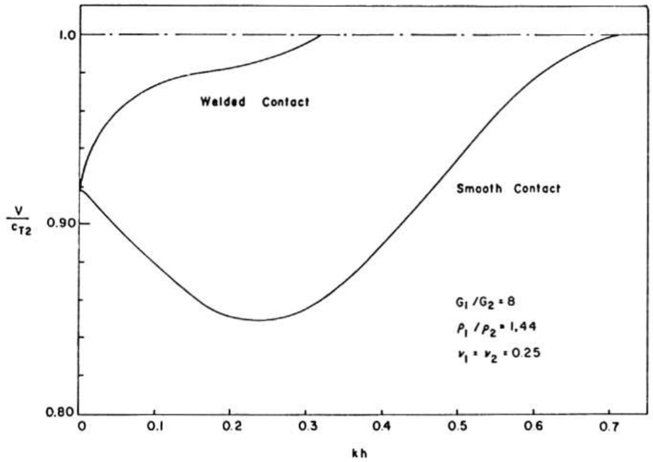

Fig. 2 Real roots of $\Delta(k, V) = 0$

$$\int_{-\infty}^{\infty} (x - a)^{-2} dx, \tag{10}$$

which do not exist, not even in the sense of principal values. For $k > 0$ the equation $\Delta(k, V) = 0$ also appears as the dispersion equation in the study of free waves in a layered half-space. For a relatively stiff and heavy layer the writer has computed the real roots of $\Delta(k, V) = 0$ for both welded and smooth contact. It is seen from Fig. 2 that for smooth contact a double zero appears for $V/c_{T2} \sim 0.85$ at $kh = 0.24$. At that critical velocity we have integrals of the type (10), which blow up, and we encounter a resonance effect. This type of resonance effect does not occur in a two-layered fluid half-space. For bonded contact a similar resonance effect appears at the velocity of Rayleigh waves *of the supporting half-space*. Since the motion *in the layer* is now not exponentially decaying, we expect the influence of layering to be of importance for load velocities close to c_{R2}. It is seen from this example that for a relatively stiff layer the responses of a two-layered fluid and an elastic half-space can be very different.

Author's Closure

The modal structure of a two-layered, elastic half-space was studied originally by Bromwich[4] and has since been studied extensively by others, notably Love.[5] Extensive discussion and references are given by Ewing, Jardetzky, and Press.[6] The mode studied by Achenbach is a particular case of a Love wave (it should be remarked, however, that the particular model considered by Achenbach, in which the upper layer is both denser and stiffer than the substratum, is rather unrealistic), and the minimum in the dispersion curve (wave speed versus wave number) is typical; higher modes also exist, and their dispersion curves exhibit multiple maxima and minima.[7] The existence of these extrema implies the existence of higher-order points of stationary phase in the integral representations of the solutions to properly posed initial-value problems, but these can be handled by standard analytical devices.

The virtue, as well as the deficiency, of the liquid model, *vis-a-vis* its elastic counterpart, is that it does not exhibit a complicated modal structure and therefore permits the explicit solution of certain boundary-value problems. The solutions for such steady-state problems as that considered by the author, namely, a surface load moving at uniform speed, provide asymptotic approximations to the corresponding solutions for elastic media. These approximations cannot be expected to be uniformly valid for moving-load speeds in the neighborhoods of any of the Love waves. It seems likely that the most important of these waves is the lowest mode, which tends to the Rayleigh wave for the upper layer as the thickness of that layer becomes large compared with the wavelength; nevertheless, the author was guilty of oversimplification in referring only to this mode. The remaining modes would certainly have to be considered in the solution of the moving-load problem for a two-layered elastic half-space; indeed, it appears likely that the existence of these modes precludes an explicit solution of this problem.

[4] T. J. I'A. Bromwich, "On the Influence of Gravity on Elastic Waves, and, in Particular, on the Vibrations of an Elastic Globe," *Proceedings of the London Mathematical Society*, vol. 30, 1898, pp. 98–128.

[5] A. E. H. Love, *Some Problems of Geodynamics*, Cambridge University Press, London, England, 1911.

