

# Stresses Due to a Sharp Notch in a Work-Hardening Elastic-Plastic Material Loaded by Longitudinal Shear

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*A work-hardening elastic-plastic stress analysis is presented for a sharp notch or, as a limiting case, a crack perturbing a remotely applied uniform stress field. Mathematical complexities are reduced through considering the kinematically simple case of antiplane longitudinal shear deformations and by employing a deformation plasticity theory rather than the more appropriate incremental theory. Consequently, a general solution is available valid for any relation between stress and strain in the work-hardening range, so long as the remotely applied stress does not exceed the initial yield stress. When a power law relates stress to a strain in the work-hardening range, the deformation theory solution is also the correct incremental solution at low applied stress levels causing yielding on a scale small compared to notch depth. For cracks, the near crack tip strain field is shown to depend on loads and geometry only through the elastic stress intensity factor when yielding is on a small scale, and the elastic-plastic boundary and lines of constant strain magnitude are circles. Extensive numerical results are tabulated for a crack, 45 deg V-notch, and 90 deg V-notch in power-law-hardening materials, and exhibited graphically for a crack.*

## Introduction

ACCURATE determinations of stress and deformation fields near various stress concentrators, and cracks in particular, employing realistic stress-strain relation are of fundamental importance for the mechanical description of fracture and fatigue. Obvious mathematical difficulties generally accompany such determinations. However, a class of problems involving cracks or sharp notches in planes of work-hardening elastic-plastic materials, subjected to uniform remotely applied stress fields, may be treated in a straightforward manner through the methods presented here, provided one is content to introduce certain simplifications which considerably reduce the mathematical complexity.

The first of these is kinematical in nature: We consider the antiplane case of longitudinal shear involving displacements only in the direction perpendicular to the cracked or notched plane. Letting the  $x$  and  $y$ -axes lie in the plane, with the  $z$ -direction perpendicular, means that the only nonvanishing displacement is the  $z$ -direction component,  $w = w(x, y)$ . As a consequence, all strain components vanish identically except for the longitudinal shears  $\gamma_{xz}$  and  $\gamma_{yz}$ . Introducing the notations  $\gamma_x = \gamma_{xz}$  and  $\gamma_y = \gamma_{yz}$ , these are given by

$$\gamma_x = \partial w / \partial x, \quad \gamma_y = \partial w / \partial y \quad (1)$$

A compatibility equation for the strain results by observing that  $\partial^2 w / \partial x \partial y$  must be independent of the order of differentiation, so that

$$(\partial \gamma_x / \partial y) - (\partial \gamma_y / \partial x) = 0 \quad (2)$$

Within the context of the usual assumptions of an isotropic material and small deformations, all stresses vanish except the longitudinal shears  $\tau_{xz}$  and  $\tau_{yz}$ . With notations  $\tau_x = \tau_{xz}$  and  $\tau_y = \tau_{yz}$ ,

Contributed by the Applied Mechanics Division and presented at the Applied Mechanics Conference, Pasadena, Calif., June 26-28, 1967, of THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS.

Discussion of this paper should be addressed to the Editorial Department, ASME, United Engineering Center, 345 East 47th Street, New York, N. Y. 10017, and will be accepted until July 15, 1967. Discussion received after the closing date will be returned. Manuscript received by ASME Applied Mechanics Division, December 30, 1965; final draft, August 3, 1966. Paper No. 67-APM-1.

the only equation of equilibrium not identically satisfied by vanishing stresses is

$$(\partial \tau_x / \partial x) + (\partial \tau_y / \partial y) = 0 \quad (3)$$

When equations (2, 3) are complimented by a stress-strain relation, generally nonlinear equations result which must be solved subject to appropriate boundary conditions. It is expected that solutions obtained, while valid for the antiplane case of longitudinal shear, will shed some light on the analogous and more important problems involving in-plane deformations due to tensile loadings. In particular, the results of elastic crack stress analysis reported by Irwin [1, 2]<sup>1</sup> and Paris and Sih [3] indicate that the stress intensity factor (coefficient of a characteristic inverse square root crack tip singularity in stresses, which is related to Irwin's energy release rate [1]) of a longitudinal shear problem usually approximates closely, and sometimes gives exactly, the factor for the tensile problem having the same geometry, when the applied shear stress term appearing in the stress intensity factor formula is replaced by a term giving the applied tensile stress. Further, McClintock and Irwin [4] have pointed out that several important observed features of plastic influence in tensile crack extension are predictable from the perfectly plastic solutions for longitudinal shear cracks as provided in [5-8].

The second simplification introduced for the treatment of cracks and notches in work-hardening materials has to do with the constitutive relation: A deformation plasticity theory is employed relating stress to total strain in an isotropic fashion, so that directions of the principal shear stress and strain coincide. An incremental theory is clearly more appropriate; however, appreciable errors are not expected for monotonic loading and, indeed, for the case where a power law relates stress to strain in the hardening range, it is shown that the deformation solution is also an incremental solution when yielding is on a small scale. It is noted that deformation plasticity solutions also provide solutions for nonlinear elastic materials. The stress-strain relation is specified by a function  $\tau = \tau(\gamma)$  relating principal stress and strain, defined by

$$\tau = (\tau_x^2 + \tau_y^2)^{1/2}, \quad \gamma = (\gamma_x^2 + \gamma_y^2)^{1/2} \quad (4)$$

<sup>1</sup> Numbers in brackets designate References at end of paper.



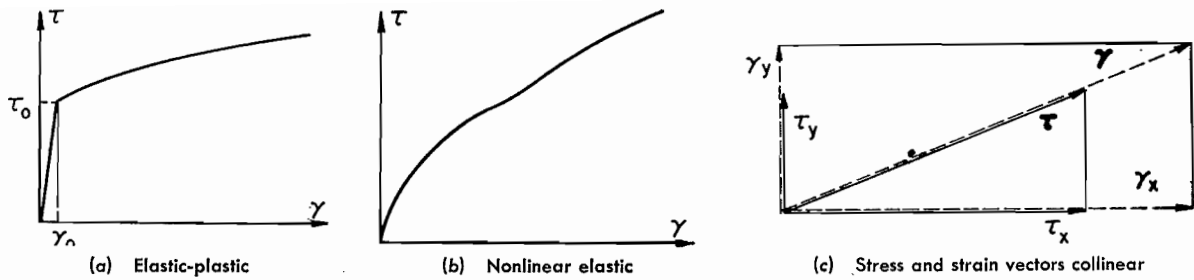


Fig. 1 Stress-strain relation

A typical hardening stress-strain relation is shown in Fig. 1(a), where a linear elastic relation prevails up to an initial yield stress  $\tau_0$ , with corresponding strain  $\gamma_0$ , and  $\tau_0/\gamma_0 = G =$  elastic shear modulus. A more general relation appropriate, say, for a nonlinear elastic material is shown in Fig. 1(b). Noting from Fig. 1(c) that  $\gamma_x/\gamma = \tau_x/\tau$ , component forms of the stress-strain relation are

$$\tau_x = \gamma_x \frac{\tau(\gamma)}{\gamma}, \quad \tau_y = \gamma_y \frac{\tau(\gamma)}{\gamma} \quad (5)$$

In terms of boundary conditions, these imply, for example, that, if  $\tau_x$  vanishes on a portion of the boundary, so does  $\gamma_x$ .

