J.R. Rice, "On the Structure of Stress-Strain Relations for Time-Dependent Plastic Deformation in Metals", Journal of Applied Mechanics, 37, 1970, pp. 728-737

# On the Structure of Stress-Strain Relations for Time-Dependent Plastic Deformation in Metals

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*The paper is concerned with the structure of multiaxial stress-strain relations in time-dependent metal plasticity, as for transient creep and rate sensitive yielding. First, a general kinematical relation is developed between the macroscopic inelastic strain tensor and microstructural slip displacements, as modeled either by continuum shearing on crystallographic planes of individual grains or by the motion of discrete dislocation lines. It is assumed that at any given slipped state, the rate of slipping on a particular system is governed by the resolved shear stress on that system (or by the local "forces" on dislocation lines). This leads to the primary result of the paper: Components of the macroscopic inelastic strain rate tensor are derivable, at each instant in the course of deformation, from a potential function of stress. General features of the flow potential surfaces in stress space are discussed, and some specific functional forms are examined. Linear viscoelasticity and time-independent plasticity are developed as limiting cases of the flow potential formulation, and the appropriateness of a potential function for stationary creep is discussed.*

## Introduction

THIS paper is concerned with the essential structure of inelastic stress-strain relations for metals, particularly in the transient creep and strain-rate sensitive ranges of behavior where time effects dominate. The considerations are primarily macroscopic, but an attempt is made to relate behavior to general features of dislocation motion. The viewpoint that plasticity[1] is inherently time-dependent has been put forth in the dislocation dynamics theory developed by Johnston and Gilman [1, 2].[2] In a sense, the present study provides the mathematical framework for extension of their theory to general stress states and loading paths. Methods employed closely parallel those of Bishop and Hill [3, 4] in studies of the time-independent idealization as based on the concept of crystalline slip within individual grains of a polycrystal, and also those of Kestin and Rice [5] in developing a thermodynamic formulation for inelastic behavior based on internal state variables. Our concern throughout will be with small strains and rotations, and with macroscopically homogeneous stress and strain fields.

**Time-Independent Plasticity.** To clarify what is intended by "essential structure"[3] in the opening sentence, consider the time-independent idealization, for which a definite elastic range (interior of current yield surface) is usually assumed to exist at each stage of the deformation history. The essential content of the theory is then expressed by the maximum plastic work inequality [6, 7]:

$$(\Sigma_{ij} - \Sigma^0_{ij})dE^p_{ij} \geqslant 0. \tag{1}$$

Here $d\mathbf{E}^p$ is a plastic strain increment under a stress $\mathbf{\Sigma}$ on the current yield surface, and $\mathbf{\Sigma}^0$ is any other stress state lying either within or on the surface. The inequality leads to normality of plastic strain increments to the yield surface at smooth points,

[1] The term "plasticity" is employed here in the physical sense, as inelastic behavior due (primarily) to slip in metals, and not in the more restricted sense as a classification of path-dependent behavior without time effects.

[2] Numbers in brackets designate References at end of paper.

Presented at the Sixth U. S. National Congress of Applied Mechanics, Harvard University, Cambridge, Mass., June 15–19, 1970.

Discussion of this paper should be addressed to the Editorial Department, ASME, United Engineering Center, 345 East 47th Street, New York, N. Y. 10017, and will be accepted until October 20, 1970. Discussion received after the closing date will be returned. Manuscript received by ASME Applied Mechanics Division, September 3, 1969; final revision, January 30, 1970.

[3] I borrow this apt phrase from Hill [4]; the spirit of the present inquiry closely parallels that of his work.

and to the requirement of convex surfaces. It is a guide to experiment, for complete incremental stress-strain relations may be written once the location of the yield surface is known in terms of the plastic distortion history. Further, it is the key element for validity of certain general principles, e.g., the limit theorems associated with the nonhardening idealization, uniqueness, and variational theorems, etc.

Different and seemingly more fundamental postulates have been introduced as a basis for the inequality. For example, Drucker [8, 7] considers a stressed body and makes a stability of material postulate that any additional set of stresses must do nonnegative work on the strains they produce. When applied to a cycle of additional stressing, beginning and ending at a stress state $\mathbf{\Sigma}^0$, the stability postulate together with the path independence of elastic strain leads to the inequality (1). In a similar way, Il'yushin [9] arrives at the inequality by postulating nonnegative work in a cycle of straining originating from an arbitrary deformed state. While such postulates may seem to be rather general thermodynamic principles, they are not. As Drucker [10] has noted, they are instead nothing more than reasonable classifications of behavior for metals, and fail, for example, when applied to materials for which inelastic behavior results from slip with frictional resistance of the Coulomb type. This connection with deformation mechanisms was implicitly recognized in the Bishop and Hill [3] derivation. They adopted the conventional continuum view of crystalline slip, with conditions for plastic straining of a particular slip system depending only on the resolved shear stress in the slip direction. This is equivalent to assuming an inequality in the form of (1) for each slip system, from which they were able to derive (1) in terms of macroscopic stress and strain. McClintock and Argon [11] have presented a parallel approximate argument for validity of the inequality through a dislocation, rather than continuum, model of crystalline slip.

**Synopsis of Present Study.** This study of the time-dependent range similarly attempts to relate stress-strain relations to deformation mechanisms. Rather than seeking postulates of an extrathermodynamic nature, such as stability of material, the emphasis is on elucidating the essential structure of macroscopic stress-strain relations as implied by plausible idealizations of slip processes at the microstructural level.

We start by deriving a kinematical relation between microstructural slip displacements and macroscopic plastic strain in the next section. The material is modeled as a generally inhomogeneous linear elastic system, such as a polycrystal, capable of internal rearrangement through slip. The relation appears to have escaped notice in earlier studies, and some errors are traced to use of a conventional formula which is really valid only for elastically homogeneous materials (e.g., single crystals). It is found instructive to separately consider a continuum model of slip within individual crystallites, and a discrete dislocation model. Both lead to the same time-dependent macroscopic structure for stress-strain relations, although a number of difficulties arise in starting a theory with the latter. Specifically, once it is accepted that the rate of shearing on a given slip system depends on local stresses only through the resolved shear in the slip direction, or that the velocity of a segment of dislocation line depends only on the local "force" in the Burgers vector direction, the following macroscopic structure emerges: For a given history of prior deformation, the current inelastic strain rate depends on current stress through being derivable from a scalar flow potential,

$$\dot{E}^p_{ij} = \frac{\partial \Omega(\mathbf{\Sigma}, \text{history})}{\partial \Sigma_{ij}}. \tag{2}$$

Constant $\Omega$ surfaces in stress space are shown to be convex. Residual stress variations on the microscale suggest a motion of the family of $\Omega$ surfaces with history in accord with Bauschinger and strain-recovery effects. Transient creep and rate sensitive behavior are discussed within the flow potential representation, with linear viscoelasticity and time-independent plasticity developed as limiting cases. Finally, we inquire as to what conditions might result in a potential function representation for long-time strain rates in materials exhibiting stationary creep at constant stress.

**Internal Variables in Irreversible Thermodynamics.** No true irreversible thermodynamics of general applicability during processes, as distinguished from end point equilibrium states, currently exists. Entropy and temperature, and thus the second law, are given operational meaning only for the latter. However, one might in some cases trace irreversible behavior to internal rearrangements of a material due to changes in certain "internal variables." If we take the point of view that these internal variables could, in concept though not in practice, be manipulated or constrained at will by imposition of appropriate forces, then they join our list of state variables in providing a full thermodynamic characterization of the constrained equilibrium states corresponding, say, to a given internal arrangement, applied stress, and temperature. This makes the equilibrium thermodynamics formalism directly applicable to processes, with the enlarged list of state variables, provided the following essential approximation can be accepted: That every achievable irreversible process may be considered as a sequence of constrained equilibrium states corresponding to the instantaneous values of the internal variables.

These ideas have been adopted by Kestin and Rice [5, section 8] in developing an elementary internal variable theory for solids which leads to the same flow potential representation as in equation (2), and which otherwise parallels the present study. They considered a set of discrete, scalar internal variables, and an equation of state which reduced to linear thermoelasticity when the internal variables were held fixed. The separation of strain into an inelastic part $\mathbf{E}^p$, depending only on internal variables, and an elastic part follows immediately. A Maxwell relation, following from the laws of thermodynamics as applied to constrained equilibrium states, is the key relation: The variation with respect to $\Sigma_{ij}$, at fixed temperature and internal variables, of the thermodynamic force associated with a particular variable = the variation of $E^p_{ij}$ with respect to that variable [5].