[6] W. Ewing, W. Jardetzky, and F. Press, *Elastic Waves in Layered Media*, McGraw-Hill Book Company, Inc., New York, N. Y., 1957, p. 189 ff.

[7] Ibid., p. 195.

Discussion: "Stresses in an Infinite Strip Containing a Semi-Infinite Crack" (Knauss, W. G., 1966, ASME J. Appl. Mech., 33, pp. 356–362)
J. C. Rice (misprint of J. R. Rice), J. Appl. Mech. **34**, 248 (1967)

Name misprinted! -- should be J. R. Rice

Stresses in an Infinite Strip Containing a Semi-Infinite Crack[1]

J. C. RICE.[2] While Professor Knauss' study provides a useful evaluation of a practical fracture testing configuration, no elaborate computations are required for determining the stress intensity factor and thus the singular crack tip stress state. In fact, from an independent estimate of the stress intensity factor, the results given in the paper are suggested to be in error.

Irwin [1][3] has defined a stress intensity factor K_I such that the stress directly ahead of the crack tip has the form (Fig. 1)

$$\sigma_y(x, 0) = K_I(2\pi x)^{-1/2} + \text{nonsingular terms.} \tag{1}$$

Then, judging from Professor Knauss' equation (33), his stress intensity factor K is related to Irwin's by

$$K = \frac{(1 - \nu^2) b K_I}{(2\pi)^{1/2} E v_0} \tag{2}$$

where ν is Poisson ratio, E is Young's modulus, b is half strip width, v_0 is vertical displacement of clamped strip boundary. Now defining $-\partial V/\partial c$ as the potential energy per unit thickness drained out of a body by a unit crack length extension, Irwin [1][3] has shown that

$$-\frac{\partial V}{\partial c} = \frac{K_I^2}{E} \tag{3}$$

for plane stress, a factor of $(1 - \nu^2)$ appearing for plane strain. Irwin set the potential energy release equal to the work done in removing tractions from the new crack surface. The procedure is clearly valid for configurations as in Fig. 1, where boundary displacements are specified so that boundary forces do no work;

[1] By W. G. Knauss, published in the June, 1966, issue of the Journal of Applied Mechanics, vol. 33, Trans. ASME, vol. 88, Series E, pp. 356–362.

[2] Assistant Professor of Engineering, Division of Engineering, Brown University, Providence, R. I. Assoc. Mem. ASME.

[3] Numbers in brackets indicate References at end of Discussion.

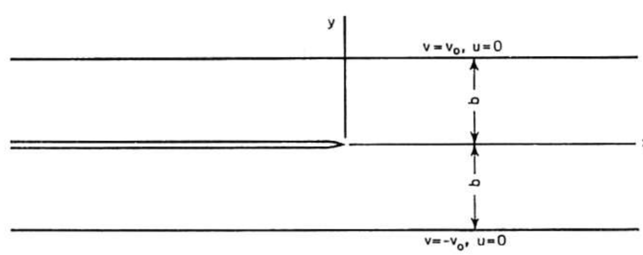

Fig. 1

the studies of Bueckner [2] and Sanders [3] have demonstrated the general validity of equation (3). Swedlow [4] has recently presented results on energy releases in biaxially stressed systems which tend to contradict equation (3), however, an error in his computations may readily be inferred as he obtains an expression for $\partial V/\partial c$ which is not negative definite. Rice and Drucker [5] proved directly from the theorem of minimum potential energy that $\partial V/\partial c \leqslant 0$ [note simply that the displacement field before crack extension is kinematically admissible after crack extension so that $V(c + dc) \leqslant V(c)$]. See [6] for an alternate proof.

As Professor Knauss noted, the energy release rate is determined immediately for the configuration of Fig. 1 as

$$-\frac{\partial V}{\partial c} = 2bW_0 \tag{4}$$

where W_0 is the elastic energy density of the region far to the right of the crack tip. For plane stress conditions and a linearly elastic material,

$$W_0 = \frac{Ev_0^2}{2(1 - \nu^2)b^2}. \tag{5}$$

Thus Irwin's stress intensity factor is, from equations (3–5),

$$K_I = \frac{Ev_0}{[(1 - \nu^2)b]^{1/2}},$$

or, according to Professor Knauss' definition, equation (2),

$$K = \left[\frac{(1 - \nu^2)b}{2\pi}\right]^{1/2} \approx 0.345b^{1/2} \quad \text{when} \quad \nu = \frac{1}{2}. \tag{6}$$

Professor Knauss reports a value of 0.707 $b^{1/2}$, high by a factor of slightly over 2.