### Basic Equations in Strain Plane

The nonlinear equations governing longitudinal shear problems may be reduced to linear equations by regarding physical coordinates as functions of the strains or, equivalently, of the stresses. That is,  $x = x(\gamma_x, \gamma_y)$  and  $y = y(\gamma_x, \gamma_y)$  or  $x = x(\tau_x, \tau_y)$  and  $y = y(\tau_x, \tau_y)$ . Noting that

$$\begin{aligned} dx &= \frac{\partial x}{\partial \gamma_x} d\gamma_x + \frac{\partial x}{\partial \gamma_y} d\gamma_y \\ &= \frac{\partial x}{\partial \gamma_x} \left( \frac{\partial \gamma_x}{dx} dx + \frac{\partial \gamma_x}{dy} dy \right) \\ &\quad + \frac{\partial x}{\partial \gamma_y} \left( \frac{\partial \gamma_y}{dx} dx + \frac{\partial \gamma_y}{dy} dy \right) \end{aligned} \quad (6)$$

with a similar equation for  $dy$ , and equating coefficients of the arbitrary  $dx$  and  $dy$ , one has a system of four linear equations for the partials of  $(\gamma_x, \gamma_y)$  with respect to  $(x, y)$ , with coefficients involving partials of  $(x, y)$  with respect to  $(\gamma_x, \gamma_y)$ . Solving for the partial derivatives involved in the compatibility equation (2) and substituting, there results in view of the homogeneity of (2)

$$\frac{\partial x}{\partial \gamma_y} - \frac{\partial y}{\partial \gamma_x} = 0 \quad (7)$$

at all points of the strain plane where the Jacobian of the transformation is nonzero (i.e., at all points where a finite area or arc of the physical plane does not map into a single point in the strain plane). Similarly, working in terms of stresses, the equilibrium equation (3) becomes

$$\frac{\partial x}{\partial \tau_x} + \frac{\partial y}{\partial \tau_y} = 0 \quad (8)$$

Such transformation techniques have been employed for perfectly plastic solutions in [5, 7, 8] and by Neuber [9] in obtaining some solutions for nonlinear materials which will appear as a limiting case of our results in a following section. Similar techniques are employed in the hodograph method [10] of fluid mechanics which reduces the nonlinear equations of two-dimensional compressible flow to linear equations in terms of velocity components. Clearly, the success of the transformation formulation depends on the ability to convert boundary conditions for the

problem of interest from the physical plane to the strain plane, and usually on the existence of a one-to-one correspondence.

It is convenient to cast the general results in a vector notation so that governing equations may be easily transformed to the strain coordinate system most appropriate for a given problem. The nonvanishing components of stress and strain form two-dimensional vector  $\tau$  and  $\gamma$  given, respectively, in Cartesian components by

$$\tau = \tau_x i_x + \tau_y i_y, \quad \gamma = \gamma_x i_x + \gamma_y i_y \quad (9)$$

with  $(i_x, i_y)$  unit vectors in the  $(x, y)$ -directions, and  $\tau = |\tau|$ ,  $\gamma = |\gamma|$ . Gradients  $\nabla_\tau$  and  $\nabla_\gamma$  are defined in the stress and strain coordinate systems, and in Cartesian form

$$\nabla_\tau = i_x \frac{\partial}{\partial \tau_x} + i_y \frac{\partial}{\partial \tau_y}, \quad \nabla_\gamma = i_x \frac{\partial}{\partial \gamma_x} + i_y \frac{\partial}{\partial \gamma_y} \quad (10)$$

Denoting by

$$r = x i_x + y i_y \quad (11)$$

the position vector of a point in the physical plane, compatibility (7) and equilibrium (8) equations become

$$\nabla_\gamma \times \cdot = 0, \quad \nabla_\tau \cdot r = 0 \quad (12)$$

The gradient in the stress coordinates must be converted to that in the strain coordinates through the stress-strain relations. For corresponding changes  $(d\gamma, d\tau)$  in the vectors  $(\gamma, \tau)$ , it is required, from the definition of a gradient, that

$$d\tau \cdot \nabla_\tau = d\gamma \cdot \nabla_\gamma \quad (13)$$

Stress (solid line) and strain (dashed line) vectors and small increments are pictured in Fig. 2(a), and our use of a deformation theory requires that vectors  $(\gamma, \gamma + d\gamma)$  be collinear with  $(\tau, \tau + d\tau)$ . With reference to the blown-up Fig. 2(b), the vector  $d\gamma$  is resolved into the sum of a vector in the direction of  $d\tau$  and a vector in the direction of  $\gamma$ . Due to the geometrical similarity, the contribution to  $d\gamma$  in the direction of  $d\tau$  is  $(\gamma/\tau)d\tau$ . From the geometry in Fig. 2(b), the magnitude of the contribution in the  $\gamma$ -direction is, to the first order,  $d\gamma - (\gamma/\tau)d\tau$ . As  $\gamma/\gamma$  is a unit

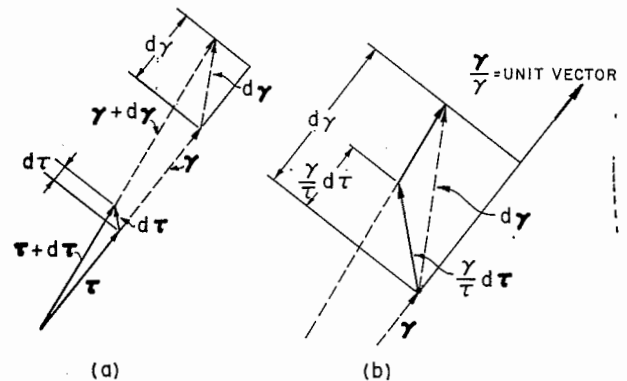


Fig. 2 Stress and strain vectors and their increments

vector in the  $\gamma$ -direction, in vector form this is  $[(d\gamma/d\tau) - (\gamma/\tau)](\gamma/\gamma)d\tau$ . Note that  $(d\tau, d\gamma)$  are not in general equal to  $(|d\tau|, |d\gamma|)$  but are rather the changes in scalar lengths of  $(\tau, \gamma)$ . Since  $\tau^2 = \tau \cdot \tau$ ,  $d\tau = (\tau/\tau) \cdot d\tau = (\gamma/\gamma) \cdot d\tau$ , and the  $\gamma$ -direction contribution may be written as

$$\left(\frac{d\gamma}{d\tau} - \frac{\gamma}{\tau}\right) \frac{\gamma}{\gamma} \left(\frac{\gamma}{\gamma} \cdot d\tau\right)$$

Summing the two contributions,

$$d\gamma = \frac{\gamma}{\tau} \left\{ d\tau + \left(\frac{\tau}{\gamma} \frac{d\gamma}{d\tau} - 1\right) \frac{\gamma}{\gamma} \left(\frac{\gamma}{\gamma} \cdot d\tau\right) \right\} \quad (14)$$

Substituting for  $d\gamma$  in equation (13), which relates the gradients,

$$d\tau \cdot \nabla_\tau = \frac{\gamma}{\tau} d\tau \cdot \left\{ \nabla_\gamma + \frac{\gamma}{\gamma} \left(\frac{\tau}{\gamma} \frac{d\gamma}{d\tau} - 1\right) \left(\frac{\gamma}{\gamma} \cdot \nabla_\gamma\right) \right\} \quad (15)$$

Since this equation must hold for arbitrary values of  $d\tau$ , the stress and strain coordinate gradients are related by, after noting  $\tau = \tau(\gamma)$  from the stress-strain relation,

$$\nabla_\tau = \frac{\gamma}{\tau(\gamma)} \left\{ \nabla_\gamma + \frac{\gamma}{\gamma} \left[ \frac{\tau(\gamma)}{\gamma\tau'(\gamma)} - 1 \right] \left(\frac{\gamma}{\gamma} \cdot \nabla_\gamma\right) \right\} \quad (16)$$

Thus the governing equilibrium and compatibility equations of (12) are given entirely in strain coordinates by

$$\nabla_\gamma \times \mathbf{r} = 0 \quad (17)$$

$$\nabla_\gamma \cdot \mathbf{r} + \left[ \frac{\tau(\gamma)}{\gamma\tau'(\gamma)} - 1 \right] \frac{\gamma}{\gamma} \cdot \left[ \left(\frac{\gamma}{\gamma} \cdot \nabla_\gamma\right) \mathbf{r} \right] = 0 \quad (18)$$

These may be reduced to a single equation by noting that (17) implies the existence of a scalar potential function  $\psi = \psi(\gamma)$  such that physical coordinates are given by

$$\mathbf{r} = \nabla_\gamma \psi \quad (19)$$

and (17) is satisfied identically. The remaining equation (18) then requires  $\psi$  to satisfy

$$\nabla_\gamma^2 \psi + \left[ \frac{\tau(\gamma)}{\gamma\tau'(\gamma)} - 1 \right] \frac{\gamma}{\gamma} \cdot \left[ \left(\frac{\gamma}{\gamma} \cdot \nabla_\gamma\right) (\nabla_\gamma \psi) \right] = 0 \quad (20)$$

