International Journal of Fracture Mechanics, 4, 1968, pp. 41-49 (also published in International Symposium on Fracture Mechanics, Wolters-Noordhoff Publ., Groningen, 1968, 41-49). Paper based on presentation at the International Symposium on Fracture Mechanics, held in Kiruna, Sweden, during week of 7 August 1967.

THE ELASTIC–PLASTIC MECHANICS OF CRACK EXTENSION

James R. Rice*

ABSTRACT

This paper briefly reviews progress in the elastic plastic analysis of crack extension. Analytical results for plane strain and plane stress deformation fields are noted, and elastic–plastic fracture instability as well as transitional behavior and combined rate and thermal effects are discussed.

INTRODUCTION

Background material for this review of work concerning the elastic–plastic analysis of crack extension is supplied largely by a survey on plasticity aspects of fracture by McClintock and Irwin(1), by a survey on crack plasticity and fatigue by Rice(2), and by several recent reports(3-6) discussing tensile plasticity problems. Since these works either have or shortly will appear in the published literature, it appears most appropriate here to attempt a brief synopsis of progress to date, viewpoints achieved, and problems remaining to be solved, without specific reference to detailed mathematical methods and results.

PLANE STRAIN DEFORMATION NEAR CRACKS

Plane strain conditions prevail when plastic regions are small and transverse contraction prohibited by constraint of surrounding elastic material (as with a small plastic zone relative to the thickness dimension of a cracked plate). This deformation mode presents the least tough configuration for a cracked body, excepting the foil range. Analysis has proven possible under the assumption of elastic as well as plastic incompressibility, and while this assumption is appropriate in highly strained regions near the crack tip, certain features of the solutions suggest the necessity of choosing realistic Poisson ratios. The plane strain slip line theory is then applicable within yielded regions of a non–hardening material(3). Consequences are: (1) a large hydrostatic stress elevation directly ahead of the crack, with a maximum tension of 2.57 times the uniaxial yield stress for a Tresca material and 2.99 times for a Mises material, (2) a strain singularity varying inversely with distance from the crack tip in centered fan regions directly above and below the tip, but no large strain concentration directly ahead of the tip (unless finite geometry changes are considered as discussed below), and (3) a displacement discontinuity at the crack tip so that the end region opens into a blunted shape. The path independent energy line integral developed by Rice(3) leads to approximate estimates of the maximum zone dimension and opening displacement as well as lower bounds.

Blunting plays an important role in creating large strains directly ahead of the crack tip, contrary to our usual notions. Slip line fields drawn for a crack with tip rounded by plastic deformation lead again to fans above and below the tip, but now the fans are non–centered and focus into a region ahead of the tip comparable in size to the opening displacement, causing intense deformation ahead of the crack as well as above and below the tip. No detailed analysis of blunting has yet been carried out (although Wang(7) has considered a similar problem for the fully plastic case). The importance of such calculations is readily seen as crack opening displacements are on the order of the initial yield strain times a linear dimension of the plastic zone, meaning that the blunting affected region of large strains extends over size ranges of 50 to 150 microns even for

---

* Assistant Professor of Engineering, Brown University, Providence, U. S. A.

fairly brittle structural metals fracturing with plastic zones in the neighborhood of a quarter to a half inch in size. Not surprisingly, the same size scale is pertinent to the mechanisms such as ductile void growth discussed at this meeting by McClintock.

Near crack tip plane strain singularities may be found for incompressible materials hardening according to an equivalent shear stress proportional to equivalent strain raised to a hardening exponent N[4, 5]. Again, the largest deformations result above and below the crack tip, although some limited strain intensification results directly ahead. Blunting remains important as the highly strained region ahead of the tip is fully comparable to dimensions of the rounded crack tip. One noteworthy feature of the solution is the very rapid increase of stress triaxiality with hardening exponent. The ratio of maximum tensile stress to equivalent uniaxial flow stress is constant directly ahead of the crack, as along any radial line emanating from the tip. Values of the stress ratio directly ahead are, for a Tresca material, 2.57 when $N = 0$ (as anticipated from the perfectly plastic slip line theory), 3.36 when $N = 0.1$, 4.21 when $N = 0.2$, and 5.85 when $N = 0.3$. Values about 15°/o higher result for a Mises material. Thus in addition to stress elevation by hardening alone, a further elevation results solely by modifying stress ratios ahead of the crack. The assumption of elastic (and perhaps even plastic) incompressibility becomes suspect, however, at such large stress ratios and predicted values are very likely somewhat in excess of actual stresses.

