uniquely drawing α, β lines within it, but neither is there any need to do so since the properties are simply those of a constant stress region moving as a rigid body.

An interesting stress discontinuity may arise when the yield contour contains a flat segment (i.e. ϕ = constant = ϕ_0, say, for a finite range of l). Suppose, for example, that a group of α lines have continuously and monotonically turning tangents over a range of ϕ values which includes ϕ_0. Then, l suffers a jump discontinuity along each α line, say from l^- to l^+ as we transverse the lines in the positive direction. If we provisionally apply (2.11_1), we expect the jump

$$\sigma^+ - \sigma^- = l^+ - l^-, \tag{2.15}$$

where σ^- and σ^+ are the discontinuous values of the mean normal stress just ahead and just beyond the point on any α line at which $\phi = \phi_0$.

Let us bring the generalization of Hencky's first theorem to bear on the matter. Consider any two infinitesimally spaced β lines which cross the group of α lines in such a way as to straddle the point $\phi = \phi_0$ on any one α line. The change in l between these two β lines is constant by the theorem, and hence equal to $l^+ - l^-$ everywhere along their length. This means that the β lines must straddle the point $\phi = \phi_0$ along *every* α line, which in turn means that the slope of the β line through the jump points is constant at $\phi_0 + \frac{1}{2}\pi$, and hence that the considered stress discontinuity must occur across a ***straight*** β line.

That the discontinuity here described from the slip line equations conforms with overall equilibrium can be verified by direct analysis. Choose the rotated x', y' system of Fig. 2 so that $\theta = \phi_0$ and thus that the axes are coincident with the α, β directions at the jump points. The equation of the flat segment will then be τ'_{xy} = constant, the flat being parallel to the $(\sigma'_x - \sigma'_y)/2$ axis in the Mohr plane (see Fig. 4 in Section 3.2). Since the jump occurs across the y' axis, coincident with the straight β line, σ'_x and τ'_{xy} must be continuous across it for equilibrium, but σ'_y does not need to be continuous. In traversing the discontinuity the stress state changes from that at one end of the flat to that at the other, so that

$$(\sigma'_x - \sigma'_y)^+/2 - (\sigma'_x - \sigma'_y)^-/2 = l^+ - l^-. \tag{2.16}$$

Thus, the stress jumps are

$$(\sigma'_x)^+ = (\sigma'_x)^-, \qquad (\tau'_{xy})^+ = (\tau'_{xy})^-, \qquad (\sigma'_y)^+ = (\sigma'_y)^- - 2(l^+ - l^-). \tag{2.17}$$

These imply (2.15) upon using (2.2). To complete the verification, we need only recall that a discontinuity of the kind considered, taking place across a straight line and consisting of a jump of constant magnitude in the normal stress component directed parallel to the line, is always in conformity with equilibrium if the stress fields on the two sides are themselves in equilibrium.

Flats and corners will typically be present on the yield contours for highly oriented materials, such as single crystals or composites, and in Section 3.2 we will see an example which incorporates some of their special features as discussed here.

3. Application

3.1 *Frictionless indentation by a flat punch*

Consider a flat-ended, rigid, frictionless punch in incipient indentation of a half-plane. This problem was discussed by Hill (1950) and Shoemaker (1963) for their respective special cases of anisotropy. Figs. 3(a),(b) show two possible slip line fields, drawn and labelled with reference to the yield curve of Fig. 1. It will be seen that one or the other of these fields is valid depending on whether the arc length around the yield curve from Q to R is least when measured around the top or bottom portion of the curve. When both arc lengths are equal, as would be the case for a yield curve exhibiting symmetry under stress reversals (i.e. if $\sigma_x, \sigma_y, \tau_{xy}$ correspond to yield, so also do $-\sigma_x, -\sigma_y, -\tau_{xy}$), both slip line patterns may exist simultaneously beneath the punch (Fig. 3(c)).