Note that, for a linear elastic material,  $\tau(\gamma) = G\gamma$  and  $\frac{\tau(\gamma)}{\gamma\tau'(\gamma)} = 1$  so that (20) reduces to the Laplace form  $\nabla_\gamma^2 \psi = 0$ . In Cartesian strain coordinates,  $\psi = \psi(\gamma_x, \gamma_y)$  and the governing equations are

$$x = \partial\psi/\partial\gamma_x, \quad y = \partial\psi/\partial\gamma_y \quad (21)$$

$$\frac{\partial^2 \psi}{\partial \gamma_x^2} + \frac{\partial^2 \psi}{\partial \gamma_y^2} + \frac{1}{\gamma^2} \left[ \frac{\tau(\gamma)}{\gamma\tau'(\gamma)} - 1 \right] \times \left[ \gamma_x^2 \frac{\partial^2 \psi}{\partial \gamma_x^2} + 2\gamma_x \gamma_y \frac{\partial^2 \psi}{\partial \gamma_x \partial \gamma_y} + \gamma_y^2 \frac{\partial^2 \psi}{\partial \gamma_y^2} \right] = 0 \quad (22)$$

The differential equation (20) takes a considerably simpler form when a polar coordinate system is employed in the strain plane, consisting of the magnitude,  $\gamma$ , of the principal shear strain and the angle,  $\phi$ , measured positive counterclockwise, between the principal shear (or stress) direction and the  $y$ -axis, as in Fig. 3. In this coordinate system, with radial and angular unit vectors  $(\mathbf{i}_\gamma, \mathbf{i}_\phi)$ ,  $\gamma = \gamma \mathbf{i}_\gamma$  and  $\nabla_\gamma = \mathbf{i}_\gamma \frac{\partial}{\partial \gamma} + \frac{1}{\gamma} \mathbf{i}_\phi \frac{\partial}{\partial \phi}$ . Taking  $\psi = \psi(\gamma, \phi)$ , coordinates of the position vector  $\mathbf{r}$  referred to radial and angular strain directions are given by (19). Converting these to Cartesian form,

$$x = -\sin \phi \frac{\partial \psi}{\partial \gamma} - \frac{\cos \phi}{\gamma} \frac{\partial \psi}{\partial \phi}, \quad y = \cos \phi \frac{\partial \psi}{\partial \gamma} - \frac{\sin \phi}{\gamma} \frac{\partial \psi}{\partial \phi} \quad (23)$$

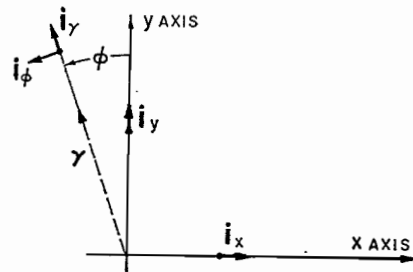


Fig. 3 Polar strain coordinate system; principal shear strain and angle with  $y$ -axis

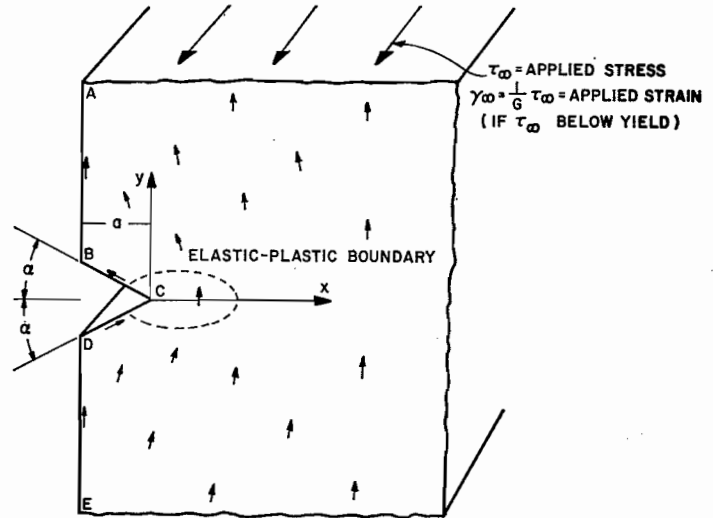


Fig. 4 Problem considered and notation; arrows denote direction of strain vector at typical points

Equation (20) governing  $\psi$  becomes

$$\frac{\tau(\gamma)}{\gamma\tau'(\gamma)} \frac{\partial^2 \psi}{\partial \gamma^2} + \frac{1}{\gamma} \frac{\partial \psi}{\partial \gamma} + \frac{1}{\gamma^2} \frac{\partial^2 \psi}{\partial \phi^2} = 0 \quad (24)$$

We shall find this polar form particularly convenient for treating cracks or sharp notches in elastic-plastic materials, so long as the plastic region does not completely traverse the notched plane.

To derive an expression for displacement,  $w$ , in terms of the potential function,  $\psi$ , note that

$$dw = \gamma \cdot d\mathbf{r} = d(\gamma \cdot \mathbf{r}) - d\gamma \cdot \mathbf{r} \quad (25)$$

Since  $\mathbf{r} = \nabla_\gamma \psi$ ,  $d\gamma \cdot \mathbf{r} = d\gamma \cdot \nabla_\gamma \psi = d\psi$  is an exact differential and (25) integrates to

$$w = \gamma \cdot \nabla_\gamma \psi - \psi + \text{const} \quad (26)$$

### Sharp Notch in a Semi-Infinite Plane

A general formulation is given here for the problem of a sharp edge notch of angle  $2\alpha$  and depth  $a$  in a semi-infinite plane, and subject to the remotely applied stress  $\tau_\infty$ , as shown in Fig. 4. Solutions to this problem in the perfectly plastic case have been given in [5, 7, 8]. It is noted that symmetry allows a solution to serve also for the case of an internal double-ended notch (mirror image of Fig. 4 added on) in an infinite plane, as  $\tau_x$  and thus  $\gamma_x = \partial w/\partial x$  vanish along the center line. This correspondence is only approximate for the analogous tensile case. Although the formulation is valid for an arbitrary nonlinear stress-strain law and applied stress, a solution has been found only for stress-strain laws of the linear elastic work-hardening plastic type, as depicted in Fig. 1(a), with the remotely applied stress,  $\tau_\infty$  (or strain,  $\gamma_\infty$ ) less than the initial yield stress,  $\tau_0$  (or strain,  $\gamma_0$ ) so that the elastic-plastic boundary (dashed line in Fig. 4) does not extend to infinity. Interestingly, this solution may be obtained in a com-

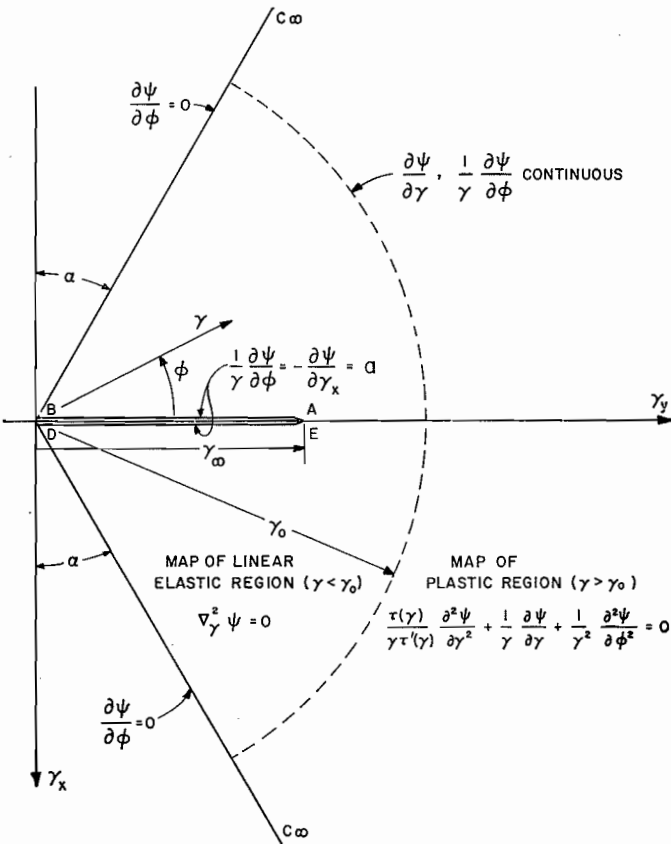


Fig. 5 Strain plane mapping of physical plane; corresponding points, boundary conditions, and governing equations shown

pletely general manner which does not rely on the assumption of some explicit type of stress-strain relation,  $\tau = \tau(\gamma)$ , in the work-hardening range.