Predicted plastic zone sizes and the general pattern of plastic deformation are in reasonable agreement with etching experiments on silicon iron reported by Rosenfield at this meeting, and also with the more nearly plane strain results obtained in as yet unpublished work by W. G. Clark at the Westinghouse Electric Corporation. Detailed comparison with experiment is presently impossible for two reasons. Only the dominant crack tip singular terms have been obtained, and other terms may result in as yet unknown complete solutions. Also, the neglect of compressibility will surely modify predicted results in the highly stressed region ahead of the crack where approximate estimates of plastic zone size from the singular terms alone are considerably smaller than the small dimensions observed.

Examination of perfectly plastic limit deformation fields for notched bodies in plane strain[1, 8] shows the lack of uniqueness inherent to near tip deformation fields in the large scale yielding range. Low stress level fractures may be correlated in terms of a single parameter since the elastic stress intensity factor controls small scale yielding solutions[1–3]; no similar single parameter will exist at loads near general yielding, at least within the perfectly plastic idealization. The hydrostatic stress elevation typical of small scale yielding persists to limit load for sufficiently deep symmetrical double edge notches; the single edge notch or internal notch, on the other hand, causes no hydrostatic elevation in limit load and the deformation field consists of localized sliding on planes at ± 45° with the notch line, rather than in the centered fans. With such significant differences in local stress and deformation fields, it is clear that fracture criteria based on integrated measures of local deformation such as crack opening displacements[9–10] can be at most succesful for limited classes of cracked geometries. Longitudinal shear calculations[11] show that strain hardening results formally in unique singular strain distributions as the crack tip is approached, but the region over which these singularities truly govern becomes unrealistically small at loads near general yielding on lightly hardening materials. Rice[6] has conjectured on the form of large scale elastic–plastic plane strain yielding at loads below limit values (and prior to the joining of plastic regions in the double edge notch case). It appears that the plastic zone elongates considerably in a direction at 45° with the crack line, with originally straight slip lines of the fan now curving and becoming extinct by contact with the elastic–plastic boundary, and with the slip line initially vertical to the crack line extending toward the outer extremity of the plastic zone so that a simple tension stress state results there. The radius of curvature of the initially vertical slip line at the crack tip would have to decrease toward zero as the limit load is approached in the single edge notch case for consistency with the limit field; this means that unloading must occur within the plastic region.

PLANE STRESS AND INTERMEDIATE TRANSVERSE CONSTRAINT

It is well known that toughness is optimized in sheet materials when the thickness is comparable to or smaller than plastic zone dimensions, so that an at least partial shear lip type fracture results from the through–the–thickness plane stress deformation mode. Two dimensional plane stress formulations present some degree of mathematical simplicity, but the problem is essentially three dimensional in nature. Hutchinson(5) has given the form of near tip two dimensional plane stress singularities through techniques identical to those of plane strain. The dependence on distance from the crack tip is the same for power law hardening materials, but the angular distribution of strains is very different and the most intense deformations result over a broad region ahead of the tip. Similar conclusions result from the numerical finite element computations by Swedlow et al.(12). The ratio of maximum stress ahead of the crack tip in plane strain to maximum stress in plane stress is approximately 2.5 for very light hardening(5), and this ratio decreases at first gradually with increasing values of the hardening exponent.