The same procedure may be employed for other boundary conditions. For example, if the clamped boundary condition ($u = 0$) is replaced by a condition of zero shear stress ($\tau_{xy} = 0$),

$$K_I = \frac{Ev_0}{b^{1/2}}. \tag{7}$$

Also, if the horizontal boundaries are stress free and loads are applied by bending moments M per unit sheet thickness acting on the cracked arms remotely far away along the negative x-axis,

$$K_I = \frac{(12)^{1/2}M}{b^{3/2}}. \tag{8}$$

The procedure given here of calculating stress intensity factors for the configuration of Fig. 1 may be directly verified in the simpler case of antiplane strain, a case for which detailed stress distributions are more readily obtained. Irwin [1] defined a stress intensity factor K_{III} for this case such that the stress ahead of the crack has the form

$$\tau_{yz}(x, 0) = K_{\mathrm{III}}(2\pi x)^{-1/2} + \text{nonsingular terms}, \tag{9}$$

and has shown that the energy release rate is then

$$-\frac{\partial V}{\partial c} = \frac{K_{\mathrm{III}}^2}{2G} \tag{10}$$

(G is shear modulus). Now, suppose Fig. 1 is a cross section of an infinitely long prismatic body and the antiplane displacement w_0 is imposed. Then the strain energy density as $x \to +\infty$ is reading computed and equations (4, 10) lead to

$$K_{\mathrm{III}} = Gw_0\left(\frac{2}{b}\right)^{1/2}. \tag{11}$$

Now it is easy enough to show that the shear stress distribution

$$\tau_{yz} + i\tau_{xz} = \frac{Gw_0}{b}\exp\left[\frac{\pi}{4b}(x + iy)\right] \times \left\{2\sinh\left[\frac{\pi}{2b}(x + iy)\right]\right\}^{-1/2} \tag{12}$$

solves the stated problem. Computing the shear stress directly ahead of the crack and comparing with equation (9), equation (11) is independently verified.

References

1 G. R. Irwin, "Fracture Mechanics," in *Structural Mechanics* (Proceedings, 1st Naval Symposium), Pergamon Press, New York, N.Y., 1960.

2 H. F. Bueckner, "The Propagation of Cracks and Energy of Elastic Deformation," Trans. ASME, vol. 80, 1958, pp. 1225–1230.

3 J. L. Sanders, "On the Griffith-Irwin Fracture Theory," Journal of Applied Mechanics, vol. 27, Trans. ASME, vol. 82, Series E, 1960, pp. 352–353.

4 J. L. Swedlow, "On Griffith's Theory of Fracture," GALCIT Report SM 63-8, California Institute of Technology, March, 1963.

5 J. R. Rice and D. C. Drucker, "Energy Changes in Stressed Bodies Due to Void and Crack Growth," Brown University Report ARPA SD-86/E24, August, 1965.

6 J. R. Rice, "An Examination of the Fracture Mechanics Energy Balance," to appear in Proceedings of the International Conference on Fracture, Sendai, Japan, 1965.

Author's Closure

Professor Rice's comment with respect to the numerical value of the stress concentration factor is correct. Indeed, one of the reviewers had suggested "an independent estimate of the stress intensity factor" without indicating the method to be used in this calculation. Interpreting the suggested check as an estimate of the error incurred in truncating the infinite product which enters the stress intensity factor, footnote 5 was added in the paper. The writer is thus indebted to Professor Rice for clarifying the order and nature of the discrepancy.

On the basis of apparently successful evaluation of similar infinite products for calculating stress intensity factors[4] it was felt that the numerical evaluation of the infinite products was sufficiently accurate. Professor Rice's calculation implies that this assumption was fallacious inasmuch as the inherent error in the roots when coupled with the slow convergence of the products could lead to a sizable error.