The geometrical shape of the strain plane image of the physical plane in Fig. 4 is shown in Fig. 5 and is determined by noting the direction of the strain vector at various points along the boundary. Note that, along  $AB$ ,  $\gamma_x = \phi = 0$ . Boundary conditions at the stress-free corner,  $B$ , require  $\gamma_x$  and  $\gamma_y$  to vanish there; point  $A$  is at infinity, where strains are uniformly  $\gamma_x = 0$  and  $\gamma_y = \gamma_\infty$ . Thus  $AB$  appears in the strain plane as the segment of the  $\gamma_y$ -axis between zero and  $\gamma_\infty$ . Along the notch boundary  $BC$ ,  $\phi = \frac{1}{2}\pi - \alpha$ . As the strain vector must abruptly change direction at point  $C$ , a singularity is required and  $BC$  makes an angle  $\alpha$  with the negative  $\gamma_x$ -axis in the strain plane, with point  $C$  at infinity. Similarly, the maps of  $DE$  and  $CD$  are found. The identification of  $AB$  and  $DE$  with top and bottom sides, respectively, of the slit in the strain plane in Fig. 5 is made by noting from Fig. 4 that  $\gamma_x$  is negative at small distances from  $AB$  and positive at small distances from  $DE$ .

To determine boundary conditions in the strain plane, note that  $x = -a$  along  $AB$  and  $DE$ . Recalling as in (19, 21, 23) that physical coordinates are related to the gradient of a strain potential,  $\psi$ ,

$$\left. \frac{\partial \psi}{\partial \gamma_x} \right|_{\gamma_x=0} = -\frac{1}{\gamma} \left. \frac{\partial \psi}{\partial \phi} \right|_{\phi=0} = -a \quad (27)$$

on  $AB$  and  $DE$ . On the notch boundary  $BC$ , one has  $y + x \tan \alpha = 0$ . Substituting for  $x$  and  $y$  from the polar form (23) and setting  $\phi = \frac{1}{2}\pi - \alpha$ , one obtains

$$\left. \frac{\partial \psi}{\partial \phi} \right|_{\phi=\frac{\pi}{2}-\alpha} = 0 \quad (28)$$

on  $BC$ . Similarly,

$$\left. \frac{\partial \psi}{\partial \phi} \right|_{\phi=-\frac{\pi}{2}+\alpha} = 0 \quad (29)$$

on  $CD$ . Noting that  $\psi$  is governed by (20, 22, or 24), the complete boundary problem is now formulated in the strain plane. The slit geometry makes solution for an arbitrary nonlinear  $\tau(\gamma)$  and arbitrary  $\gamma_\infty$  generally difficult. However, when an initially linear elastic material is considered as in Fig. 1(a),  $\psi$  is governed by the Laplacian form  $\nabla_{\gamma^2} \psi = 0$  for  $\gamma < \gamma_0$  ( $\gamma = \gamma_0$  being the map of the elastic-plastic boundary); if  $\gamma_\infty < \gamma_0$  so that the slit is entirely within the Laplacian region, the powerful methods of analytic function theory may be employed to handle this otherwise difficult geometry.

The solution for  $\psi$  is reduced to determination of an arbitrary analytic function in the elastic region ( $\gamma < \gamma_0$ ) through successive use of a conformal transformation, introduction of an image plane, and reduction to a linear Hilbert arc problem. A separation of variables technique is employed in the plastic region ( $\gamma > \gamma_0$ ), expressing  $\psi$  as an infinite sum of solutions to (24), with constant coefficients to be determined. Finally, the two solutions are matched on the elastic-plastic boundary ( $\gamma = \gamma_0$ ), leading to an infinite system of equations for the various unknowns, these being solved through series developments in the ratio of remotely applied strain to initial yield strain. Details of the calculations follow.

#### Solution in Elastic Region ( $\gamma < \gamma_0$ )

Since  $\nabla_{\gamma^2} \psi = 0$  in this portion of the strain plane,  $\psi$  may be represented as the imaginary part of an analytic function of  $\gamma_y - i\gamma_x = \gamma e^{i\phi}$ . A conformal transformation  $\zeta = \xi + i\eta = \zeta(\gamma_y - i\gamma_x)$  allows  $\psi$  to be represented as  $\psi = \text{Im} \{h(\zeta)\}$  and insures that normal derivatives appearing in boundary conditions transform to normal derivatives in the  $\zeta$ -plane. The transformation

$$\zeta = \xi + i\eta = \left( \frac{\gamma_y - i\gamma_x}{\gamma_0} \right)^\lambda = \left( \frac{\gamma}{\gamma_0} \right)^\lambda e^{i\lambda\phi} \quad (30)$$

with  $\lambda$  related to half notch angle,  $\alpha$ , by

$$\lambda = \frac{\pi}{\pi - 2\alpha} \quad (31)$$

maps the slit sector comprising the elastic region in Fig. 5 into a slit unit semicircle in the  $\zeta$ -plane as in Fig. 6, where the slit extends on the  $\zeta$ -axis over  $0 \leq \xi \leq s$ ,  $s$  being the dimensionless applied loading

$$s = \left( \frac{\gamma_\infty}{\gamma_0} \right)^\lambda = \left( \frac{\tau_\infty}{\tau_0} \right)^\lambda \quad (32)$$

Physical coordinates are given by first derivatives of  $\psi$  with

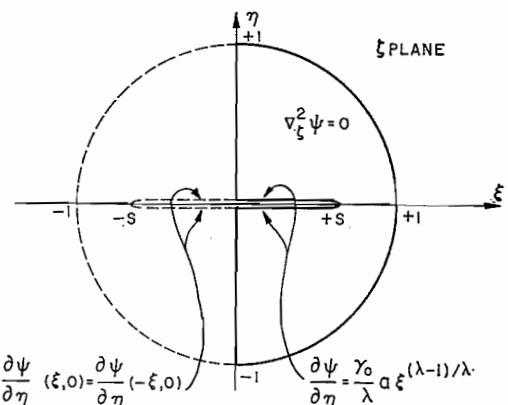


Fig. 6 Conformal mapping of elastic region of strain plane onto split unit semicircle in  $\zeta$ -plane; image region shown by dashed lines

respect to  $\gamma_x$  and  $\gamma_y$ . In terms of  $\zeta$ , with  $\psi = \text{Im} \{h(\zeta)\}$ , Cauchy-Riemann conditions may be shown to lead to

$$\begin{aligned} -x + iy &= -\frac{\partial\psi}{\partial\gamma_x} + i\frac{\partial\psi}{\partial\gamma_y} = e^{-i\phi} \left( \frac{1}{\gamma} \frac{\partial\psi}{\partial\phi} + i\frac{\partial\psi}{\partial\gamma} \right) \\ &= \frac{\lambda}{\gamma_0} \zeta^{(\lambda-1)/\lambda} \left( \frac{\partial\psi}{\partial\eta} + i\frac{\partial\psi}{\partial\xi} \right) \\ &= \frac{\lambda}{\gamma_0} \zeta^{(\lambda-1)/\lambda} h'(\zeta) \end{aligned} \quad (33)$$

Using (33) to relate derivatives in the strain plane and  $\zeta$ -plane, boundary conditions (28) and (29) on the inclined lines in Fig. 5 become, on the  $\eta$ -axis in Fig. 6,

$$\frac{\partial\psi}{\partial\xi} \Big|_{\xi=0} = 0 \quad (34)$$

and (27) on the slit in Fig. 5 becomes, on its map along the  $\xi$ -axis slit in Fig. 6,

$$\frac{\partial\psi}{\partial\eta} \Big|_{\eta=0} = \frac{\gamma_0}{\lambda} \xi^{-(\lambda-1)/\lambda} a, \quad 0 < \xi < s \quad (35)$$

Thinking geometrically of  $\psi$  as the height of some surface measured perpendicularly to the  $\xi, \eta$ -plane, condition (35) corresponds to lowering the surface on the top side of the slit and raising on the bottom side through applying the rotation  $\frac{\partial\psi}{\partial\eta} \Big|_{\xi=0}$  about the  $\xi$ -axis, while (34) requires that the rotation about the  $\eta$ -axis, at points on this axis, be zero. This same distortion (and thus an identical solution) is obtained by considering a circular region identical to that in Fig. 6 for  $\xi > 0$ , but containing a slit over the length  $-s < \xi < +s$  with the same rotation of the  $\psi$ -surface at  $\xi$  as at  $-\xi$ . Thus introduction of an image slit, shown by the dotted line in Fig. 6, with  $(\partial\psi/\partial\eta)(\xi, 0) = (\partial\psi/\partial\eta) \times (-\xi, 0)$ , identically satisfies (34).