A very different mode of plane stress plastic deformation is envisioned in the model proposed by Dugdale(13). Here plastic flow is presumed confined to a narrow region ahead of the tip of height on the order of sheet thickness, in coincidence with through–the–thickness slip as elaborated by Hahn and Rosenfield(14). Analysis is carried out according to methods reviewed by Barenblatt(21), with the crack length imagined as extended into the plastic region where yield level stresses resist opening and the plastic zone size determined by a boundedness condition. Displacements of the extended crack surfaces are interpreted as averaged plastic strain times a dimension on the other of sheet thickness. Rice(13) has recently shown that solutions are obtainable for the crack tip opening displacement when the restraining stress is any function of separation distance that may include hardening and subsequent necking. For small scale yielding, Irwin's elastic energy release rate is equal to the area under the stress–separation distance curve out to a distance equal to the opening displacement. The Dugdale model gives no detail (other than an approximate estimate of opening displacement) in the region very near the tip; etching observations(14) suggest a truly three dimensional state of affairs and further analysis would be useful.

Experiments on different materials tend to support both patterns predicted by two–dimensional plane stress theory and by the Dugdale model. Gerberich(15) has seen patterns of the former type in experiments on cracked aluminum plates and Hahn and Rosenfield(14) have revealed Dugdale patterns in silicon iron. This suggests that actual results are strongly governed by the three dimensional features of the problem. Local necking will tend to shift observed results toward the Dugdale zone, and this may be helped both by light strain hardening and by conditions under which the plastic zone is large compared to the thickness dimension. Another feature of the plane stress problem which has received little attention is the influence of the actual shear lip shape assumed by the fracture. A combination of tensile and anti–plane stresses act relative to the inclined crack plane, and both modes of deformation are presumably present.

STABLE CRACK EXTENSION AND FINAL INSTABILITY

Plastic stress–strain relations are incremental in nature so that stress and strain are not uniquely related, but depend on the history of deformation. This has important implications for the problems of extending cracks, which are clearly a more proper focus of interest in fracture studies than stationary cracks. Analysis has been carried through in reasonably complete form only for the case of perfectly plastic longitudinal shear in work by McClintock[1, 16]. In that case the stationary crack solution results in strains ahead of the crack becoming infinite inversely with distance from the tip. Now if a fracture criterion (say, for convenience, a critical strain at some small characteristic distance ahead of the tip) is met, the crack will start to advance into material which has been less highly strained. But in contrast to elastic behavior, the material is now not free to adjust its strains to the values which would have been achieved by monotonic loading at the new crack length. Thus an additional load increment is required, at least initially, to preserve the meeting of the fracture criterion and a period of stable crack growth begins under steadily increasing load. An instability finally results, in the sense that the load required to maintain quasi–static extension ceases to increase with increasing crack length, and rapid extension ensues. The amount of stable growth and ratio of instability load to initiation load increase rapidly with increasing ductility, and depend also on the ratio of initial crack length to the characteristic structural length. Detailed computations and approximate formulae for instability have been given[1,16].

A seemingly alternate analysis of stable extension has been proposed by Krafft et al. [17]. Here it is assumed that the Irwin energy release rate required to extend a crack is a universal function of the change in length. Attempts at following such a resistance curve lead ultimately to the necessity for a load drop, and instability ensues. Rice[6] has recently shown that the McClintock theory is completely equivalent to the resistance curve approach at low stress levels, excepting that predictions of the shape of the resistance curve may be made in the former case.

Judging from results on aluminum alloys reported by Broek[22] and by as yet unpublished results on silicon iron and a medium strength steel by E. J. Wessel and W. G. Clark of Westinghouse Electric Corporation stable growth (prior to pop–in) becomes negligible as idealized plane strain conditions are increasingly approached. The latter investigators failed to detect growth in limiting cases of large thickness to plastic zone ratios through an acoustical technique sensitive to one mil, so it may tentatively be presumed that any extension is limited to the slight length increments resulting in the plastic blunting of the tip. Analysis of extending cracks in plane strain, based on a moving slip line field, predicts significant reductions of strain in the centered fans above and below the tip[6]. But this result is perhaps not inconsistent with observations, since the important factor leading to growth effects in longitudinal shear is the focusing of large plastic strain concentrations in toward the tip from the region directly ahead, and this does not occur in plane strain. The focusing does result under plane stress conditions where growth effects are most pronounced. Unfortunately, analysis has progressed slower for this case and the extending crack problem has not yet been studied. The gradual change of the fracture surface from the flat to shear lip geometry is another important feature of the problem which will complicate analysis, particularly when extension begins as a plane strain pop–in.

