

**B. BUDIANSKY**

Professor,  
Division of Engineering  
and Applied Physics,  
Harvard University,  
Cambridge, Mass.  
Mem. ASME

**J. R. RICE**

Professor,  
Division of Engineering,  
Brown University,  
Providence, R. I.  
Assoc. Mem. ASME

# Conservation Laws and Energy-Release Rates

*New path-independent integrals recently discovered by Knowles and Sternberg are related to energy-release rates associated with cavity or crack rotation and expansion. Complex-variable forms are presented for the conservation laws in the cases of linear, isotropic, plane elasticity. A special point concerning plastic stress distributions around cracks is discussed briefly.*

## Introduction

THE well-known  $J$  integral of fracture mechanics [1, 3, 5]<sup>1</sup> has been related to potential-energy-release rates associated with moving or extending cracks in linear or nonlinear elastic materials. Some new path-independent integrals (or conservation laws) have recently been discovered by Knowles and Sternberg [2]. In this paper these new laws are related to energy release rates associated with cavity or crack rotation and expansion rates. In addition, the conservation laws are displayed in complex-variable form for the case of linear, isotropic, plane elasticity. Finally, an implication concerning plastic stress distributions around cracks is discussed briefly.

## Conservation Laws

Consider a two-dimensional deformation field for which the displacement vector  $\mathbf{u}$  depends only on  $x_1$  and  $x_2$ . The  $J$  integral is

$$J = \oint_C (W dx_2 - T_i u_{i,1} dl)$$

where  $C$  is a closed curve in the  $x_1, x_2$  plane,  $W$  is the energy

density, and  $T_i$  is the stress vector acting on the outer side of  $C$ . The  $J$  integral is actually the first component of the vector

$$J_k = \oint_C (W n_k - T_i u_{i,k}) dl \quad (1a)$$

where  $\mathbf{n}$  is the unit outward normal to  $C$ , lying in the same plane. Each component of  $\mathbf{J}$  vanishes (trivially for  $J_3$ ) for all closed paths bounding a region in which  $W$  depends only on the strain  $\eta_{ij} = (u_{i,j} + u_{j,i})/2$ , and in which the stresses  $\sigma_{ij}$ , related to  $T_j$  on  $C$  by  $\sigma_{ij} n_i$ , satisfy

$$\sigma_{ij} = \frac{1}{2} \left( \frac{\partial W}{\partial \eta_{ij}} + \frac{\partial W}{\partial \eta_{ji}} \right) \text{ and } \sigma_{ij,i} = 0.$$

This implies that  $J_\alpha (\alpha = 1, 2)$  has the same value not necessarily zero, for all paths that enclose a hole or crack.

The new Knowles-Sternberg integrals, in 2- $D$ , are

$$L = \oint_C \epsilon_{ij} (W x_j n_i + T_i u_j - T_k u_{k,i} x_j) dl \quad (2a)$$

and

$$M = \oint_C (W x_i n_i - T_k u_{k,i} x_i) dl \quad (3a)$$

where  $\epsilon_{ijk}$  is the alternating tensor. Under the same conditions specified for  $\mathbf{J}$ ,  $L$  vanishes if, in addition,  $W$  depends only on the scalar invariants of  $\eta_{ij}$ . For  $M$  to vanish it is necessary that  $W$  be a quadratic function of the components of  $\eta_{ij}$ .

The results apply not only for any combination of plane and antiplane straining, but also for the 2- $D$  theory of generalized plane stress in which  $\mathbf{u}$  and  $\boldsymbol{\sigma}$  denote thickness averages. Also, with suitable redefinitions of the basic variables,  $J_\alpha$  and  $L$  vanish when geometrical nonlinearity is admitted [2, 3]; and if  $W$  is a homogeneous function of degree  $m$  in the strain components,  $M$  will still vanish if it is redefined by adding the quantity  $(m - 2)T_i u_i$  to its integrand.

In three dimensions the integrals generalize to

<sup>1</sup> Numbers in brackets designate References at end of paper.  
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$$J_k = \int_S (W n_k - T_i u_{i,k}) dS \quad (1b)$$

$$L_k = \int_S \epsilon_{kij} (W x_j n_i + T_i u_j - T_i u_{i,j} x_j) dS \quad (2b)$$

$$M = \int_S (W x_i n_i - T_j u_{j,i} x_i - \frac{1}{2} T_i u_i) dS \quad (3b)$$

where  $S$  is a closed surface with outer normal  $\mathbf{n}$ , and analogous conservation theorems have been shown to hold by Knowles and Sternberg.

### Energy-Release Rates

Eshelby [3, 4] and Rice [5] have shown that  $J_k$  can be interpreted as the energy-release rate when a void or a crack tip is translated in position relative to a material body. We show that  $L_k$  and  $M$  have similar interpretations here.