Extending  $h(\zeta)$  analytically into  $\xi < 0$  and noting  $2(\partial\psi/\partial\eta) = h'(\zeta) + \bar{h}'(\bar{\zeta})$ , boundary conditions along the slit and its image as applied to the top side lead to

$$[h'(\xi)]^+ + [\bar{h}'(\xi)]^- = \frac{2\gamma_0}{\lambda} |\xi|^{-(\lambda-1)/\lambda} a, \quad -s < \xi < +s \quad (36)$$

with the notations  $[q(\xi)]^+$ ,  $[q(\xi)]^-$  denoting the values of  $q(\zeta)$  as  $\zeta$  approaches the  $\xi$ -axis from the top and bottom sides, respectively. Similarly, conditions on the bottom side become

$$[h'(\xi)]^- + [\bar{h}'(\xi)]^+ = \frac{2\gamma_0}{\lambda} |\xi|^{-(\lambda-1)/\lambda} a, \quad -s < \xi < s \quad (37)$$

Subtracting (37) from (36), one obtains

$$[h(\xi) - \bar{h}(\xi)]^+ = [h(\xi) - \bar{h}(\xi)]^-, \quad -s < \xi < s \quad (38)$$

which implies that  $h(\zeta) - \bar{h}(\bar{\zeta})$  is single valued along the slit and thus analytic everywhere within the unit circle  $|\zeta| \leq 1$ , including points along the slit. But symmetry requires that  $(\partial\psi/\partial\xi)(\xi, 0) = h(\xi) - \bar{h}(\xi)$  (for  $1 > |\xi| > s$ ) be zero; analyticity then requires  $h(\zeta) - \bar{h}(\bar{\zeta})$  to vanish everywhere in the unit circle and  $h(\zeta) = \bar{h}(\bar{\zeta})$ . Thus (36) or (37) becomes

$$[h'(\xi)]^+ + [h'(\xi)]^- = \frac{2\gamma_0}{\lambda} |\xi|^{-(\lambda-1)/\lambda} a, \quad -s < \xi < s \quad (39)$$

Following Muskhelishvili [11], we introduce the function  $(\zeta^2 - s^2)^{1/2}$ , taking on the values  $\pm i(s^2 - \xi^2)^{1/2}$  on top and bottom sides so that

$$\begin{aligned} &[(\xi^2 - s^2)^{1/2} h'(\xi)]^+ - [(\xi^2 - s^2)^{1/2} h'(\xi)]^- \\ &= \frac{2i\gamma_0}{\lambda} |\xi|^{-(\lambda-1)/\lambda} (s^2 - \xi^2)^{1/2} a, \quad -s < \xi < +s \end{aligned} \quad (40)$$

As single valuedness of  $(\xi^2 - s^2)^{1/2}$  for  $|\xi| > s$  and previous remarks on symmetry require the difference to vanish on  $1 > |\xi| > s$ , a particular solution,  $h_p'(\zeta)$ , to (40) is given by [11]

$$(\xi^2 - s^2)^{1/2} h_p'(\zeta) = \frac{\gamma_0 a}{\lambda\pi} \int_{-s}^{+s} \frac{(s^2 - \mu^2)^{1/2} |\mu|^{-(\lambda-1)/\lambda}}{\mu - \zeta} d\mu \quad (41)$$

The complete solution to (40) is obtained by adding the most general homogeneous solution,  $h_H'(\zeta)$ , which is simply that  $(\zeta^2 - s^2)^{1/2} h_H'(\zeta)$  be analytic everywhere in the unit circle. In order to satisfy symmetry conditions of  $(\partial\psi/\partial\eta)(\xi, 0) = (\partial\psi/\partial\eta) \times (-\xi, 0)$  outside the slit ( $1 > |\xi| > s$ ), this takes the form

$$(\xi^2 - s^2)^{1/2} h_H'(\zeta) = \zeta g(\zeta) \quad (42)$$

where  $g(\zeta)$  is analytic everywhere for  $|\zeta| \leq 1$  and has a Taylor expansion containing only even powers of  $\zeta$ .

Adding (41) and (42), recalling (33) that physical coordinates are given by  $-x + iy = (\lambda/\gamma_0) \zeta^{(\lambda-1)/\lambda} [h_p'(\zeta) + h_H'(\zeta)]$ , and substituting  $t = (\mu/s)^{-\lambda}$  for the integration variable in (41),

$$\begin{aligned} -x + iy &= \frac{\lambda}{\gamma_0} \zeta^{(2\lambda-1)/\lambda} (\zeta^2 - s^2)^{-1/2} \\ &\times \left\{ g(\zeta) + \frac{2}{\pi} \gamma_0 a s^{(1+\lambda)/\lambda} \int_0^1 \frac{(1-t^{2\lambda})^{1/2} dt}{s^2 t^{2\lambda} - \zeta^2} \right\} \end{aligned} \quad (43)$$

If the material under consideration remained linear elastic for  $\gamma > \gamma_0$ , (43) would be valid also for all  $|\zeta| > 1$ , and the requirement of  $-x + iy \rightarrow 0$  as  $\zeta \rightarrow \infty$  (the crack tip  $x = y = 0$  is the point at infinity in the strain plane) would yield  $g(\zeta) = 0$ . However, for an elastic-plastic material,  $g(\zeta)$  must be determined, as will be shown subsequently, from the requirement that  $x$  and  $y$  be continuous functions of  $\gamma$  and  $\phi$  at the elastic-plastic boundary  $\gamma = \gamma_0$ .

#### Solution in Plastic Region ( $\gamma > \gamma_0$ )

Equation (20) for  $\psi$  holds in the plastic region with  $\tau = \tau(\gamma)$  being the work-hardening stress-strain law. Equations (28, 29) suggest the polar form of equation (24) as convenient, and a solution through separation of variables is

$$\psi = \sum_{k=1}^{\infty} D_k f_k(\gamma) \sin(2k-1)\lambda\phi \quad (44)$$

with  $\lambda$  defined by (31) and  $f_k(\gamma)$  satisfying

$$\frac{\tau(\gamma)}{\gamma\tau'(\gamma)} f_k''(\gamma) + \frac{1}{\gamma} f_k'(\gamma) - \frac{(2k-1)^2\lambda^2}{\gamma^2} f_k(\gamma) = 0 \quad (45)$$

with the conditions imposed that

$$f_k(\gamma_0) = 1, \quad f_k'(\infty) = 0 \quad (46)$$

The latter of (46) is required as  $x = y = 0$  (at the crack tip) for infinite strains; the first is imposed merely to make the functions  $f_k(\gamma)$  definite.

Quite general forms for  $\tau(\gamma)$  are permissible and, indeed, it is not necessary that the slope  $\tau'(\gamma)$  be continuous in the work-hardening range. The requirement of continuity of  $x$  and  $y$  as functions of  $\gamma$  and  $\phi$  requires that, in addition to satisfying (46),  $f_k(\gamma)$  and  $f_k'(\gamma)$  be continuous at points of discontinuity of  $\tau'(\gamma)$ . It is of interest and of particular importance for fracture mechanics applications that, for a crack [ $\lambda = 1$  by (31) as  $\alpha = 0$ ], a solution may be found for  $f_1(\gamma)$  in terms of a single quadrature involving  $\tau(\gamma)$ . However, the completion of our solution for the unknowns  $D_k$  will be seen to involve only the set  $f_k'(\gamma_0)$  and to be otherwise independent of the functional form of  $f_k[\gamma]$ . This permits the finding of a general solution valid for any arbitrary work-hardening stress-strain relation.