The reader may see that equation (6) of the next section, relating slip displacements to macroscopic plastic strain, or its variants (9) and (13) for the two slip models, contains the same information. They next assumed a local dependence, in that at a given internal state, the rate of change of a particular variable would depend only on the thermodynamic force associated with that variable. Together with the Maxwell relation, this leads to the flow potential representation of equation (2) with $\Omega$ viewed as a function of $\mathbf{\Sigma}$, temperature, and current internal variables (or history). Our subsequent equations (19) or (29) embody the same local dependence assumption, and lead directly to the flow potential.

We shall not pursue some other thermodynamic consequences of the internal variable theory in [5] and its analog here, since the present study aims only at a "mechanical" description with no reference to temperature changes. We shall, however, question the appropriateness of the sequence of constrained equilibrium states assumption when discussing the discrete dislocation model of slip.

## Relationship Between Microstructural Slip and Macroscopic Plastic Strain

**Preliminaries.** Consider a macroscopic sample of material occupying a region of volume $V$ and having the external bounding surface $A_{\text{ext}}$. The sample, in detailed composition, may be either a single crystal, polycrystal, or any inhomogeneous combination of different phases. It is assumed to be large enough for stress-strain behavior to be representative of other samples chosen from material with the same chemical composition, forming history,

heat-treatment, etc. We choose some arbitrary state of the sample, for which no tractions act on its external surface, as a reference state from which strains and displacements are to be measured. The material and achievable stress levels are assumed to be such that linear elasticity describes the deformation. But inelastic behavior is allowed, in that it is possible for relative internal slip displacements to occur. $A_{slip}$ will denote the collection of surfaces active in deformation from the reference state, and slip displacements are denoted by $\Delta \mathbf{u} = \mathbf{u}^+ - \mathbf{u}^-$, where we arbitrarily label one side of $A_{slip}$ by (+) and the other by (−).

The sample is subjected to a homogeneous macroscopic stress state $\boldsymbol{\Sigma}$. Precisely, we shall take this to mean that surface tractions on $A_{ext}$ are given by $\Sigma_{ij}n_j$ where $\mathbf{n}$ is the outward unit normal. The macroscopic strain tensor $\mathbf{E}$ associated with any set of surface displacements $\mathbf{u}$ is defined by requiring that $\Sigma^*_{ij}E_{ij}$ be the specific work in carrying tractions corresponding to any stress state $\boldsymbol{\Sigma}^*$ through those displacements:

$$(\Sigma^*_{ij}E_{ij})V = \int_{A_{ext}} (\Sigma^*_{ij}n_j)u_i dA. \tag{3}$$

Validity for any $\boldsymbol{\Sigma}^*$ and symmetry of the strain tensor then implies that

$$E_{ij} = \frac{1}{V}\int_{A_{ext}} \frac{1}{2}(n_i u_j + n_j u_i) dA. \tag{4}$$

It is noted that the subsequent development could equally well follow by assuming the sample is subjected to surface displacements derivable from a homogeneous strain field $\mathbf{E}$, with $\boldsymbol{\Sigma}$ defined in the work sense in terms of actual stresses on $A_{ext}$. With either boundary condition, it is easy to show that $\boldsymbol{\Sigma}$ and $\mathbf{E}$ are volume averages of the stresses and strains in the sample. In fact, for more general boundary conditions simulating homogeneous deformation, we may choose $\boldsymbol{\Sigma}$ and $\mathbf{E}$ as volume averages and the work increment is still $\Sigma_{ij}dE_{ij}$ within an approximation that Hill [4] suggests is increasingly more accurate with increasing ratios of sample size to a characteristic length such as grain size.

A fundamental role is played by the elastic stress field created in the material by isothermal application of a macroscopic stress $\boldsymbol{\Sigma}$. This field is denoted by $\bar{\boldsymbol{\sigma}}$. It is computed by assuming that no slip occurs during the stress application, and its definition does not rely on whether its actual imposition would cause further slip. Further, because of the superposition principle, $\bar{\boldsymbol{\sigma}}$ does not depend on the slipped state, but rather only on geometrical shapes and elastic moduli of constituent parts of the material. It is a homogeneous linear function of $\boldsymbol{\Sigma}$ at each point of the material. A macroscopic elastic strain $\mathbf{E}^e$ may be defined as the response to $\boldsymbol{\Sigma}$, computed as if no slip occurred. Again, by superposition, the linear relation between $\boldsymbol{\Sigma}$ and $\mathbf{E}^e$ is independent of the slipped state. The inelastic strain $\boldsymbol{\Sigma}^p$ is defined simply as the surplus: $\mathbf{E}^p = \mathbf{E} - \mathbf{E}^e$.

**The Relation.** With these preliminaries, we now turn to the relation between inelastic strain and slip displacements. But there is an important qualification! For a given set of current slip displacements $\Delta\mathbf{u}$, we compute the inelastic strain that would result if the material sample were in static equilibrium with relative displacements fixed at these current values on $A_{slip}$. We shall have to assume that the actual inelastic strains in a process differ negligibly from these values, which is the same as the approximation of a sequence of constrained equilibrium states in the internal variable formulation. In other words, a quasi-static description of slip is assumed to be adequate. Hence $\mathbf{E}^p$ is properly interpreted as a permanent strain in the sense that it would be the entire strain remaining upon unloading if no further slip occurred.

The reciprocal theorem in elasticity is the key: The work of load set (*) on displacement set (**) equals the work of load set (**) on displacement set (*), where (*) and (**) denote any two solutions of the elastic field equations. For our case, we must note that work on both $A_{ext}$ and $A_{slip}$ must be included. Hence

$$(\Sigma^*_{ij}E^{**}_{ij})V - \int_{A_{slip}} \sigma^*_{ij}n_j\Delta u_i^{**}dA = (\Sigma^{**}_{ij}E^*_{ij})V - \int_{A_{slip}} \sigma^{**}_{ij}n_j\Delta u_i^* dA \tag{5}$$

The minus sign appears because $\mathbf{n}$ is taken here as the unit normal on that side of $A_{slip}$ labeled (+), directed positive away from $A_{slip}$, since it is more natural to think in terms of tractions exerted by the material on the slip surface. Let us take (*) as the elastic field associated with an arbitrary stress $\boldsymbol{\Sigma}^*$, so that we write $\bar{\boldsymbol{\sigma}}^*$ for $\boldsymbol{\sigma}^*$ and note that $\Delta\mathbf{u}^*$ is zero. We choose (**) as the residual field left on unloading from the current state with $\Delta\mathbf{u}^{**}$ held fixed at the current values $\Delta\mathbf{u}$ on $A_{slip}$. Hence, $\mathbf{E}^{**}$ is $\mathbf{E}^p$ and $\boldsymbol{\Sigma}^{**}$ vanishes. The right side of equation (5) is zero, and we therefore have

$$\Sigma^*_{ij}E^p_{ij} = \frac{1}{V}\int_{A_{slip}} \bar{\sigma}^*_{ij}n_j\Delta u_i dA. \tag{6}$$

Since the relation holds for arbitrary stresses $\boldsymbol{\Sigma}^*$ and associated elastic fields $\bar{\boldsymbol{\sigma}}^*$, it gives the required relation between a set of slip displacements and the associated macroscopic plastic strain.[4] For example, the left side is $E^p_{11}$ if we take $\bar{\boldsymbol{\sigma}}^*$ to correspond to a macroscopic stress $\Sigma^*_{11}$ = unity, with all other components zero. Similarly, $E^p_{12}$ results when $\bar{\boldsymbol{\sigma}}^*$ corresponds to $\Sigma^*_{12} = \Sigma^*_{21}$ = a half, with other components zero. Hence, the elastic field may be interpreted as an "influence function" in converting slip displacements to inelastic strain. It is now convenient to examine two different levels at which crystalline slip may be brought into the development—continuum and dislocation.