TRANSITIONAL FRACTURE BEHAVIOR

Current views of static fracture strength transitions with temperature recognize that two transitions are in fact present. Fracture initiation mechanisms in steels may vary with temperature from nearly pure cleavage to combined cleavage and ductile joining of microcracks to ductile mechanisms involving void growth, but at the same time a geometry dependent transition from plane strain type constraint with accompanying hydrostatic stress elevations to less constrained conditions such as plane stress is present. While the great differences in local stress elevations mean that a brittle–ductile transition will generally involve transitions of both types, it is well to note that misleadingly low transition temperatures will occur if the increasing toughness of fracture mechanisms necessitates plastic zone dimensions allowing loss of constraint. The point has been emphasized in presentations by Smith(23) and Tetelman(24). Rate sensitivity of the flow stress enters strongly into the possibility of brittle behavior for running cracks. Brittle mechanisms are favored by local stress elevations and in some cases by achievement of plane strain conditions in configurations where static yield stress values would result in plane stress deformation. Here the cracking of locally embrittled regions can serve as the critical event, as in the studies by Mylonas(18) on embrittlement by previous mechanical and thermal treatment.

In addition to any rate dependence of the flow stress, local heat generation by plastic deformation enters for very rapid loadings on stationary cracks and particularly for running cracks. Consideration of the latter case is important in determining toughness levels at which fracture arrest will be possible. Eftis and Krafft(19) indicate the importance of rapid quenching by surrounding elastic material in modifying the normal adiabatic stress–strain relations of fast rate tension or compression tests, and have achieved some success in correlating fracture toughness in terms of the strain hardening exponent anticipated for conditions at the tip. Preliminary studies of local temperature fields have been conducted by N. Levy at Brown University. Examination of Dugdale, anti–plane strain, and plane strain perfect plasticity models shows that temperatures predicted at the tip are not greatly sensitive to the model employed. In fact, crack tip temperatures are finite and the maximum temperature occurs away from the tip (except in the infinite rate adiabatic limit) due to quenching by adjacent elastic material. This suggests that some degree of uncertainty in the exact pattern of plastic deformation is acceptable. Approximate estimates of tip temperatures were made for mild steel, 2024 Aluminum, and 6 Al–V Titanium alloys, by employing room temperature yield stresses and thermal properties and by taking the maximum plastic zone dimension as 1 cm. Titanium is by far the more heat sensitive. Loading times slower than 10^{-3} sec. resulted in temperature elevations well below 25°C for steel and aluminum, but of approximately 300°C for titanium. Running crack results, neglecting inertia terms, suggest 60 to 125°C temperature rises for steel and aluminum for a speed range from approximately 50 to 500 meters per sec., whereas a 1000°C temperature rise is predicted in titanium at about 50 meters per sec. With high local temperatures viewed as increasing ductility as if the specimen were uniformly at the crack tip temperature, the increasing toughness with loading rate in titanium is readily understood, but it is unclear as to whether thermal considerations could explain limiting speeds and minimum toughness levels in mild steel. Further refinements of calculations accounting for rate and temperature dependence of yield stress and thermal properties are required.

ACKNOWLEDGEMENT

Financial support of the Advanced Research Projects Agency under contract SD–86 with Brown University is gratefully acknowledged. This paper is a written version of a lecture at the International Symposium on Fracture Mechanics at Kiruna, Sweden, August 1967.

Received September 8, 1967.

REFERENCES

1. F. A. McClintock and G. R. Irwin, 'Plasticity Aspects of Fracture Mechanics', In *Symp. on Fracture Toughness Testing and Applications,* ASTM–STP–381, 1965.

2. J. R. Rice, 'The Mechanics of Crack Tip Deformation and Extension by Fatigue', in *Symp. on Fatigue Crack Growth*, ASTM–STP–415, 1967.

3. J. R. Rice, 'A Path Independent Integral and the Approximate Analysis of Strain Concentration by Notches and Cracks', Brown Univ. ARPA SD–86 Report E39, May 1967.

4. J. R. Rice and G. F. Rosengren, 'Plane Strain Deformation Near a Crack Tip in a Power Law Hardening Material', July 1967, to appear in J. Mech. Phys. Sol.