Equations (23) for physical coordinates may be put in the compact form

$$-x + iy = e^{-i\phi} \left( \frac{1}{\gamma} \frac{\partial \psi}{\partial \phi} + i \frac{\partial \psi}{\partial \gamma} \right) \quad (47)$$

Evaluating the derivatives on the elastic-plastic boundary  $\gamma = \gamma_0$  and recalling that  $f_k(\gamma_0) = 1$ ,

$$\begin{aligned} -x + iy|_{\gamma=\gamma_0} &= \frac{e^{-i\phi}}{2\gamma_0} \sum_{k=1}^{\infty} D_k \{ [(2k-1)\lambda \\ &+ \gamma_0 f_k'(\gamma_0)] e^{i(2k-1)\lambda\phi} + [(2k-1)\lambda - \gamma_0 f_k'(\gamma_0)] e^{-i(2k-1)\lambda\phi} \} \end{aligned} \quad (48)$$

### Matching of Solutions

Expressions for physical coordinates have been obtained for the elastic ( $\gamma < \gamma_0$ ) and plastic ( $\gamma > \gamma_0$ ) regions of the strain plane. We now determine the unknown function  $g(\zeta)$  and constants  $D_k$  involved by requiring these solutions to agree on the elastic-plastic boundary  $\gamma = \gamma_0$ . Consider  $\sigma = e^{i\lambda\phi}$ , which is the value of  $\zeta$  on  $\gamma = \gamma_0$ . Equating (43) with  $\zeta = \sigma$  to (48) and solving for  $g(\sigma)$  results in

$$\begin{aligned} g(\sigma) &= \frac{2}{\pi} \gamma_0 a s^{(1+\lambda)/\lambda} \sigma^{-2} \int_0^1 \frac{(1-t^{2\lambda})^{1/2} dt}{1 - (s^2 \sigma^{-2}) t^{2\lambda}} \\ &+ \frac{1}{2\lambda} (1 - s^2 \sigma^{-2})^{1/2} \sum_{k=1}^{\infty} D_k \{ [(2k-1)\lambda + \gamma_0 f_k'(\gamma_0)] \sigma^{2(k-1)} \\ &+ [(2k-1)\lambda - \gamma_0 f_k'(\gamma_0)] \sigma^{-2k} \} \end{aligned} \quad (49)$$

as the required continuity condition. We observe that the expression on the right contains both positive and negative powers of  $\sigma^2$ ; but it is recalled that  $g(\zeta)$  is analytic in the elastic region  $|\zeta| \leq 1$  and thus its boundary value,  $g(\sigma)$ , may contain only nonnegative powers of  $\sigma^2$ . Requiring that coefficients of  $\sigma^{-2}$ ,  $\sigma^{-4}$ ,  $\dots$ ,  $\sigma^{-2k}$ ,  $\dots$  vanish, one obtains a coupled infinite system of linear equations for  $D_1, D_2, \dots, D_k, \dots$ . Once these are determined, substitution into (44), together with (47), gives the physical coordinates having strains in the plastic region. Substituting  $D_k$  back into (49) and replacing  $\sigma$  with  $\zeta$ ,  $g(\zeta)$  is determined so that (43) gives physical coordinates having strains in the elastic range. A closed-form method of solution for  $D_k$  has not occurred to the author, and therefore the development of solutions in powers of the dimensionless applied load,  $s = (\gamma_\infty/\gamma_0)^\lambda = (\tau_\infty/\tau_0)^\lambda$ , is employed.

We represent  $g(\zeta)$  in the form

$$g(\zeta) = -\frac{1}{2} \gamma_0 a s^{(1+\lambda)/\lambda} \sum_{j=1}^{\infty} G_j(\zeta) s^{2j} \quad (50)$$

where each  $G_j(\zeta)$  is independent of  $s$  and analytic in the unit circle  $|\zeta| \leq 1$ . Each constant coefficient  $D_k$  is represented as

$$D_k = -\frac{\lambda \gamma_0 a s^{(1+\lambda)/\lambda}}{2(k-1)\lambda - \gamma_0 f_k'(\gamma_0)} \sum_{j=0}^{\infty} d_{k,j} s^{2j} \quad (51)$$

with each  $d_{k,j}$  independent of  $s$ , and the notation

$$C_k = \frac{(2k-1)\lambda + \gamma_0 f_k'(\gamma_0)}{(2k-1)\lambda - \gamma_0 f_k'(\gamma_0)} \quad (52)$$

is employed. The integral appearing in (49) is Laurent expandible for  $s < 1$ , and we let

$$\sum_{j=0}^{\infty} B_j \sigma^{-2j} s^{2j} = \frac{4}{\pi} \int_0^1 \frac{(1-t^{2\lambda})^{1/2} dt}{1 - (s^2 \sigma^{-2}) t^{2\lambda}} \quad (53)$$

so that each  $B_j$  is given in terms of gamma functions by

$$B_j = \frac{4}{\pi} \int_0^1 t^{2j\lambda} (1-t^{2\lambda})^{1/2} dt = \frac{\Gamma\left(j + \frac{1}{2\lambda}\right)}{\lambda \sqrt{\pi} \Gamma\left(j + \frac{1}{2\lambda} + \frac{3}{2}\right)} \quad (54)$$

With these substitutions, (49) becomes

$$\begin{aligned} \sum_{j=0}^{\infty} G_j(\sigma) s^{2j} &= -\sum_{j=0}^{\infty} B_j \sigma^{-2(j+1)} s^{2j} \\ &+ (1 - s^2 \sigma^{-2})^{1/2} \sum_{j=0}^{\infty} s^{2j} \left\{ \sum_{k=1}^{\infty} d_{k,j} [C_k \sigma^{2(k-1)} + \sigma^{-2k}] \right\} \end{aligned} \quad (55)$$

and we solve for the unknowns by equating coefficients of each power of  $s$ . Equating coefficients of  $s^0$ , one has

$$G_0(\sigma) = -B_0 \sigma^{-2} + \sum_{k=1}^{\infty} d_{k,0} [C_k \sigma^{2(k-1)} + \sigma^{-2k}] \quad (56)$$

Since negative powers of  $\sigma$  must not appear by analyticity of  $G_0(\zeta)$ , the solution for  $d_{k,0}$  is

$$\begin{aligned} d_{1,0} &= B_0 \\ d_{k,0} &= 0, \quad k \geq 2 \end{aligned} \quad (57)$$

and substituting back into (56),

$$G_0(\zeta) = B_0 C_1 \quad (58)$$

Next, equating coefficients of  $s^2$ , (55) results, after using (57), in

$$\begin{aligned} G_1(\sigma) &= -B_1 \sigma^{-4} - \frac{1}{2} \sigma^{-2} \{ B_0 [C_1 + \sigma^{-2}] \} \\ &+ \sum_{k=1}^{\infty} d_{k,1} [C_k \sigma^{2(k-1)} + \sigma^{-2k}] \end{aligned} \quad (59)$$

Again, since negative powers must not appear,

$$\begin{aligned} d_{1,1} &= \frac{1}{2} B_0 C_1 \\ d_{2,1} &= B_1 + \frac{1}{2} B_0 \\ d_{k,1} &= 0, \quad k \geq 3 \end{aligned} \quad (60)$$

and inserting into (62),

$$G_1(\zeta) = \frac{1}{2} B_0 C_1^2 + (B_1 + \frac{1}{2} B_0) C_2 \zeta^2 \quad (61)$$

Continuing in this manner,  $J_{k,2}$  and  $G_2(\zeta)$ ,  $d_{k,2}$  and  $G_3(\zeta)$ , and so on, may be determined. However, a recursive scheme may be developed to solve for  $G_j(\zeta)$  and  $d_{k,j}$  in terms of  $d_{p,j-1}$ ,  $d_{q,j-2}$ , and so on. Let

$$\begin{aligned} \sum_{p=1}^{\infty} \beta_p \sigma^{-2p} s^{2p} &= 1 - (1 - s^2 \sigma^{-2})^{1/2} \\ \beta_1 &= 1/2, \quad \beta_2 = 1/8, \quad \beta_3 = 1/16, \quad \beta_4 = 5/128 \quad (62) \\ \beta_p &= \frac{1}{4^{p-1}} \frac{(2p-3)!}{(p-2)! p!}, \quad p \geq 2 \end{aligned}$$

the coefficients  $\beta_p$  being obtained through the binomial theorem. When this series is inserted for  $(1 - s^2 \sigma^{-2})^{1/2}$  in (55) and the coefficient of  $s^{2j}$  is identified after multiplying the infinite series, there results, for  $j \geq 1$ ,

$$\begin{aligned} G_j(\sigma) &= -B_j \sigma^{-2(j+1)} + \sum_{k=1}^{\infty} d_{k,j} [C_k \sigma^{2(k-1)} + \sigma^{-2k}] \\ &- \sum_{i=0}^{j-1} \beta_{j-i} \left\{ \sum_{k=1}^{\infty} d_{k,i} [C_k \sigma^{2(k-1-j+i)} + \sigma^{-2(k+j-i)}] \right\} \end{aligned} \quad (63)$$