**Continuum Slip.** Here the viewpoint is that slip within individual grains may be considered as deformation in the usual continuum sense, and that the inelastic strain at each point is resolvable into a set of simple shearing strains corresponding to a discrete, finite set of permissible slip planes and directions. We let $\mathbf{n}$ and $\mathbf{s}$ denote, respectively, the normal to a particular slip plane and a permissible slip direction, and let $\gamma$ be the permanent shearing strain ("engineering" definition) on that system. Then, considering a large number of closely spaced parallel slip surfaces, each accomplishing a small slip displacement so as to add up to $\gamma$, $\Delta u_i dA$ of equation (6) is replaced by $\gamma s_i dV$ in the limit and

$$\Sigma^*_{ij}E^p_{ij} = \frac{1}{V}\int_{V_{slip}} \bar{\sigma}^*_{ij}n_i s_j \gamma dV. \tag{7}$$

Here $V_{slip}$ must be viewed as including all slip systems operative at a point, so that a physical volume element is to be integrated over separately for each operative system. The shear stress on a particular system is

$$\tau = \sigma_{ij}n_i s_j, \tag{8}$$

so that

$$\Sigma^*_{ij}E^p_{ij} = \frac{1}{V}\int_{V_{slip}} \bar{\tau}^*\gamma dV, \tag{9}$$

where $\bar{\tau}^*$ denotes shear stress from the linear elastic stress field $\bar{\boldsymbol{\sigma}}^*$ associated with $\boldsymbol{\Sigma}^*$. Similarly, for the rate of strain

[4] The corresponding equation (9), to follow, for the continuum slip model may be inferred from Hill's work, upon proper combination of equations (8) and (9) of reference [4]. An earlier derivation was given by the author in the unpublished Brown University Technical Report ARPA SD-86/E31, "Some Mechanical and Thermodynamical Aspects of Plasticity," Apr. 1966. Also, Dr. Hill has brought the work of J. Mandel (*Mechanique des Mileaux Continus*, Gauthier-Villars, Paris, 1966, Vol. II, pp. 720–721) to my attention, through which the continuum slip version may also be inferred.

$$\Sigma^*_{ij}\dot{E}^p_{ij} = \frac{1}{V}\int_{V_{\text{slip}}} \bar{\tau}^*\dot{\gamma}dV. \tag{10}$$

**Dislocation Slip.** At the level of discrete dislocation lines, $\Delta\mathbf{u}$ in equation (6) is either an appropriate Burgers vector $\mathbf{b}$, or a sum of integer multiples of operative Burgers vectors if several dislocation lines of different types have crossed a given point on $A_{\text{slip}}$. Each particular slip system may conveniently be thought of as having two Burgers vectors, one the opposite of the other, to simply handle relative displacements in either direction. Thus

$$\Sigma^*_{ij}E^p_{ij} = \frac{1}{V}\int_{A_{\text{slip}}} \bar{\sigma}^*_{ij}n_jb_idA, \tag{11}$$

where now a given patch of area on $A_{\text{slip}}$ is counted once in the integration for every dislocation line that crosses it, and where the choice of $\mathbf{b}$ depends on the type of dislocation involved. In accord with conventional terminology, we may call

$$\bar{f} = \bar{\sigma}_{ij}n_jb_i \tag{12}$$

the force on the dislocation due to the linear elastic stress field, and

$$\Sigma^*_{ij}E^p_{ij} = \frac{1}{V}\int_{A_{\text{slip}}} \bar{f}^*dA. \tag{13}$$

Differentiating with respect to time, the rate of change of slipped area due to motion of a particular dislocation line segment $dL$ is $vdL$, where $v$ is the velocity of the line normal to itself and is positive if in a direction so as to increase the slipped area associated with that dislocation. Thus

$$\Sigma^*_{ij}\dot{E}^p_{ij} = \frac{1}{V}\int_{L_{\text{disl}}} \bar{f}^*vdL, \tag{14}$$

where $L_{\text{disl}}$ denotes the collection of all dislocation lines in the material sample.

It is clear that $\bar{\boldsymbol{\sigma}}$, $\bar{\tau}$, or $\bar{f}$ are not the total stress tensor at a point, shear stress on a slip system, or force on a dislocation. Other contributions come from the presence of a slipped state. The barred quantities are, however, the only contributions which vary with applied stress $\boldsymbol{\Sigma}$ when the slipped state is constrained at current relative displacement values. In particular, when $\boldsymbol{\Sigma}$ is zero the barred quantities vanish, so that the other contributions may be interpreted as a residual stress tensor, residual shear stress, or residual dislocation force corresponding to the current slipped state.

**Elastic Homogeneity.** A particularly simple form results for elastically homogeneous material samples (say, a single crystal, or a single-phase polycrystal with isotropic elastic constants in each grain). In that case, $\bar{\boldsymbol{\sigma}}$ is spatially uniform and equal to $\boldsymbol{\Sigma}$ itself. Thus equation (6) leads to

$$E^p_{ij} = \frac{1}{V}\int_{A_{\text{slip}}} \frac{1}{2}(n_i\Delta u_j + n_j\Delta u_i)dA \tag{15}$$

for the general relation, or to

$$E^p_{ij} = \frac{1}{V}\int_{V_{\text{slip}}} \frac{1}{2}(n_is_j + n_js_i)\gamma dV;$$

$$E^p_{ij} = \frac{1}{V}\int_{A_{\text{slip}}} \frac{1}{2}(n_ib_j + n_jb_i)dA \tag{16}$$

for the two idealizations of slip. The macroscopic permanent strain is therefore the volume average of permanent strains within the sample.

The restriction to elastically homogeneous materials is frequently overlooked. For example, the dislocation version is often "derived" on purely geometric grounds and applied indiscriminantly [2, 12], in spite of Cottrell's [13] caution that the usual geometric derivation is incorrect, and that the linear elasticity and homogeneity assumptions are essential. A similar error occurs in a recent paper [14] discussing the derivation of the maximum plastic work inequality from the conventional Schmid law characterization of crystalline slip. The author employs equation (16) for macroscopic plastic strain, and goes on to conclude that the inequality need not apply for an aggregate of elastically anisotropic crystals. The same conclusion would not have been reached had the proper equation (9) been used. While the correct expressions for $\mathbf{E}^p$ are clear, it must be admitted that the elastic field $\bar{\boldsymbol{\sigma}}$ would be too difficult to determine in practice for complicated, elastically inhomogeneous microstructures.

The relation between the two models of slip is given by equations (16). Evidently, the permanent shear on a particular slip system in a region of volume $V$ is

$$\gamma = \frac{bA_{\text{slip}}}{V}, \tag{17}$$

where $b$ is the magnitude of the Burgers vector and $A_{\text{slip}}$ the total area swept out (patches of area being counted positive or negative and once or more, as appropriate) by dislocations in that system. The permanent strain rate is

$$\dot{\gamma} = \frac{b\dot{A}_{\text{slip}}}{V} = b\frac{L_{\text{disl}}}{V}\langle v\rangle = b\rho\langle v\rangle, \tag{18}$$

where $L_{\text{disl}}$ is the total length of mobile dislocation lines in the system, $\rho = L_{\text{disl}}/V$ the mobile density, and $\langle v\rangle$ the line average mobile dislocation velocity.

## Potential Function for Inelastic Strain Rate

Our point of view is that slip, and thus macroscopic inelastic deformation, is inherently time-dependent. A *rate* of inelastic strain, rather than an instantaneous inelastic strain, is associated with every state of stress. The dependence on stress may be very abrupt and, at a given stress, the inelastic strain rate may in some cases rapidly diminish to zero with accumulating strain. In fact, we shall think of the time-independent idealization as resulting from pressing "abrupt" and "rapidly" to a singular limiting rate dependence, with zero inelastic strain rate for a range of stresses (interior of current yield surface) and an unbounded strain rate at the limits of this range.

We suppose that a certain definite slipped state has been achieved in a prior history of deformation, and focus attention on the following question: How does the current inelastic strain rate, $\dot{\mathbf{E}}^p$, depend on the current stress, $\boldsymbol{\Sigma}$?