Consider a 3-D elastostatic boundary-value problem associated with the material contained within the surface  $S + s$ , for which the portion  $s$  of the boundary is traction-free, and external loading is imposed only by tractions on  $S$ . Without changing the boundary conditions of  $S$ , contemplate the continuously varying sequence of static solutions for the displacements  $u$  generated as the spatial specification of  $s$  is varied with a timelike parameter  $t$ . The potential energy of the system at any time is

$$\Pi = \int_{V(t)} W dV - B[\mathbf{u}]$$

where  $V(t)$  is the volume enclosed by  $S + s(t)$  and  $B[\mathbf{u}]$  is the potential of the loading specified on  $S$ . At each point in  $V(t)$ ,  $W$  is a function of the time-varying strains compatible with  $\mathbf{u}$ . Then

$$\frac{d\Pi}{dt} \equiv \dot{\Pi} = \int_{V(t)} \dot{W} dV - \frac{d}{dt} \{B[\mathbf{u}]\} + \int_{s(t)} W v_i m_i ds$$

where  $v_i$  denotes the "velocity" of points on  $s$  and  $m_i$  is the current outward normal to  $s$ . (Note that only the normal component of  $v_i$  is determined uniquely by a given motion of the cavity boundary.) But by the principle of virtual work the first two terms in  $\dot{\Pi}$  cancel (assuming that  $d\mathbf{u}/dt$  is an admissible function in  $V(t)$ ), so that

$$\dot{\Pi} = \int_s W v_i m_i ds$$

This result has been derived, less concisely, by Rice and Drucker [6].

Next, suppose that  $s$  is the boundary of a cavity, and let  $v_j = \delta_{ij}$ ; this corresponds to a conceptual translation of the cavity with a unit velocity in the  $i$ -direction. Let  $n_i = -m_i$  be the unit inward normal on the cavity surface; then  $\xi_i \equiv -\dot{\Pi}$  is the rate of energy release per unit of cavity translation in the  $i$ -direction, and is given by

$$\xi_i = \int_s W n_i ds$$

But by the first conservation law (1b), we have

$$\xi_i = J_i \quad (1c)$$

wherein the integral in  $J_i$  can be calculated on any closed surface surrounding the cavity. This last equation is Eshelby's [3] result, and takes the same form as his equation for "forces" on point defects in solids [4].

Next, consider a unit angular velocity about the  $i$ -axis of the specified cavity wall; then  $v_j = -\epsilon_{ijk} x_k$ , and this leads to the result

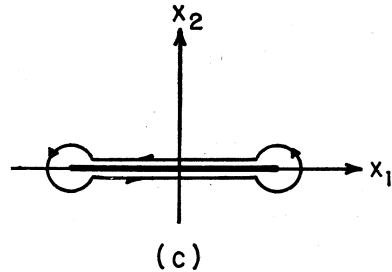
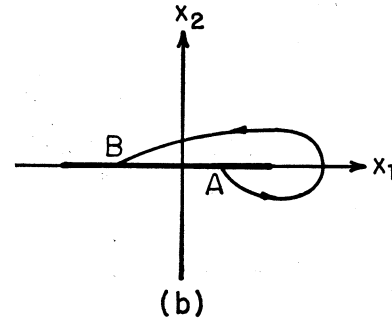
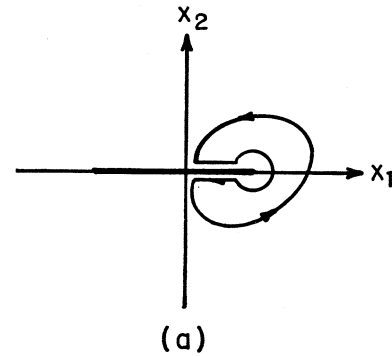


Fig. 1 Integration paths

$$\xi_i = -L_i \quad (2c)$$

for the rate of energy release per unit cavity rotation about the  $i$ -axis (with the usual right-hand rule sign convention for rotation.)

Finally, let the cavity boundary expand uniformly according to the rule  $v_i = x_i$ . This gives the energy-release rate

$$\xi = M \quad (3c)$$

Here the rate is with respect to relative scale change  $dl/l$ , where  $l$  is any characteristic length of the cavity.

The 2-D versions of these energy-release relations, for plane or antiplane conditions, are self-evident.

Since the final results are expressible as integrals over surfaces or curves off the cavity boundary, there is no reason to doubt their validity when the cavity is a crack.

In 2-D crack studies, the  $J(=J_i)$  integral has been exploited for closed paths of the type shown in Fig. 1(a). Since the integrand vanishes on the crack edges, it follows that the  $J$  integral has the same value for all open paths connecting any points  $A, B$  on opposite sides of the crack, as shown in Fig. 1(b). In this case, the integral provides the energy-release rate per unit crack tip extension [1, 5].