It turns out that the infinite product need not be evaluated in order to calculate the stress intensity factor. In fact, it follows from equation (*20*)[5] and the calculations in the Appendix of the paper that the function

$$F(\omega) = (1 - \nu^2)\,\frac{\omega}{2}\,\frac{(3 - \nu)\sinh^2\omega + (1 + \nu)\omega^2 + 4/(1 + \nu)}{(3 - \nu)\sinh\omega\cosh\omega - (1 + \nu)\omega} \tag{1}$$

can be factored into

$$\left.\begin{aligned} F_-(\omega) &= \left\{\frac{1 - \nu^2}{2}\right\}^{1/2}(i\omega)^{1/2} + iO\left(\frac{1}{\omega}\right) \\ F_+(\omega) &= \left\{\frac{1 - \nu^2}{2}\right\}^{1/2}(-i\omega)^{1/2} + iO\left(\frac{1}{\omega}\right) \end{aligned}\right\}\begin{aligned} &|\omega| \cdot \to \infty \\ &|\mathrm{Arg}\,\omega| < \frac{\pi}{2} \end{aligned} \tag{2}$$

Substitution into (*26*) gives for the stress σ_y on the crackline and ahead of the crack as $x \to 0$

$$\sigma_y(x, 0) = \frac{iEv_0}{1 - \nu^2}\,\frac{1}{\pi}\int_0^\infty \left\{\frac{1 - \nu^2}{2}\right\}^{1/2}\left\{\frac{1}{i\omega}\right\}^{1/2} \exp(-i\omega x)d\omega + O(1) \tag{3}$$

or

[4] R. A. Westmann, "Pressurized Star Crack," *J. Mech. and Phys.*, vol. 43, 1964, pp. 191–198.

[5] Italicized equation numbers refer to the original paper.

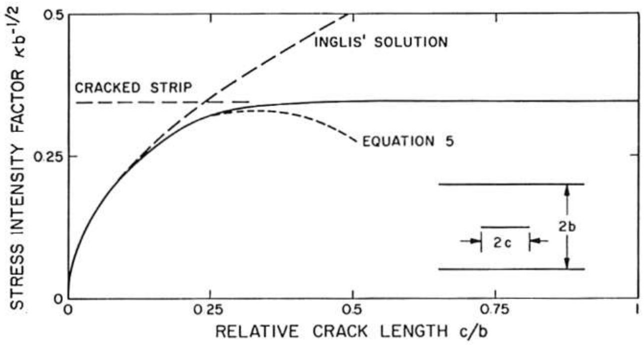

Fig. 2 Stress intensity factor K as a function of crack length, $\nu = 1/2$

$$\frac{1-\nu^2}{Ev_0}\,\sigma_y(x, 0)_{x\to 0} = \left\{\frac{1-\nu^2}{2\pi}\right\}^{1/2} x^{-1/2} + O(1) \tag{4}$$

Upon defining the stress intensity factor K as in (*33*), one has

$$K = \left\{\frac{1-\nu^2}{2\pi}\right\}^{1/2}$$

which is in agreement with equation (6) in Professor Rice's comment (unit strip half-width).

Professor Rice's calculations also suggest a way in which equation (*34*), derived for the case of shear free boundaries, may be modified to account for clamped boundaries. Note that the stress intensity factors for the clamped strip may be obtained by replacing b by $(1 - \nu^2)b$ in the expression derived for the shear force strip [Professor Rice's equations for K_{I} following his equation (5) and equation (7)]. Upon replacing b in equation (*34*) by $(1 - \nu^2)b$, one obtains thus

$$\frac{K}{b^{1/2}} = \left[\frac{c}{2b}\right]^{1/2}\left\{1 - \frac{\pi^2}{24}\left[\frac{c}{(1-\nu^2)b}\right]^2 + O\left[\frac{c}{b}\right]^4\right\} \tag{5}$$

Observe that for $c/b \ll 1$, Inglis' solution is obtained as one would expect on physical grounds and that the effect of the boundary condition (clamped versus shear free boundary) is felt only as $c/b \rightarrow O(1)$. Accordingly, Fig. 5 in the paper appears modified as shown in Fig. 2 in this Closure.