**Potential Function.** We seek to answer the question, in general terms, by first considering the continuum level idealization of slip. The important assumption embodies the conventional picture of crystalline deformation: At a given slipped state, the rate of permanent shearing $\dot{\gamma}$ on a particular slip system, active at a point, depends on the stress at that point only through the shear stress $\tau$ acting on the system. Symbolically,

$$\dot{\gamma} = \dot{\gamma}(\tau, \text{current state}) = \dot{\gamma}(\tau) \tag{19}$$

where the dependence on "current state" is meant to indicate that latent hardening and allied cross effects are included. Equivalently, in terms of equation (18) and the Johnston-Gilman theory, the mean dislocation velocity $\langle v\rangle$ on a given system depends only on the shear stress for a given slipped state. We shorten the notation to $\dot{\gamma}(\tau)$ since only the dependence on stress is of present interest. Let $\dot{\mathbf{E}}^p(\boldsymbol{\Sigma})$ denote the dependence of inelastic strain rate on stress at the current slipped state, and let us use a stress increment $d\boldsymbol{\Sigma}$ for the arbitrary stress $\boldsymbol{\Sigma}^*$ in equation (10). Then

$$\dot{E}^p_{ij}(\boldsymbol{\Sigma})d\Sigma_{ij} = \frac{1}{V}\int_{V_{\text{slip}}} [\dot{\gamma}(\tau)d\bar{\tau}]dV$$

$$= \frac{1}{V}\int_{V_{\text{slip}}} [\dot{\gamma}(\tau)d\tau]dV. \quad (20)$$

We replace $d\bar{\tau}$ with $d\tau$ in the last step since a variation in macroscopic stress at a fixed slipped state changes only the contribution to $\tau$ from the linear elastic field.[5] But the last integral has an exact differential for its integrand, and hence $\dot{E}^p_{ij}(\boldsymbol{\Sigma})d\Sigma_{ij}$ is an exact differential. This means that there must exist a potential function of stress, say $\Omega(\boldsymbol{\Sigma})$, at each slipped state, such that components of the inelastic strain rate are given by its derivatives with respect to stress

$$\dot{E}^p_{ij}(\boldsymbol{\Sigma}) = \frac{\partial\Omega(\boldsymbol{\Sigma})}{\partial\Sigma_{ij}}. \quad (21)$$

As indicated by our earlier version of this equation in the Introduction, the detailed form of the flow potential will, of course, depend on the prior history of deformation. In general, the determination of appropriate functional forms must rely on macroscopic experiment. Nevertheless, approximate forms and general trends could be deduced from the models for polycrystals considered by many authors, notably Lin and Ito [15], Budiansky and Wu [16], and Hutchinson [17]. For such studies, the potential $\Omega$ can be expressed in terms of minipotentials $\omega$ for each slip system

$$\Omega(\boldsymbol{\Sigma}) = \frac{1}{V}\int_{V_{\text{slip}}} \omega(\tau)dV; \qquad \omega(\tau) = \int_0^\tau \dot{\gamma}(\tau)d\tau. \quad (22)$$

**Plastic Work Rate Inequality and Geometry of $\Omega$ Surfaces.** It appears plausible to assume that, at a given slipped state, $\dot{\gamma}$ is a nondecreasing function of $\tau$

$$[\dot{\gamma}(\tau^B) - \dot{\gamma}(\tau^A)][\tau^B - \tau^A] \geqslant 0. \quad (23)$$

A similar inequality must then hold for the tensorial macroscopic quantities. For if we consider two stress states $\boldsymbol{\Sigma}^B$ and $\boldsymbol{\Sigma}^A$, then by equation (10)

$$[\dot{E}^p_{ij}(\boldsymbol{\Sigma}^B) - \dot{E}^p_{ij}(\boldsymbol{\Sigma}^A)][\Sigma^B_{ij} - \Sigma^A_{ij}]$$

$$= \frac{1}{V}\int_{V_{\text{slip}}} [\dot{\gamma}(\tau^B) - \dot{\gamma}(\tau^A)][\bar{\tau}^B - \bar{\tau}^A]dV$$

$$= \frac{1}{V}\int_{V_{\text{slip}}} [\dot{\gamma}(\tau^B) - \dot{\gamma}(\tau^A)][\tau^B - \tau^A]dV, \quad (24)$$

where the bars are removed in the last step since the "$A$" and "$B$" stresses are considered at the same slipped state. Hence,

$$[\dot{E}^p_{ij}(\boldsymbol{\Sigma}^B) - \dot{E}^p_{ij}(\boldsymbol{\Sigma}^A)][\Sigma^B_{ij} - \Sigma^A_{ij}] \geqslant 0. \quad (25)$$

A related inequality follows from the potential function representation. First note that if we integrate $\dot{\mathbf{E}}^p(\boldsymbol{\Sigma}) - \dot{\mathbf{E}}^p(\boldsymbol{\Sigma}^A)$ along a straight line in stress space from $\boldsymbol{\Sigma}^A$ to $\boldsymbol{\Sigma}^B$ (keeping in mind that the slipped state is considered fixed during the integration), the integral is nonnegative

$$\int_A^B [\dot{E}^p_{ij}(\boldsymbol{\Sigma}) - \dot{E}^p_{ij}(\boldsymbol{\Sigma}^A)]d\Sigma_{ij} \geqslant 0 \quad (26)$$

This is because $d\boldsymbol{\Sigma}$ always has the direction of $\boldsymbol{\Sigma} - \boldsymbol{\Sigma}^A$, so that each infinitesimal increment is nonnegative by (25). But the integral is path independent since $\dot{E}^p_{ij}(\boldsymbol{\Sigma})d\Sigma_{ij} = d\Omega(\boldsymbol{\Sigma})$, and the inequality applies for all paths. Evaluating the integral, we have

$$\Omega(\boldsymbol{\Sigma}^B) - \Omega(\boldsymbol{\Sigma}^A) + (\Sigma^A_{ij} - \Sigma^B_{ij})\dot{E}^p_{ij}(\boldsymbol{\Sigma}^A) \geqslant 0. \quad (27)$$

Now, let us choose $\boldsymbol{\Sigma}^B$ to lie in the tangent plane to the surface $\Omega = \text{const}$ passing through the point $\boldsymbol{\Sigma}^A$ (i.e., in the "supporting plane" at $A$). Since $\dot{\mathbf{E}}^p(\boldsymbol{\Sigma}^A)$ is normal to this plane, the last term vanishes, and $\Omega(\boldsymbol{\Sigma}^B) \geqslant \Omega(\boldsymbol{\Sigma}^A)$. This implies that the surfaces of constant flow potential are convex, for the supporting planes at all points of such a surface never intersect the surface.

Finally, if we choose the stress state $\boldsymbol{\Sigma}^B$ to lie either on or within the surface $\Omega = \text{const}$ passing through $\boldsymbol{\Sigma}^A$, then $\Omega(\boldsymbol{\Sigma}^B) \leqslant \Omega(\boldsymbol{\Sigma}^A)$, and therefore

$$(\Sigma^A_{ij} - \Sigma^B_{ij})\dot{E}^p_{ij}(\boldsymbol{\Sigma}^A) \geqslant 0. \quad (28)$$

This is identical in form to the maximum plastic work inequality (1) of the time-independent theory and would, by itself, imply both normality and convexity.

## On Starting From Dislocation Model of Slip

The same essential macroscopic structure can follow from the more suitable model of slip at the discrete dislocation level. But, as we shall see, the simplest viewpoint is not without its difficulties and many problems remain to be resolved. For example, we may assume that the velocity of a given segment of dislocation line depends only on the local force active there (i.e., zero effective mass). This is the same as saying that the local shear stress (apart from the singular, but symmetric, self-stress term) in the direction of the Burgers vector governs, and is the conventional assumption in internal friction studies as summarized by Mason [18]. Thus

$$v = v(f, \text{current state}) = v(f) \quad (29)$$

Current state still enters since the functional form would depend on the orientation of the line segment (for example, different forms for edge versus screw), as well as on the presence of geometric barriers such as forest dislocations, stable networks, alloying atoms, grain and second-phase particle boundaries, etc.