Now, after a considerable bit of rearrangement of the summations involved, it is found possible to identify the coefficient of each negative power of  $\sigma$  in the right side of (63). Further, upon setting these coefficients equal to zero, it is found possible to solve for each  $d_{k,j}$  ( $k = 1, 2, 3, \dots$ ) in terms of  $d_{r,j-1}$ ,  $d_{r,j-2}, \dots, d_{r,0}$  ( $r = 1, 2, 3, \dots$ ). The result is a recursive solution, valid for  $j \geq 1$ , so that, starting with the set  $d_{r,0}$  as given by (57), one computes the set  $d_{r,1}$ ; then starting with two sets  $d_{r,0}$  and  $d_{r,1}$ , one computes the set  $d_{r,2}$ ; and so on. The recursive equations are

$$d_{1,j} = \sum_{i=0}^{j-1} \beta_{j-i} d_{j-i,i} C_{j-i}$$

$$d_{k,j} = \sum_{i=j+1-k}^{j-1} \beta_{j-i} d_{k-j+i,i} + \sum_{i=0}^{j-k} \beta_{j-i} d_{j-k+1-i,i} C_{j-k+1-i}, \text{ for } 2 \leq k \leq j \quad (64)$$

$$d_{j+1,j} = B_j + \sum_{i=0}^{j-1} \beta_{j-i} d_{1+i,i}$$

$$d_{k,j} = \sum_{i=0}^{j-1} \beta_{j-i} d_{k-j+i,i}, \text{ for } j+2 \leq k < \infty$$

A very useful fact, since series solutions for  $D_k$  in terms of  $d_{k,j}$  (51) must be finitely terminated in performing computations, is that

$$d_{k,j} = 0, \text{ if } k \geq j+2 \quad (65)$$

Observe from (57) and (60) that this is true for  $d_{k,0}$  and  $d_{k,1}$ . It is then readily induced from the last recursive formula of (64) that (65) holds for all  $j$ . Suppose one is willing to neglect, in comparison to unity, terms involving  $s$  raised to a higher power than  $2J$  [or, by (32),  $\gamma_\infty/\gamma_0$  to a higher power than  $2\lambda J$ ] in expressions for  $D_k$ . Then, by (51) and (65), replacing by zero all neglected higher-order terms,

$$D_k = -\frac{\lambda \gamma_0 a s^{(1+\lambda)/\lambda}}{(2k-1)\lambda - \gamma_0 f_k'(\gamma_0)} \sum_{j=k-1}^J d_{k,j} s^{2j}, \quad k = 1, 2, \dots, J+1 \quad (66)$$

$$D_k = 0, \quad k = J+2, J+3, \dots$$

Thus a solution for physical coordinates corresponding to a given strain is given accurate to within an error of order  $(\gamma_\infty/\gamma_0)^{2\lambda(J+1)}$  by retaining the first  $(J+1)$ -terms in equation (44) for  $\psi$ , calculating the coefficients according to (66) from a finite number of  $d_{k,j}$  as obtained recursively from (64). It is also of interest to note, after reviewing earlier work, that a solution accurate to any arbitrarily chosen order satisfies all boundary conditions and governing equations in the strain plane in Fig. (6), except that continuity along the arc  $\gamma = \gamma_0$  is satisfied only to the chosen order. In terms of the physical plane in Fig. 4, this means that stress-free boundary conditions along the notched surface are exactly satisfied and stress equilibrium, strain compatibility, and stress-strain equations are exactly satisfied in the elastic and plastic regions of the physical plane corresponding to the maps of regions  $\gamma < \gamma_0$  and  $\gamma > \gamma_0$  in the strain plane. The approximation of (66) appears in the fact that physical coordinates carrying given strains are correct within an error of order  $(\gamma_\infty/\gamma_0)^{2\lambda(J+1)}$ , and a disparity of this order appears in the location of the elastic-plastic boundary in the physical plane as computed separately from solutions for the elastic and plastic regions and, thus, in the true location of the boundary.

The analytic functions  $G_j(\zeta)$  appearing in (50) for  $g(\zeta)$  are obtained from (63). After recognizing that the solution for  $d_{k,j}$  has removed all negative powers of  $\sigma$ , one obtains, using (65),

$$G_j(\zeta) = \sum_{q=0}^j d_{q+1,j} C_{q+1} \zeta^{2q} + \sum_{p=0}^{j-1} \left( \sum_{i=0}^{j-1} \beta_{j-i} d_{j+p+1-i,i} C_{j+p+1-i} \right) \zeta^{2p}, \quad \geq 1 \quad (67)$$

so that  $G_j(\zeta)$  is a polynomial of degree  $2j$  in  $\zeta$ .

#### Solution to Within Error of Order $s^{12}$

Expressions for  $D_k$  neglecting terms of higher order than  $s^{10}$  are given subsequently, so that physical coordinates corresponding

to given strains in the plastic region may be computed from (45) and (48). For a crack,  $\alpha = 0$  and  $\lambda = 1$  so that the error in physical coordinates is order  $(\gamma_\infty/\gamma_0)^{12}$ ; for a 90 deg notch,  $\alpha = \pi/4$  and  $\lambda = 2$  so that the error is order  $(\gamma_\infty/\gamma_0)^{24}$ . Solving for  $d_{k,j}$  ( $j = 1, 2, \dots, 5$ ) from (65) and inserting into (66),

$$D_1 = -\frac{\lambda \gamma_0 a s^{(1+\lambda)/\lambda}}{\lambda - \gamma_0 f_1'(\gamma_0)} \left\{ B_0 + \frac{1}{2} B_0 C_1 s^2 + \frac{1}{4} B_0 C_1^2 s^4 + \frac{1}{8} \left[ \left( B_1 + \frac{1}{2} B_0 \right) C_2 + B_0 C_1^3 \right] s^6 + \frac{1}{16} \left[ \left( B_1 + \frac{5}{4} B_0 \right) C_2 C_1 + B_0 C_1^4 \right] s^8 + \frac{1}{16} \left[ \left( B_2 + \frac{1}{2} B_1 + \frac{3}{8} B_0 \right) C_3 + \frac{1}{2} (B_1 + 2B_0) C_1 C_1^2 + \frac{1}{2} B_0 C_1^5 \right] s^{10} \right\}$$

$$D_2 = -\frac{\lambda \gamma_0 a s^{(1+\lambda)/\lambda}}{3\lambda - \gamma_0 f_2'(\gamma_0)} \left\{ \left( B_1 + \frac{1}{2} B_0 \right) s^2 + \frac{3}{8} B_0 C_1 s^4 + \frac{3}{16} B_0 C_1^2 s^6 + \frac{1}{8} \left[ \left( B_1 + \frac{1}{2} B_0 \right) C_2 + \frac{3}{4} B_0 C_1^3 \right] s^8 + \frac{3}{64} \left[ \left( B_1 + \frac{3}{2} B_0 \right) C_1 C_2 + B_0 C_1^4 \right] s^{10} \right\}$$

$$D_3 = -\frac{\lambda \gamma_0 a s^{(1+\lambda)/\lambda}}{5\lambda - \gamma_0 f_3'(\gamma_0)} \left\{ \left( B_2 + \frac{1}{2} B_1 + \frac{3}{8} B_0 \right) s^4 + \frac{5}{16} B_0 C_1 s^6 + \frac{5}{32} B_0 C_1^2 s^8 + \frac{15}{128} \left[ \left( B_1 + \frac{1}{2} B_0 \right) C_2 + \frac{2}{3} B_0 C_1^3 \right] s^{10} \right\}$$

$$D_4 = -\frac{\lambda \gamma_0 a s^{(1+\lambda)/\lambda}}{7\lambda - \gamma_0 f_4'(\gamma_0)} \times \left\{ \left( B_2 + \frac{1}{2} B_2 + \frac{3}{8} B_1 + \frac{5}{16} B_0 \right) s^6 + \frac{35}{128} B_0 C_1 s^8 + \frac{35}{256} B_0 C_1^2 s^{10} \right\}$$

$$D_5 = -\frac{\lambda \gamma_0 a s^{(1+\lambda)/\lambda}}{9\lambda - \gamma_0 f_5'(\gamma_0)} \left\{ \left( B_4 + \frac{1}{2} B_3 + \frac{3}{8} B_2 + \frac{5}{16} B_1 + \frac{35}{128} B_0 \right) s^8 + \frac{63}{256} B_0 C_1 s^{10} \right\}$$

$$D_6 = -\frac{\lambda \gamma_0 a s^{(1+\lambda)/\lambda}}{11\lambda - \gamma_0 f_6'(\gamma_0)} \left\{ \left( B_5 + \frac{1}{2} B_4 + \frac{3}{8} B_3 + \frac{5}{16} B_2 + \frac{35}{128} B_1 + \frac{63}{256} B_0 \right) s^{10} \right\}$$

$$D_k = 0, \quad k = 7, 8, 9, \dots$$

Comparing these equations with (66), the  $d_{k,j}$  may be read off so that  $g(\zeta)$  may be computed from (50, 67) to the same order of accuracy, should the solution in the elastic region be desired. Also, the accuracy of the plastic region solution may be improved through adding on terms of order  $s^{12}$ ,  $s^{14}$ , and so on, after computing the required  $d_{k,6}$ ,  $d_{k,7}$ , and so on, from (64); however, the foregoing equations are sufficiently accurate except when the remotely applied strain,  $\gamma_\infty$ , is very near the initial yield strain,  $\gamma_0$ , so that  $s$  is near unity (all of our series diverge at  $s = 1$ , as may be inferred physically from the fact that the elastic-plastic boundary then extends to infinity).