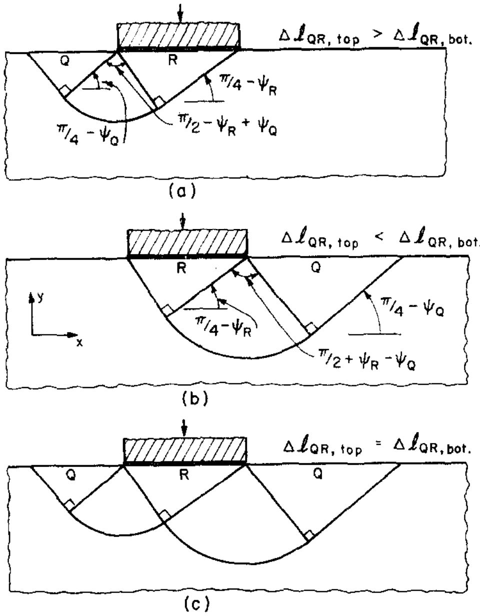

FIG. 3. Punch indentation: general case.

If yield occurs directly beneath the punch, it is to be expected that σ_y is the greatest compressive stress, and hence, since $\tau_{xy} = 0$ on the boundary, the stress state there corresponds to point R on the yield contour of Fig. 1. Similarly, if yield occurs to the right or left of the punch, it is expected that σ_x is the greatest compression and since $\tau_{xy} = 0$ on the boundary, the stress state along the boundary corresponds to point Q on the yield contour.

Assume tentatively that yield occurs to the left as in Fig. 3(a). Since σ_y vanishes on the boundary, we have

$$\sigma_x = (\sigma_x - \sigma_y)_Q, \qquad \sigma_y = \tau_{xy} = 0, \tag{3.1}$$

and thus

$$\sigma = \tfrac{1}{2}(\sigma_x - \sigma_y)_Q, \quad \phi = \pi/4 - \psi_Q, \quad l = l_Q \tag{3.2}$$

there. Now any β line (such as the outermost slip line shown) emanating from this yielding region to the left and terminating along the punch boundary ends in a region for which

$$\phi = 3\pi/4 - \psi_R, \quad l = l_R. \tag{3.3}$$

Thus, from (2.11₂), the mean normal stress along the punch boundary is given by

$$\sigma + l_R = \tfrac{1}{2}(\sigma_x - \sigma_y)_Q + l_Q, \tag{3.4}$$

and hence the stress state along the punch boundary is

$$\left.\begin{aligned} \sigma_x &= \tfrac{1}{2}(\sigma_x - \sigma_y)_R + \tfrac{1}{2}(\sigma_x - \sigma_y)_Q - (l_R - l_Q), \\ \sigma_y &= -\tfrac{1}{2}(\sigma_x - \sigma_y)_R + \tfrac{1}{2}(\sigma_x - \sigma_y)_Q - (l_R - l_Q), \\ \tau_{xy} &= 0. \end{aligned}\right\} \tag{3.5}$$

The constant stresses along the punch boundary and the portion of half-plane boundary to its left determine constant stress fields within their triangular domains of influence, namely, R and Q in Fig. 3(a), with angles as indicated. As is also shown, these constant stress domains are joined by a centered fan in which the radial lines (not shown in the figure) emanating from the edge of the punch are α lines. Note that in general the fan angle differs from $\frac{1}{2}\pi$ unless the material exhibits symmetry under stress reversals so that $\psi_R = \psi_Q$. As for the corresponding velocity field, it is easy to see that (2.13) admits the solution $v_\alpha = 0$, $v_\beta =$ constant everywhere throughout the region. This is consistent with a vertically downward motion of the punch without rotation, and also meets the necessary conditions of non-penetration of the rigid region and flow in the direction of the *outward* normal to the yield contour.

Hence, on the basis of the tentative assumption that flow occurs to the left of the punch, the indentation pressure ($-\sigma_y$ from (3.5)) is

$$p = \tfrac{1}{2}(\sigma_x - \sigma_y)_R - \tfrac{1}{2}(\sigma_x - \sigma_y)_Q + \Delta l_{QR,\,bot.}, \tag{3.6}$$

where $\Delta l_{QR,\,bot.} \equiv l_R - l_Q$. This notation is preferred to indicate that the arc length around the yield curve, from Q to R, is that measured off along the *bottom* portion of the yield contour of Fig. 1. This is so because the succession of stress states encountered as a β line is traversed from the left of the punch to the punch boundary correspond to non-decreasing values of ϕ; the centered fan contains all stress states along the bottom portion of the yield contour.