Equation (30) is analogous to equation (19) of the continuum slip treatment, and the remaining steps are similar. Thus equation (14) leads to

$$\dot{E}^p_{ij}(\boldsymbol{\Sigma})d\Sigma_{ij} = \frac{1}{V}\int_{L_{\text{disl}}} [v(f)d\bar{f}]dL = \frac{1}{V}\int_{L_{\text{disl}}} [v(f)df]dL \quad (30)$$

Only the contribution to $f$ from the linear elastic field is variable, the dislocated state being considered as fixed, so that $d\bar{f} = df$. Again, we have an exact differential, and the same macroscopic potential function representation results with

$$\Omega(\boldsymbol{\Sigma}) = \frac{1}{V}\int_{L_{\text{disl}}} \omega(f)dL; \qquad \omega(f) = \int_0^f v(f)df. \quad (31)$$

Adding the assumption of a nondecreasing velocity-force relation, the convexity inequalities (25), (27), (28) also follow through steps identical to their derivation for continuum slip.

**Some Difficulties.** Perhaps the simple picture adopted previously does indeed reflect the salient physical features of dislocation motion. Certainly, it is not in discord with the macroscopic critical shear stress law for single crystals, or with the related continuum slip model treated earlier. However, the model is at least partly deficient for a number of reasons. First, our considerations have been quasi-static with motion considered as a

[5] Recall that for our purposes here, the slipped state is viewed as *fixed* at whatever distribution of slips has been attained in deformation up to the current time. Thus, when considering the stress variation within the material sample when the macroscopic stress is changed by $d\boldsymbol{\Sigma}$, with the slipped state viewed as fixed, we are assured by superposition that the variation in the residual stress field (due to slip) is zero so that only the linear elastic stress field varies. Hence the bar may be dropped. An alternate viewpoint: One may consider the stress increment $d\boldsymbol{\Sigma}$ to be applied instantaneously. The progression of slip at a finite rate permits no instantaneous variation in the residual stress field, the instantaneous variation occurring solely in the linear elastic field.

stable phenomenon (i.e., the sequence of constrained equilibrium states assumption). This may be more or less appropriate for single crystals, relatively free of slip obstacles, and deformed at moderate rates (for example, LiF as studied by Johnston and Gilman [1]). It would also seem appropriate for internal friction, say, as treated by the Granato-Lücke string model [18], for the damping of nominally elastic oscillations. Further, the picture does not preclude dislocation multiplication as a straightforward geometric consequence of the motion of a pinned segment, as in either a Frank-Read or screw cross-slip source. However, structurally useful metals are so by virtue of having many slip obstacles, and deform at stresses which are enormous compared to pure single-crystal strength levels. Presumably, dislocations on an uncluttered region of a slip plane would often move at velocities approaching elastic wave speeds. Rather than being stable and quasi-static, motion is perhaps better modeled as a series of unstable jumps from obstacle to obstacle.[6] Indeed, measurement of acoustic emission [19] has become a sensitive technique for detecting plastic deformation.

Thermally assisted breakaway from barriers, rather than the force-velocity relation per se for isolated dislocations, would thus appear to be the rate controlling mechanism [9]. This poses another, but less serious, difficulty. Conditions for initiation of an unstable jump might still be thought of as depending on the local force, but certainly the waiting time is a random function of force. Dislocation climb over barriers through diffusion cannot be rigorously included in the framework of slip developed here. However, the model of dislocations as singular lines in an elastic continuum is already rather macroscopic at this level, so that perhaps thermally activated processes involving climb need not be singled out for special consideration.

Evidently, a statistical treatment would be necessary in any serious attempt to overcome these difficulties. If for no other reason, statistics is required because one hardly wishes to admit that the "current slipped state" has changed with rather drastic localized changes in the dislocation structure having a negligible macroscopic consequence. For example, generation of a hundred dislocation loops of a 50 micron mean radius certainly changes conditions locally, but results in plastic strains on the order of only $10^{-10}$ in a cubic centimeter size specimen. Volume averaging is, of course, already included in our equations. Perhaps an as yet undeveloped theory would free the formulation of erratic details by appealing to ensemble averages over specimens with identical macroscopic histories. For example, if a potential function $\Omega$ could be defined as either a definite or random function of $\mathbf{\Sigma}$ for each member of the ensemble, the average inelastic strain rate would still be derivable from a definite potential function of stress which would be the average $\Omega$ for the ensemble.

These difficulties with the discrete dislocation model are not resolved here. One might instead attempt to think in terms of aggregates of dislocations and their densities and average velocities at a "point" in the material sample. Here point is taken to mean a region (if existent) large compared to the mean dislocation spacing but small in comparison to characteristic dimensions such as grain size. This is, however, nothing but the continuum slip viewpoint, complemented by the recognition that $\dot{\gamma} = b\rho\langle v\rangle$, equation (18). It will therefore lead again to the potential function representation if $\langle v\rangle$ depends only on $\tau$ at each given slipped state.

## Tendency Toward Bauschinger and Strain-Recovery Effects

Without more detailed considerations, very little can be predicted on the dependence of the $\Omega$ surfaces on the history of deformation. However, it can be shown that in a certain average sense, residual stresses tend to increase in a direction opposing that of slip, so that Bauschinger and strain-recovery effects are to be expected.

Recall that for a linear elastic system, the work product of any change in loads times the associated change in displacements is nonnegative (by positive definiteness of the matrix of elastic constants). We shall let $\delta$ before a quantity stand for any such change, and note as in deriving equation (5) that loads and displacements on slip surfaces must also be included. Thus

$$(\delta\Sigma_{ij}\delta E_{ij})V - \int_{A_{\text{slip}}} [\delta\sigma_{ij}n_j\delta(\Delta u_i)]dA \geqslant 0 \tag{32}$$

Let us write the stress at a point within the material as

$$\boldsymbol{\sigma} = \bar{\boldsymbol{\sigma}} + \boldsymbol{\sigma}^{\text{res}} \tag{33}$$

where $\bar{\boldsymbol{\sigma}}$ is the elastic field introduced earlier, so that $\boldsymbol{\sigma}^{\text{res}}$ is properly identified as a residual stress field, in that it would remain upon unloading with the current slipped state fixed. Letting the $\delta$'s denote changes from one residual field to another, we have $\delta\mathbf{\Sigma} = 0$ and $\delta\boldsymbol{\sigma} = \delta\boldsymbol{\sigma}^{\text{res}}$. Thus

$$\int_{A_{\text{slip}}} [\delta\sigma_{ij}^{\text{res}}n_j\delta(\Delta u_i)]dA \leqslant 0. \tag{34}$$

In terms of the continuum slip model[7]

$$\int_{V_{\text{slip}}} [\delta\tau^{\text{res}}\delta\gamma]dV \leqslant 0. \tag{35}$$

We conclude that the residual stresses (which must average to zero) tend to preferentially increase in a direction opposite to slip in those areas of the material most fertile in plastic deformation. This would suggest a biasing of the location of the family of constant $\Omega$ surfaces away from the origin in stress space, in the general "direction" of prior plastic deformation.

## Linear Viscoelasticity

We consider linear viscoelasticity here, as a limiting case of the general flow potential framework. For metals, this may be a suitable approximation in the anelastic damping of nominal elastic vibrations, or in the microcreep range. The theory results in a different viewpoint toward viscoelasticity, and is suggestive of appropriate behavioral idealizations in the nonlinear range.

Linearity may result at two levels. First, $\dot{\gamma}$ may vary linearly with $\tau$ for each slip system at any given slipped state, and second, full linearity in the superposition of stress histories sense may result. The first requires each minipotential $\omega$ to be quadratic in $\tau$, equation (22) and, since $\tau$ is linear in $\mathbf{\Sigma}$ at each slipped state (through $\bar{\tau}$), $\Omega$ is a quadratic function of $\mathbf{\Sigma}$. Thus

$$\dot{E}_{ij}^{p} = \frac{\partial\Omega}{\partial\Sigma_{ij}} = \beta_{ijkl}\Sigma_{kl} - A_{ij} \tag{36}$$

where $\boldsymbol{\beta}$ and $\mathbf{A}$ depend, in general, on the current slipped state and in addition to the trivial symmetry implied by stress and strain tensor symmetry, $\boldsymbol{\beta}$ has the symmetry of elastic constants: $\beta_{ijkl} = \beta_{klij}$. Introducing the notion of a "rest stress" $\mathbf{R}$ at each state, as defined by

$$R_{ij} = \beta_{ijkl}^{-1}A_{kl}, \tag{37}$$

we have

$$\dot{E}_{ij}^{p} = \beta_{ijkl}(\Sigma_{kl} - R_{kl}). \tag{38}$$

Thus, as illustrated in Fig. 1, the constant $\Omega$ surfaces are concentric ellipsoids in stress space with centers at the rest stress $\mathbf{R}$

[6] This seems to reflect the viewpoint put forth by Prof. F. A. McClintock in his unpublished lecture at the Colloquium on Physical Aspects of Plasticity, ASME Applied Mechanics Conference, Brown University, June 1968.