## Complex Variable Forms

In isotropic plane elasticity, the standard complex potentials  $\phi(z)$  and  $\psi(z)$  are analytic functions of  $z = x + iy$  related to the stresses by

$$\begin{aligned}\sigma_{11} + \sigma_{22} &= 2(\phi' + \bar{\phi}') \\ \sigma_{22} - \sigma_{11} + 2i\sigma_{12} &= 2(\bar{z}\phi'' + \psi')\end{aligned}$$

The following results have been derived for plane stress:

$$J_1 + iJ_2 = -\frac{2i}{E} \left[ \oint_C (\phi')^2 dz - 2 \oint_C \phi' \psi' dz \right] \quad (1d)$$

$$\begin{aligned}L &= \frac{4}{E} \operatorname{Re} \oint_C z \psi \phi'' dz \\ &= \frac{4}{E} \operatorname{Re} \oint_C \psi' (\phi - z\phi') dz\end{aligned} \quad (2d)$$

$$M = \frac{4}{E} \operatorname{Im} \oint_C z \phi' \psi' dz \quad (3d)$$

where  $E$  is Young's modulus. For plane strain  $E$  should be replaced by  $E/(1 - \nu^2)$ , where  $\nu$  is Poisson's ratio. In the derivation of each formula, it was assumed that the region within  $C$  was free of any resultant forces, so that the potentials  $\phi$  and  $\psi$  were single-valued.

If the integration path in (1a) is open, extending from  $A$  to  $B$ , the extra quantity

$$\frac{2i}{E} [z(\bar{\phi}')^2]_A^B$$

must be appended to (1d). Thus the  $J$  integral, when taken around a crack tip, is just

$$J = \frac{2}{E} \operatorname{Im} \left\{ \int_A^B [(\phi')^2 + 2\phi' \psi'] dz + [z(\bar{\phi}')^2]_A^B \right\}$$

Similarly, extra terms appear in (2d) and (3d) if the integration paths in (2a) and (3a) are open.

In the case of antiplane shear, the relations

$$\begin{aligned}\sigma_{32} + i\sigma_{31} &= \omega'(z) \\ u_3 &= \frac{1}{2iG} (\omega - \bar{\omega})\end{aligned}$$

provide the stresses and the displacement in terms of an analytic function  $\omega(z)$ ;  $G$  is the shear modulus. Then formulas (1a), (2a), and (3a) can be written as

$$J_1 - iJ_2 = -\frac{i}{2G} \oint_C (\omega')^2 dz$$

and

$$L + iM = -\frac{1}{2G} \oint_C z(\omega')^2 dz$$

The same formulas also hold for open paths.

## Crack-Tip Stresses in the Plastic Range

In conjunction with the simple deformation theory of plasticity, the  $J_1$  integral has been used to calculate asymptotic plastic results [5, 7-9] for so-called "small-scale yielding" near the tips of cracks loaded in Modes I, II, and III. (Mode I, in the jargon of fracture mechanics, means crack opening, and Mode II is shearing parallel to the crack; in each case, the loading and geometry is symmetrical about the crack. Mode III is antiplane shear.) In these solutions, the dominant part of the singular solution near the tip in the far plastic range is determined to within a scalar

factor; the factor is then found from the invariance of  $J_1$  evaluated for paths around the crack tip at small and large radii, where conditions are, respectively, purely plastic and purely elastic. But this method has been successful only for loadings that are purely in one of the three modes mentioned. Thus, for a mixture of Modes I and II, two scalar quantities are needed to establish the stress distribution near the crack tip, and the  $J_1$  integral does not supply enough information for their determination.

Unfortunately, contrary to some initial hopes, use of the new  $L$  integral does not appear to supply the missing data in mixed-mode cases. It does, however, provide an unexpected result. The earlier pure-mode solutions indicated that  $W$  may be expected to vary inversely with the distance  $r$  from the crack tip. Now consider the evaluation of  $L$ , for a mixed-mode situation, on the path shown in Fig. 1(c). Since  $L$  has a bounded unique value for all paths enclosing the crack, and since the contributions to  $L$  due to the small circles remain bounded as their radii shrink to zero, it follows that

$$\int_{\text{crack}} x[W_+ - W_-] dx$$

must be bounded, where  $W_+$  and  $W_-$  are evaluated on the top and bottom faces of the crack, respectively. This means, therefore, that the singular  $(1/r)$  parts of  $W$  must be equal at opposing points on either side of the crack. This is surprising when one considers arbitrary mixtures of Modes I and II. On the crack,  $W$  depends only on  $|\sigma_x|$  in simple deformation theory. Hence, the equality of  $W_+$  and  $W_-$  near the crack tip implies that the ratio of the dominant singularities in  $(\sigma_x)_+$  and  $(\sigma_x)_-$  must, in all mixed-mode cases, be either 1 or  $-1$ . This ratio is  $+1$  for Mode I and  $-1$  for Mode II. If, then, one imagines the loading mode to change continuously from I to II, this ratio will have to jump suddenly from 1 to  $-1$  at some particular mode mixture.

The equality of the  $(1/r)$  contributions to  $W$  on opposite sides of the crack must persist even if some Mode III, which has a  $\tau_{xz}$  contribution to  $W$ , is also present. It may be noted, finally, that these conclusions could have been reached by a consideration of the  $J_2$  integral.

## Acknowledgment

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