An entirely parallel analysis, based on the assumption that flow occurs to the right of the punch as in Fig. 3(b), leads to the indentation pressure

$$p = \tfrac{1}{2}(\sigma_x - \sigma_y)_R - \tfrac{1}{2}(\sigma_x - \sigma_y)_Q + \Delta l_{QR,\,top}. \tag{3.7}$$

Here, the change in arc length, from R to Q, is that measured off round the top of the yield contour, the fan then containing all stress states along the top portion.

Now, since a proper flow field may be associated with both these slip line solutions, both are upper bounds on the indentation pressure and hence that giving the smallest pressure is to be chosen. Accordingly, Fig. 3(a) and (3.6) are to be chosen if $\Delta l_{QR,\,bot.}$ is the smaller arc length, whereas Fig. 3(b) and (3.7) are chosen if $\Delta l_{QR,top}$ is smaller. If the two arc lengths coincide, both slip line fields may be considered to act simultaneously as in Fig. 3(c). The writer has not pursued the question as to whether the slip line solution presented can be extended statically into the rigid region, which is necessary to a proof that the "solution" is more than an upper bound. It is, however, known that the isotropic version of these fields may be so extended for a plane of sufficient extent (BISHOP, 1953), and it would seem that the same technique could be adopted in general.

Let us consider the following interpretation of equations (3.6, 7). Setting $\theta = \pi/4$ in Fig. 1, we see that $-\frac{1}{2}(\sigma_x - \sigma_y)_Q$ and $\frac{1}{2}(\sigma_x - \sigma_y)_R$ are simply the shear yield strength magnitudes when the material is subjected to an in-plane shear stress τ'_{xy} of positive and negative sign, respectively. For a material exhibiting a yield contour with symmetry under stress reversals, these two strength magnitudes coincide and it is then easy to remember that *the indentation pressure is twice the shear strength for the orientation $\theta = \pi/4$ plus half the circumferential arc length around the yield contour.* The isotropic result $p = (2+\pi)k$ has, of course, the same interpretation, which is best seen by re-writing it as $p = 2k + \frac{1}{2}(2\pi k)$.

3.2 *Examples of highly anisotropic materials*

Consider a highly anisotropic material having a yield contour which appears as a rectangle of side semi-lengths a and b, as in the $(\sigma_y - \sigma_y)/2$, τ_{xy} plane of Fig. 4(a). The primed coordinate system, at angle θ in the x, y plane and 2θ in the stress plane, is chosen to align with principal directions of anisotropy, as shown.

The inserts of Fig.4(b) represent two distinct materials which may be thus modelled.

That on the left of Fig. 4(b) shows a unit cube of a NaCl-type ionic single crystal. The plane of deformation coincides with one of the cube faces and the x', y' axes are chosen in the directions of the in-plane cube diagonals. In-plane slip along such diagonal planes constitutes the normally observed slip system, and evidently b is to be interpreted as the shear strength for such slipping. If these were the only possible slip systems, we would have $a \to \infty$ so that the circumferential arc length around the yield surface would be unbounded and thus also would be the indentation pressure. Hence, it is necessary to assume that a secondary slip system is activated

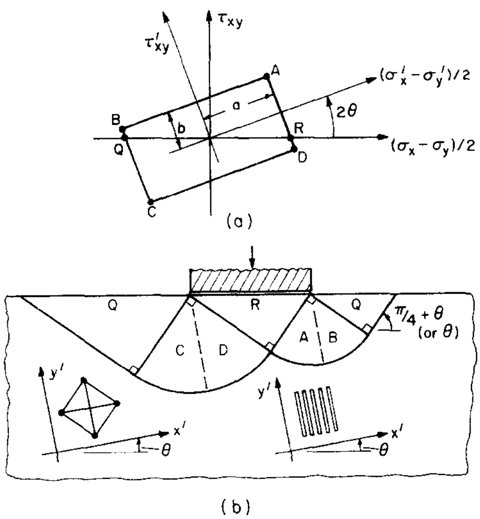

FIG. 4. Indentation of highly anisotropic material.

in indentation. This is assumed to involve in-plane slip along the faces of the cube, and a (*which will normally be much greater than* b) is therefore to be identified as the shear strength of the cube faces. This choice of secondary slip system has been proposed by GILMAN (1973), who has adopted the writer's solution† to the problem under discussion as an explanation of the anomalous indentation hardness of NaCl-type crystals.