[7] It must be remembered that this model cannot include residual stress effects at the discrete dislocation level—Bauschinger effects occur also in homogeneously deformed single crystals with no continuum level residual shear stresses on their slip systems.

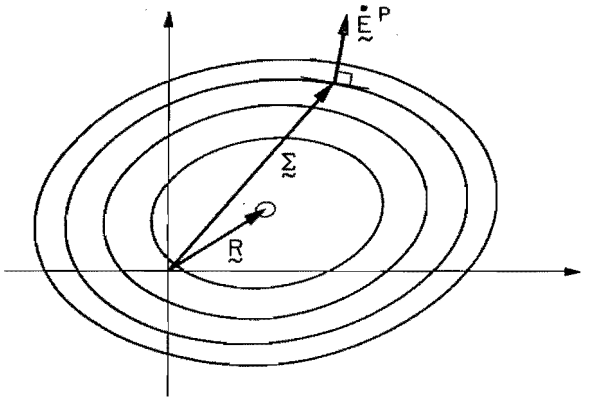


**Fig. 1 Linear viscoelasticity. Surfaces of constant flow potential $\Omega(\Sigma)$ are concentric ellipsoids in stress space**

$$\Omega = \frac{1}{2}\beta_{ijkl}(\Sigma_{ij} - R_{ij})(\Sigma_{kl} - R_{kl}) \tag{39}$$

For full linearity (i.e., in the superposition of history sense also), the $\beta$ matrix must be constant so that the orientation of the ellipsoids in stress space is fixed, and they translate as the rest stress $\mathbf{R}$ moves in stress space according to a linear functional of past stress or plastic-strain history. This description of viscoelasticity is quite reminiscent of kinematic hardening in the time-independent idealization.

To establish the connection with the creep compliance representation of stress-strain relations, let us consider a stress history starting from zero at $t = 0$ and write

$$E^p_{ij}(t) = \int_0^t C_{ijkl}(t - \tau)\dot{\Sigma}_{kl}(\tau)d\tau, \tag{40}$$

where $C_{ijkl}(t)$ is the $ij$ inelastic strain component at time $t$ due to a unit $kl$ stress component abruptly applied at $t = 0$. Note that $\mathbf{C}(0) = \mathbf{0}$, the instantaneous elastic response having been split off here into the elastic strain $\mathbf{E}^e$. After time differentiation and an integration by parts, the inelastic strain rate is

$$\dot{E}^p_{ij}(t) = \dot{C}_{ijkl}(0)\Sigma_{kl}(t) + \int_0^t \ddot{C}_{ijkl}(t - \tau)\Sigma_{kl}(\tau)d\tau. \tag{41}$$

Comparing with equations (36)–(39), we see that the $\beta$ matrix, which gives the orientation and shapes of the constant $\Omega$ ellipsoids, is just the initial slope of the creep compliance matrix, $\dot{\mathbf{C}}(0)$. Thus $\dot{C}_{ijkl}(0) = \dot{C}_{klij}(0)$. The rest stress is related to the creep compliance representation by

$$R_{ij} = -[\dot{C}_{ijkl}(0)]^{-1}\int_0^t \ddot{C}_{klmn}(t - \tau)\Sigma_{mn}(\tau)d\tau. \tag{42}$$

Long-time stationary creep under constant stress $\mathbf{\Sigma}$ is or is not possible according to whether the rest stress point in Fig. 1 does not or does ultimately drift to the current stress point. If stationary creep does result, the rate is

$$\dot{E}^p_{ij} = \dot{C}_{ijkl}(\infty)\Sigma_{kl}. \tag{43}$$

This raises an interesting question, which we shall examine in general terms later: Given that stationary creep does ultimately result in a particular material, is the rate derivable from a potential function of stress? No affirmative answer can be given on the basis of our considerations thus far. For the linear case, equation (43), the condition is that $\dot{C}_{ijkl}(\infty) = \dot{C}_{klij}(\infty)$, but such a symmetry has only been established for the initial slope of the creep compliance matrix. Of course, for isotropic linear behavior it is trivially so that a potential function representation results.

## Time-Independent Plasticity Idealization

Let us now consider a metal under conditions which might, in some appropriate limit, qualify for being "time-independent." We imagine that a certain slipped state has been achieved, and examine the current geometry of constant $\Omega$ surfaces in Fig. 2. Fig. 1 was prepared so that the change in $\Omega$ between any two of the neighboring constant $\Omega$ ellipsoids drawn there is constant. We shall adopt the same convention here, so that the plastic strain rate is in inverse proportion to the spacing between neighboring surfaces. Also, we shall use shading in regions of stress space where the spacing becomes too fine to be seen in our diagrams.

Fig. 2($a$) is a qualitative representation of the situation in the high homologous temperature creep range. A highly nonlinear dependence of strain rate on stress is implied by the rapidly decreasing spacing between neighboring $\Omega$ surfaces, as compared to Fig. 1. One might refer to the inner surfaces as representing transient creep and to the outer surfaces, clustered together in the shaded region, as representing rate sensitive yielding. However, this distinction is not essential. A somewhat lower temperature is envisioned in Fig. 2($b$). Strain rates are quite small at low stresses as compared to values near the shaded region. Indeed, if loading times are fast in comparison to characteristic times in the low stress region, but slow in comparison to characteristic times in the shaded region, we might refer to the thin shell in stress space enclosing the discrete lines to shading transition as a "yield surface."

This interpretation of the yield surface is adopted in Fig. 2($c$). Temperature is supposed to be low enough so that negligible strain rates accompany stresses within the surface, and enormous rates accompany those outside. The magnified circle shows the $\Omega$ surfaces, and we interpret the yield surface as a demarcation between regions of distant and fine spacing. Thus, within the approximation of considering the transition region infinitesimally thin, and with the rate of loading appropriately restricted, nor-

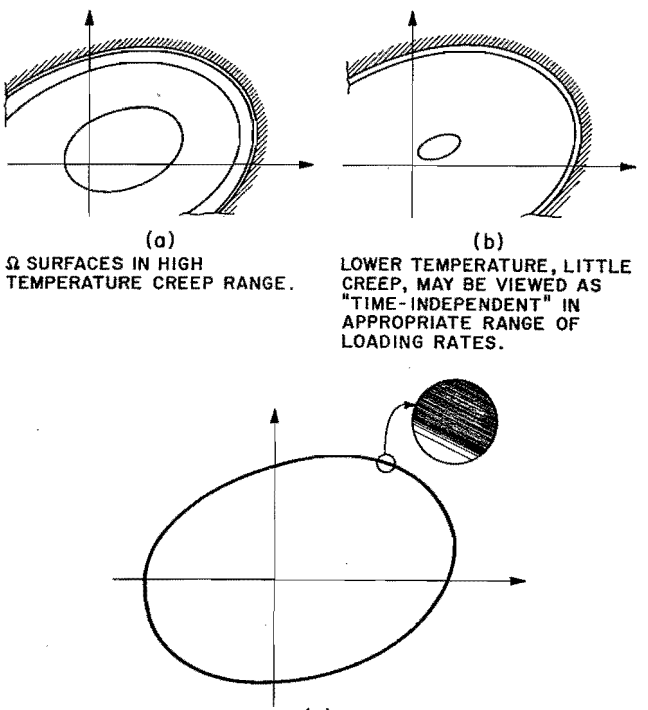


**Fig. 2 Time-independent plasticity as a limiting case of flow potential formulation for time-dependent behavior**

mality of plastic strain rates to the yield surface and convexity of the yield surface follow since both are true for constant flow potential surfaces.