Natural Frequencies of Vibration of an All-Clamped Rectangular Sandwich Panel[1]

C. W. BERT.[2] The author is to be congratulated for presenting an interesting analysis. However, the agreement with experimental results is not as good as might be desired. In fact, the percent difference computed as follows:

(Theoretical lower bound − Exp.)/Exp.

decreased fairly systematically from 10.4 percent at the lowest natural frequency to 5.2 percent at the highest frequency. The fact that the difference decreased with increasing frequency suggests that the deficiency in the analysis is associated with the pure bending contribution (and the normal-direction inertia) rather than the pure shear contribution (and the rotatory inertia), since the pure shear modes have much higher natural frequencies than the bending modes. The writer agrees with the author that the assumption of core inextensibility in the thickness direction is reasonable for practical ranges of facing thickness to core depth and identical facings. The ratio of $G_{xz}/G_{yz} = 2.6$ seems to be rather high for hexagonal-cell honeycomb core material; values ranging from 1.5 to 2.0 would be more typical.

[1] By C. E. S. Ueng, published in the September, 1966, issue of the Journal of Applied Mechanics, vol. 33, Trans. ASME, vol. 88, Series E, pp. 683–684.

[2] Professor of Aerospace and Mechanical Engineering, University of Oklahoma, Norman, Okla. Mem. ASME.

In view of these considerations, one would be inclined to agree with the author's explanation that the discrepancy is due to the lack of complete edge clamping in the experiments, except for the following point: The curve of natural frequency ω versus edge rotational spring constant K_r would be expected to have an "upper plateau" region in which large changes in K_r have a negligible effect on ω. (This is the case for a homogeneous isotropic circular plate with elastic rotational edge restraint.[3])

Author's Closure

The author thanks Professor Bert for his comments and the interest he has shown in the paper. The author agrees with Professor Bert on the point that the deviation between the analytic results and experimental values may have been caused by not only the incomplete fixity on the edges but also some other reasons, such as a possible deficiency in the analysis associated with the bending contribution. From the interesting article by Kantham,[3] it seems that this possibility exists. However, it is rather difficult to use the idea given in that article to measure the fixity for the problem reported in the paper.

[3] C. L. Kantham, "Bending and Vibration of Elastically Restrained Circular Plates," *Journal of The Franklin Institute*, vol. 265, 1958, pp. 483–491.

Three-Dimensional Stress Distribution Around an Elliptical Crack Under Arbitrary Loadings[1]

W. T. CHEN.[2] The writer of this Discussion has also treated the problem of a flat elliptical crack under shear and his results have recently been published.[3] In his work the elliptical crack occurs in a transversely isotropic elastic solid, and lies on a plane perpendicular to the material axis. It is interesting to observe the differences in the formal formulations and the mathematical approaches, although both of the physical problems can be treated by either method.

In his paper, this writer has not provided expressions for the stress intensity factors defined in the authors' paper as k_2 and k_3. It may be a useful addendum to the authors' work to record these expressions for the transversely isotropic elastic solid.

It has been found that k_2 and k_3 may be written down using slightly altered versions of the authors' equations (25) and (33). Define the two constants A and B [authors' equations (25*a*) and (25*b*)] by

$$A = \frac{ab^2k^2q\cos\omega}{(k^2-\nu)E(k) + \nu k'^2K(k)}, \tag{1}$$

$$B = \frac{ab^2k^2q\sin\omega}{(k^2+\nu k'^2)E(k) - \nu k'^2K(k)}. \tag{2}$$

All the functions and parameters are the same as in the paper except for ν, which is now defined as

$$\nu = 1 - \frac{\beta}{\alpha}, \tag{3}$$

where

[1] By M. K. Kassir and G. C. Sih, published in the September, 1966, issue of the Journal of Applied Mechanics, vol. 33, Trans. ASME, vol. 88, Series E, pp. 601–611.

[2] Department of Advanced Technology, IBM Corporation, Endicott, N. Y.

[3] W. T. Chen, "On Some Aspects of a Flat Elliptical Crack Under Shear Stress," *Journal of Mathematics and Physics*, vol. 45, June, 1966, pp. 213–223.