5. J. W. Hutchinson, 'Singular Behavior at the End of a Tensile Crack in a Hardening Material', July 1967, to appear in J. Mech. Phys. Sol.

6. J. R. Rice, 'Mathematical Analysis in the Mechanics of Fracture', Aug. 1967, to appear as chapter of *Treatise on Fracture*, (ed. H. Liebowitz), Academic Press.

7. A.J. Wang, 'Plastic Flow in a Deeply Notched Bar With a Semi–circular Root', Quart. Appl. Math., vol. XI, 1953.

8. D. C. Drucker, 'A Continuum Approach to the Fracture of Metals', in *Fracture of Solids* (eds. Drucker and Gilman), Wiley, 1963.

9. B. A. Bilby, A. H. Cottrell, and K. H. Swinden, 'The Spread of Plastic Yield From a Notch', Proc. Roy. Soc. A., vol. 272, 1963.

10. A. A. Wells, 'Application of Fracture Mechanics at and Beyond General Yielding', Brit. Weld. Jour., Nov. 1963.

11. J. R. Rice, 'Stresses Due to a Sharp Notch in a Work Hardening Elastic–Plastic Material Loaded by Longitudinal Shear', Jour. Appl. Mech., vol. 34, 1967.

12. J. L. Swedlow, W. H. Yang, and M.L. Williams, 'Elasto–Plastic Stresses and Strains in Cracked Plates', in *Proc. Int'l. Conf. on Fracture* (1965), vol. 1, Sendai, 1966.

13. D. S. Dugdale, 'The Yielding of Steel Sheets Containing Slits', Jour. Mech. Phys. Sol., vol. 8, 1960.

14. A. R. Rosenfield, P. K. Dai, and G. T. Hahn, 'Crack Extension and Propagation Under Plane Stress', in *Proc. Int'l. Conf. on Fracture* (1965), vol. 1, Sendai, 1966.

15. W. W. Gerberich, 'Plastic Strains and Energy Density in Cracked Plates: 1. Experimental Techniques and Results', Exper. Mech., vol. 4, 1964.

16. F. A. McClintock, 'Ductile Fracture Instability in Shear', Jour. Appl. Mech., vol. 25, 1958.

17. J. M. Krafft, A. M. Sullivan, and R. W. Boyle, 'Effect of Dimensions on Fast Fracture Instability of Notched Sheets', in *Proc. Crack Propagation Symp.*, Cranfield College of Aeronautics, Sept. 1961.

18. C. Mylonas, 'The Mechanics of Brittle Fracture', in *Proc. Eleventh Int'l. Congr. Appl. Mech.*, Munich, 1964.

19. J. Eftis and J. M. Krafft, 'A Comparison of the Initiation With the Rapid Propagation of a Crack in a Mild Steel Plate', J. Basic Engr., vol. 87, 1965.

20. A. R. Rosenfield and G. T. Hahn, Intern. Jour. Fract. Mech., Vol. 4, 1968.

21. G. I. Barenblatt, Intern. Jour. Fract. Mech., Vol. 4, 1968.

22. D. Broek, Intern. Jour. Fract. Mech., Vol. 4, 1968.

23. E. Smith, Intern. Jour. Fract. Mech., Vol. 4, 1968.

24. A. S. Tetelman, Intern. Jour. Fract. Mech., Vol. 4, 1968.

RÉSUMÉ – Le mémoire décrit brièvement les progrès réalisés dans l'analyse élasto–plastique de l'extension d'une fissure.

Les résultats analytiques pour les champs de déformation en état plan de tension et en état plan de déformation sont discutés. On étudie également le comportement de transition ainsi que les effets combinés de la vitesse et de la température.

ZUSAMMENFASSUNG – In dieser Abhandlung wird kurz der Erfolg von elastisch–plastischen Analysen bei Rissausdehnung. Es wurden analytische Ergebnisse für planierte Anspannungsverformungsfelder und planierte Druckverformungsfelder festgestellt. Ausserdem wurden elastisch–plastische Frakturunbeständigkeit, Übergangsverhalten und Mass– und Thermaleffekte diskutiert.