## Some Forms for Flow Potential

Fig. 2 represents "snapshots" of the $\Omega$ surfaces at a given slipped state. An extended program of research, focusing on detailed models for slip, will be necessary to elucidate their precise functional form and variation with history. Requirements for stress analysis applications are less stringent, however, and some forms are examined here on the basis of macroscopic considerations only. In general, we may write

$$\Omega = \Omega(\boldsymbol{\Sigma}, T\mathbf{E}^p) \tag{44}$$

where $T\mathbf{E}^p$ stands for the *time* history of plastic deformation. As in the time-independent theory, useful forms in particular applications will result from characterizing history by simple parameters such as accumulated "equivalent" strain increments in isotropic hardening or yield surface center locations in kinematic hardening.

One approximation is to suppose that $\Omega$ depends on history only through the *path* history of $\mathbf{E}^p$, in the sense introduced by Pipkin and Rivlin [20] to distinguish from more general dependence on the *time* history or rate of path traversal. Thus

$$\Omega = \Omega(\boldsymbol{\Sigma}, P\mathbf{E}^p) \tag{45}$$

where $P\mathbf{E}^p$ denotes path history. This is identical to the "mechanical equation of state" approximation in monotonic simple tension (i.e., $\dot{E}^p_{11}$ depends only on $\Sigma_{11}$ and $E^p_{11}$). To see that it cannot be exact, consider two different stresses $\boldsymbol{\Sigma}^A$ and $\boldsymbol{\Sigma}^B$ which result in the same strain increment $d\mathbf{E}^p$ in respective time increments $dt^A$ and $dt^B$. If changes in the slipped state are to be identical in the two cases, as suggested by equation (45), it will be necessary that $\dot{\gamma}(\tau^A)dt^A$ and $\dot{\gamma}(\tau^B)dt^B$ are identically distributed among all slip systems, and therefore that the ratio of $\dot{\gamma}(\tau^A)$ to $\dot{\gamma}(\tau^B)$ be the same for each system. This cannot be exactly true if $\dot{\gamma}$ is nonlinear in $\tau$, or even when $\dot{\gamma}$ is linear if we recognize residual stress contributions to $\tau$ in addition to that part depending on $\boldsymbol{\Sigma}$.

An approximate representation of constant $\Omega$ surfaces for an initially isotropic material may be based on the Mises equivalent stress form (or second invariant)

$$J(\boldsymbol{\Sigma}) = \frac{1}{2}\Sigma'_{ij}\Sigma'_{ij} \qquad (\boldsymbol{\Sigma}' = \text{deviatoric part of } \boldsymbol{\Sigma}). \tag{46}$$

With increasing plastic deformation, Bauschinger or strain-recovery effects may be included by considering the $\Omega$ surfaces as concentric Mises ellipsoids centered at some deviatoric rest stress $\mathbf{R}$, dependent on the previous deformation. Thus $\Omega$ depends on $J(\boldsymbol{\Sigma} - \mathbf{R})$, and the value of $\Omega$ attached to a particular value of $J$ may be assumed to depend on the history of prior deformation, say, through an equivalent strain parameter of the form

$$\eta^p = \int(dE^p_{ij}dE^p_{ij}). \tag{47}$$

Ideas from kinematic and isotropic hardening in time-independent plasticity have been incorporated. Then, if we adopt the approximation of dependence on path history only, we have

$$\Omega = \Omega[J(\boldsymbol{\Sigma} - \mathbf{R}), \eta^p] \tag{48}$$

where $R$ depends only on the path of $E^p$. For example,

$$dR_{ij} = F(\eta^p, R_{kl}dE^p_{kl}/d\eta^p)dE^p_{ij} \tag{49}$$

would translate the $\Omega$ surfaces in the direction of the outward normal at the stress point causing the strain increment. Also, the second argument of $F$ allows a distinction between the kinematic translation for continued straining in the direction of $\mathbf{R}$ and in the reverse or some other direction. The associated strain rate is

$$\dot{E}^p_{ij} = \frac{\partial\Omega[J(\boldsymbol{\Sigma} - \mathbf{R}), \eta^p]}{\partial J}(\Sigma'_{ij} - R_{ij}). \tag{50}$$

It must, of course, be remembered that specific functional forms chosen in equations (48), (49) can at best fit observed behavior over a limited range of deformation paths.

**Rate Sensitivity.** We use "rate sensitivity" here, as distinguished from "transient creep," to denote time-dependent plasticity when the material, temperature, and time scale of the intended application are such that a definite elastic range of stress space may be assumed to exist at each slipped state. Thus $\Omega$ will have a constant value (which we take as $\Omega = 0$) for stresses within the yield surface bounding this range, and $\Omega > 0$ for stresses outside. The "overstress" theory of Malvern [21] for rate sensitivity, as well as its generalization by Perzyna [22] for nonhardening behavior under combined stresses, fall into this class. In fact, Perzyna assumed the existence of a flow potential, but did not provide a justification as in the present study. Employing the Mises form, equation (48), $\Omega$ would be taken as zero for all $J$ less than a critical value corresponding to the current yield surface, and perhaps as a power function [22] of the difference between $J$ and this critical value, for larger $J$.

**Time-Independent Limit.** Let us consider a rate-sensitive material in the limit of very slow loading rates, i.e., time-independent behavior. One does not wish to perform calculations for this limit on a real time basis for slow but finite rates, and a separate treatment is desirable. This might be regarded as redundant in view of prior studies which obtain stress-strain relations from a stipulation of the yield surface and its motion with history (the yield surface is simply $\Omega = 0^+$ in our case). However, it has been stated explicitly in the review article by Naghdi [23] that some additional postulate, beyond the normality requirement, is necessary to obtain incremental linearity (versus homogeneity of degree one) between $d\mathbf{E}^p$ and $d\boldsymbol{\Sigma}$ for time-independent behavior, and this point can be clarified here. It is assumed that the yield surface is free of corners (see the following). We then show that if the yield surface location is a functional of the path history of plastic strain only, then incremental linearity *must* result. Note that this does not preclude dependence of the hardening caused by a given strain increment on the stress under which it occurs, for by convexity the stress is itself uniquely determined (at least to within a portion doing zero work) by the direction of the strain increment.

The consistency requirement is the key. Let

$$N_{ij} = \frac{\partial\Omega}{\partial\Sigma_{ij}}\Big/\left(\frac{\partial\Omega}{\partial\Sigma_{ij}}\frac{\partial\Omega}{\partial\Sigma_{ij}}\right)^{1/2} \tag{51}$$

be the unit normal to an $\Omega$ surface, or in the limit as $\Omega \to 0^+$, to the yield surface (note that $\partial\Omega/\partial\Sigma_{ij} = 0$ on the yield surface, the strain rate being zero there). If $\boldsymbol{\Sigma}$ is on the yield surface at one time, and if a loading stress increment $d\boldsymbol{\Sigma}$ is (slowly) applied, then $\boldsymbol{\Sigma} + d\boldsymbol{\Sigma}$ is on the new yield surface. Thus $N_{ij}d\Sigma_{ij}$ is the displacement of the yield surface normal to itself, and since the yield surface location depends only on the path history of plastic strain, we must have

$$N_{ij}d\Sigma_{ij} = H(P\mathbf{E}^p, d\mathbf{E}^p) \tag{52}$$

where $H$ is a homogeneous function of degree one (not necessarily linear) in the plastic strain $d\mathbf{E}^p$ created in the stress increment. But in view of the normality requirement,

$$dE^p_{ij} = N_{ij}d\eta^p, \text{ where } d\eta^p = (dE^p_{ij}dE^p_{ij})^{1/2}, \tag{53}$$

and $H$ is *linear* in the equivalent strain increment $d\eta^p$:

$$N_{ij}d\Sigma_{ij} = H(P\mathbf{E}^p, \mathbf{N})d\eta^p. \tag{54}$$

Employing this and the prior equation, we now have the incrementally *linear* form

$$dE^p_{ij} = [H(P\mathbf{E}^p, \mathbf{N})]^{-1} N_{ij} N_{kl} d\Sigma_{kl} \tag{55}$$

which proves the point.