The second material is illustrated in the insert on the right of Fig. 4(b). This is taken to be a ductile matrix material reinforced by stronger, yet ductile, fibers. We interpret b as the shear strength parallel to the fibers and a as the shear strength in a direction at $\pi/4$ to the fiber axes. In this case it cannot be rigorously asserted that the yield contour is rectangular, but the rectangle is at least an outer bound to the actual yield contour, touching it along its principal axes, if the latter is convex. It is,

† Communicated privately to Dr. J. J. Gilman in 1971.

of course, assumed in both cases that plastic flow rather than local fracturing is the mechanism of deformation during indentation, and this will not always be the case.

The slip line field of Fig. 4(b) has been constructed according to the general procedure of Fig. 3. The case shown corresponds to an angle θ which is sufficiently small so that the corner D of the rectangle has not yet reached the $(\sigma_x-\sigma_y)/2$ axis of Fig. 4(a). In this case, $\psi_Q = \psi_R = -\theta$ and hence the rightmost outer slip line has the inclination $\pi/4+\theta$ as shown. This defines the entire mesh geometry in view of the right angles. If θ is increased beyond this critical value of $\frac{1}{2}\arctan(b/a)$, $\psi_Q = \psi_R = \pi/4-\theta$ and the inclination of the rightmost slip line becomes θ, with the mesh then having its greatest extent to the right.

Half of the circumferential arc length around the yield contour is $2(a+b)$, and the distance from the origin of the stress plane to Q or R is $a/\cos(2\theta)$ for the angular range corresponding to Fig. 4(a). Hence, the indentation pressure is

$$p = \frac{2a}{\cos(2\theta)} + 2(a+b) \quad \text{for } 0 \leqslant \theta \leqslant \tfrac{1}{2}\arctan\left(\frac{b}{a}\right). \tag{3.8}$$

If θ is increased beyond this upper limit, the distance from the origin to Q or R is $b/\sin(2\theta)$, and

$$p = \frac{2b}{\sin(2\theta)} + 2(a+b) \quad \text{for } \tfrac{1}{2}\arctan\left(\frac{b}{a}\right) \leqslant \theta \leqslant \frac{\pi}{4}. \tag{3.9}$$

These results determine p as a function of θ for the angular range $0 \leqslant \theta \leqslant \pi/4$ and hence for any range since p is symmetrical about $\theta = 0$ and $\pi/4$. The variation of p with θ is most readily visualized by rotating the rectangle in Fig. 4(a), remembering that $p-2(a+b)$ is twice the distance from the origin to the point R.

Indeed, this result reveals that for strongly anisotropic materials ($a \gg b$) there is a small angular range centered on $\theta = 0$ for which high strengths of order $p = 4a$ occur, whereas there is a broad angular range centered on $\theta = \pi/4$ with lower strengths of order $p = 2a$. Of course, at all orientations it is the higher strength a rather than b which governs.

The slip line pattern of Fig. 4(b) has been drawn in conformity with the general solution of Section 3.1. However, the yield contour involves both corners and flats and it is interesting to note that the regions marked A, B, C, D within the centered fans correspond to the corners A, B, C, D on the yield contour. According to what has been said earlier, each of these regions is a region of constant stress. In fact, the entire slip line field consists of constant stress regions. Each is separated by a straight line of stress discontinuity, of the type discussed at the end of Section 2.4, arising from the flat segments between Q and C, C and D, D and R, R and A, and so on.

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Note

Since completion of my manuscript, a paper has been published by Booker, J. R. and Davis, E. H., *J. Mech. Phys. Solids* **20**, 239–25 (1972) which also obtains the plastic stress equilibrium equations (2.11) and applies them to a general solution of the punch problem. Indeed, they obtained (2.11) as a special case of a more general formulation for pressure-sensitive anisotropic materials. However, there are significant differences between their presentation and mine so that the present paper stands on its own. For example, particularly, I refer to the discussion of the role of the maximum work inequality in setting the form of the reduced yield criterion, to the simple and direct derivation of the plastic equilibrium equations in Section 2.2, and to the discussion of discontinuities associated with the corners and flats which are typical of highly anisotropic materials.

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