The quantity $H$ in the last equation can be formally calculated from the flow potential. Let us write

$$d\Omega(\mathbf{\Sigma}, P\mathbf{E}^p) = \frac{\partial \Omega(\mathbf{\Sigma}, P\mathbf{E}^p)}{\partial \Sigma_{ij}} d\Sigma_{ij} + H_\Omega(\mathbf{\Sigma}, P\mathbf{E}^p, d\mathbf{E}^p) \tag{56}$$

where $H_\Omega$ is homogeneous of degree one in $d\mathbf{E}^p$, and represents the variation of $\Omega$ when the path history is changed by $d\mathbf{E}^p$ with $\mathbf{\Sigma}$ held fixed. We cannot immediately obtain the consistency condition, as in the usual derivation, by setting $\Omega = 0^+$ and $d\Omega = 0$ for stress states following the yield surface, since $\partial\Omega/\partial\Sigma_{ij}$ also vanishes on the yield surface. The equation does, however, indicate that $H_\Omega$ vanishes on the yield surface, even if a finite value is substituted for $d\mathbf{E}^p$. Now, let us consider time-independent behavior as the limit of finite rate loading corresponding to an arbitrarily small fixed positive value of $\Omega$. Then $d\Omega = 0$ in equation (56), and we recover equation (55) with $N_{ij}$ being the limit of equation (51) as $\Omega \to 0^+$ and with

$$H(P\mathbf{E}^p, \mathbf{N}) = - \lim_{\Omega \to 0^+} \frac{H_\Omega(\mathbf{\Sigma}, P\mathbf{E}^p, \mathbf{N})}{\left(\dfrac{\partial \Omega}{\partial \Sigma_{ij}} \dfrac{\partial \Omega}{\partial \Sigma_{ij}}\right)^{1/2}} \tag{57}$$

**Corners.** Can a surface of constant flow potential contain a corner? Let us assume that, at a given slipped state, $\dot{\gamma}$ is a continuous function of $\tau$ for each slip system. Consistent with the assumed possibility of an elastic range in stress space, this may include $\dot{\gamma}$ being zero for a range of value of $\tau$ and nonzero outside this range. However, $\dot{\mathbf{E}}^p$ is simply a weighted average of $\dot{\gamma}$, equation (10) and, for a given slipped state, $\tau$ is a continuous function of $\mathbf{\Sigma}$ at each point of the material. Thus $\dot{\mathbf{E}}^p$ is a continuous function of $\mathbf{\Sigma}$. But since $\dot{\mathbf{E}}^p$ is normal to an $\Omega$ surface, we conclude that constant $\Omega$ surfaces have continuously varying normals, and hence cannot contain corners. There is one very important exception: if $\dot{\mathbf{E}}^p$ is zero at a particular point of stress space, continuity does not require that the $\Omega$ surface have a unique normal at that point. Thus, for rate-sensitive behavior, corners may occur on the yield surface $\Omega = 0^+$. They may not occur on $\Omega$ = constant $> 0$.

This raises an interesting possibility: The time-independent limit may not be unique if a corner is present, but time-dependent deformation at any arbitrarily slow but finite loading rate leads to unique behavior! Hill [4] has shown that the continuum slip model for any polycrystal or composite leads to corners (or vertices) on the yield surface. This is because the macroscopic yield surface is a polyhedron of an enormous number of faces, each face being the plane in stress space representing the critical shear stress of a single slip system of a single crystallite or element. Since faces corresponding to all active slip systems must share the current stress point during plastic deformation, their envelope results in a vertex on the yield surface at the loading point. This behavior cannot, of course, be included in approximate descriptions of yield surfaces which depend on a finite number of parameters such as an equivalent strain and set of rest stresses, as in the foregoing. The proper role of corners in a macroscopic theory remains unclear, particularly when it is recalled that polycrystalline analyses, which base yield on a small but finite plastic strain, lead to an appreciable smoothing of corners into rounded "noses" on subsequent yield surfaces [15].

**Transient Creep.** Here it is supposed that inelastic deformation is permissible at all stress levels; there is no elastic range. The flow potential form with dependence on path history then includes the constitutive equations proposed by Rabotnov [24], who again provided no justification for a potential representation. Particular functional forms may permit either long-time stationary creep under constant stress or limited straining at an ever-decreasing rate.

There appears to be no rigorous basis for separation of inelastic strains into creep and time-independent components, as is customary in structural analysis. However, if the strain-rate dependence on stress is highly nonlinear, as suggested by the fine spacing of the outer $\Omega$ surfaces in Fig. 2(*a*), an apparently instantaneous inelastic strain may result over a wide range of loading times, all long compared to characteristic times in the region of fine spacing. Thus a yield surface for instantaneous plastic strain, as an outer limit to the family of flow potential surfaces, may provide a useful idealization. But it is not to be expected that uncoupled constitutive equations will result for the creep and time-independent strain components.

## Does a Potential Function Exist for Stationary Creep?

Consider a metal at elevated temperature, and suppose that a constant rate of inelastic strain results as a unique long-time limit for every fixed stress state. We inquire here as to the conditions under which a potential function representation

$$\dot{E}^p_{ij} = \frac{\partial \Lambda(\mathbf{\Sigma})}{\partial \Sigma_{ij}} \tag{58}$$

is valid for the creep rate. The question is of considerable interest in structural analysis, since variational theorems and associated approximate methods may then be formulated for problems of stationary creep (in analogy to the potential or complementary energy theorems of elasticity [7]). Note that the existence of a flow potential $\Omega$ does not imply the existence of a stationary creep potential $\Lambda$. Different histories, and hence different families of constant $\Omega$ surfaces, are associated with each stationary state. Of course, for an isotropic material governed by the Odqvist form,

$$\dot{E}^p_{ij} = F[J(\mathbf{\Sigma})]\Sigma'_{ij}, \tag{59}$$

it is trivially so that a stationary creep potential exists. However, this is not the most general isotropic form.

To set sufficient conditions for existence of a stationary creep potential, let us adopt the continuum slip model and suppose that stationary states correspond to inelastic deformation with time-wise constant stresses within the material sample. That is, we suppose that each active slip system undergoes a stationary inelastic shearing rate, the collection of which forms a compatible strain-rate set. The assumption would be untenable at the discrete dislocation level, except in some average sense.

Let $\dot{\gamma}$ represent the shear rates on slip systems and $\dot{\mathbf{E}}^p$ the corresponding macroscopic strain rate for some stationary state. Then by the principle of virtual work,

$$(\Sigma^*_{ij}\dot{E}^p_{ij})V = \int_{V_{\text{slip}}} \tau^* \dot{\gamma} dV, \tag{60}$$

where $\mathbf{\Sigma}^*$ is an arbitrary macroscopic stress field and $\tau^*$ represents shear stresses on slip systems due to *any* stress field in equilibrium with $\mathbf{\Sigma}^*$. Note that our earlier equation (10), of similar form, employed the elastic stress field $\bar{\tau}^*$ associated with $\Sigma^*$. That is because the earlier equation applied for any set of shear rates, compatible or noncompatible, whereas the collection of shear rates corresponding to a stationary state are compatible by hypothesis, and this allows use of virtual work in the foregoing form.

Let $\dot{\mathbf{E}}^p(\mathbf{\Sigma})$ denote the stationary creep rate associated with a stress $\mathbf{\Sigma}$, and let $\tau$ denote corresponding shear stresses on slip systems. Then

$$\dot{E}^p_{ij}(\mathbf{\Sigma}) d\Sigma_{ij} = \frac{1}{V} \int_{V_{\text{slip}}} [\dot{\gamma} d\tau] dV. \tag{61}$$

Thus we see that a sufficient condition for the left-hand side to be an exact differential, leading to a stationary creep potential $\Lambda$, is that the stationary shearing rate $\dot{\gamma}$ on each slip system depend only on the shear stress $\tau$ on that system. This is indeed a rather special requirement, not likely to be true for single crystals since no distinction is made between stationary states of single slip and of multislip. Perhaps less stringent conditions could be set, or special circumstances could be found justifying this as an approximation. However, we must conclude that no firm basis, comparable to that for the flow potential $\Omega$, is available for a stationary creep potential.

## Acknowledgment

This study was supported by the Office of Naval Research, under Contract N00014-67-A-0191-0003 with Brown University. I am grateful for comments on the original manuscript by the reviewer and by Drs. R. Clifton, R. Hill, J. Kestin, and J. Willis.

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*Reprinted from the September 1970*
*Journal of Applied Mechanics*