## Small-Scale Yielding Near Cracks and Notches

When the dimensionless remotely applied stress,  $s = (\tau_\infty/\tau_0)^\lambda$ , is small in comparison to unity so that  $s^2$  is negligible, a state of small-scale yielding exists for which the dimensions of the plastic zone are negligible in comparison to notch depth,  $a$ . In this case, the solution given in the last section reduces to that offered by Neuber [9], with the important difference that the solution of [9] did not clearly relate a constant introduced to the applied load and notch geometry. Neglecting all terms in (68) of order  $s^2$  or higher in comparison to unity, one has

$$D_1 = -\frac{\lambda\gamma_0 a B_0 s^{(1+\lambda)/\lambda}}{\lambda - \gamma_0 f_1'(\gamma_0)} \quad (69)$$

$$D_k = 0, \quad k = 2, 3, 4 \dots$$

so that, by (44),

$$\psi = -\frac{\lambda\gamma_0 a B_0 s^{(1+\lambda)/\lambda}}{\lambda - \gamma_0 f_1'(\gamma_0)} f_1(\gamma) \sin \lambda\phi \quad (70)$$

Thus, through (23), physical coordinates  $(x, y)$  in the plastic region corresponding to a given strain,  $\gamma$ , with direction at angle  $\phi$  with the  $y$ -axis, are for small-scale yielding

$$x = \frac{\lambda\gamma_0 a B_0 s^{(1+\lambda)/\lambda}}{\lambda - \gamma_0 f_1'(\gamma_0)} \left[ \frac{\lambda f_1(\gamma)}{\gamma} \cos \phi \cos \lambda\phi + f_1'(\gamma) \sin \phi \sin \lambda\phi \right] \quad (71)$$

$$y = \frac{\lambda\gamma_0 a B_0 s^{(1+\lambda)/\lambda}}{\lambda - \gamma_0 f_1'(\gamma_0)} \left[ \frac{\lambda f_1(\gamma)}{\gamma} \sin \phi \cos \lambda\phi - f_1'(\gamma) \cos \phi \sin \lambda\phi \right]$$

Setting  $\gamma = \gamma_0$ , these give parametrically, in terms of  $\phi$ , the position of the elastic plastic boundary. Noting that, on the  $x$ -axis ahead of the notch tip, one has  $\phi = 0$ , strain may be related to distance from tip by the first of (71).

One may inquire as to which, if any, hardening stress-strain relations  $\tau = \tau(\gamma)$  employed in our deformation plasticity theory will lead to a solution which is also correct for an incremental theory. This is the case if, at points in the plastic zone, the direction of the strain vector is independent of applied load. This means that, upon solving for  $\phi$  in terms of  $x$  and  $y$ , the resulting expression is independent of  $s$ . Examining the small-scale yielding solution of (71), one easily justifies that this requirement is equivalent to having  $\phi$  depend only on the ratio  $x/y$  so that, conversely, the ratio  $x/y$  from equations (71) must be independent of  $\gamma$ . One then may show that  $x/y$  does not depend on  $\gamma$ , if, and only if,  $f_1(\gamma)$  is proportional to some power of  $\gamma$ . Substituting into (46), one finds that the small-scale yielding solution (71) is an incremental plasticity theory solution only if the work-hardening stress-strain relation is of the form

$$\tau(\gamma) = \tau_0 \left( \frac{\gamma}{\gamma_0} \right)^N, \quad \text{for } \gamma > \gamma_0 \quad (72)$$

Clearly, any stress-strain relation leading to agreement with incremental theory when yielding is not on a small scale must also agree for small-scale yielding, so that none but power laws of the form (72) has to be considered. Upon examining equations of the last section, one finds agreement beyond the small-scale yielding occurs only for  $N = 0$  (the case of perfect plasticity for which the present solution may be shown to agree with those of [5, 8]) and  $N = 1$  (the linear elastic case). However, as will be pointed out subsequently, the dominant notch tip strain singularity depends only on  $f_1(\gamma)$  and, although agreement with incremental theory does not occur everywhere in the plastic zone for  $N \neq 0, 1$ , the dominant singular strain term does give  $\phi$  to be independent

of  $s$ . Thus a good approximation to an incremental solution is expected, even for large-scale yielding, when a work-hardening power law stress-strain relation is employed.

### Small-Scale Yielding Near a Crack

When the notch angle  $\alpha = 0$ , one has a crack of length  $a$  subjected to a uniform stress field. In this case, by (31),  $\lambda = 1$ , and it is found that equations (45, 46) may be solved for  $f_1(\gamma)$ , which appears in (70, 71), in terms of a single quadrature. It may be verified by substitution that the solution is

$$f_1(\gamma) = \frac{\gamma}{\gamma_0} \left[ \int_{\gamma_0}^{\infty} \frac{du}{u^2 \tau(u)} \right]^{-1} \int_{\gamma}^{\infty} \frac{du}{u^2 \tau(u)} \quad (73)$$

Substituting this result into (70), setting  $\lambda = 1$  and, by (54),  $B_0 = 1$ ,

$$\psi = -\gamma_0 \tau_0 s^2 a \left[ \gamma \int_{\gamma}^{\infty} \frac{du}{u^2 \tau(u)} \right] \sin \phi \quad (74)$$

It is of interest to note [1, 3] that

$$K_w = \tau_\infty \sqrt{\pi a} = \tau_0 s \sqrt{\pi a} \quad (75)$$

is the elastic stress intensity factor for the present crack problem in the sense that the linear elastic solution has a crack tip stress singularity of the form

$$\tau_y - i\tau_x = \frac{K_w}{[2\pi(x - iy)]^{1/2}} \quad (76)$$

Comparing (75) and (74), it is seen that the plastic solution depends on remotely applied stress and crack length only through the square of the elastic stress intensity factor. It is now shown that the small-scale yielding solution for any longitudinal shear crack problem, involving loads symmetrical with respect to the crack line, depends only on the elastic stress intensity factor appropriate for the particular crack and external boundary geometry and manner of load application considered. The complete linear elastic solution to such problems contains [1, 3] a singular term of the form given by (76) to which other terms, all vanishing at the crack tip, must be added for a complete solution. As the plastic zone is presumed negligible in comparison to geometric dimensions, on the scale of lengths comparable to plastic zone dimensions, the crack appears to be semi-infinite, as in Fig. 7(a), and the appropriate boundary conditions are that the elastic solution (76) is approached for large  $|x + iy|$ . The map into the strain plane of the semi-infinitely slit physical plane is shown in Fig. 7(b), along with appropriate boundary conditions of  $\partial\psi/\partial\phi = 0$  at  $\phi = \pm\pi/2$ .

Since  $\tau_y - i\tau_x = (\tau_0/\gamma_0)\gamma e^{i\phi}$  in the elastic region  $\gamma < \gamma_0$  and  $-x + iy = e^{-i\phi} \left( \frac{1}{\gamma} \frac{\partial\psi}{\partial\phi} + i \frac{\partial\psi}{\partial\gamma} \right)$ , the asymptotic condition

$$\tau_y - i\tau_x \rightarrow \frac{K_w}{[2\pi(x - iy)]^{1/2}} \quad \text{as } |x + iy| \rightarrow \infty \quad (77)$$

is equivalent to

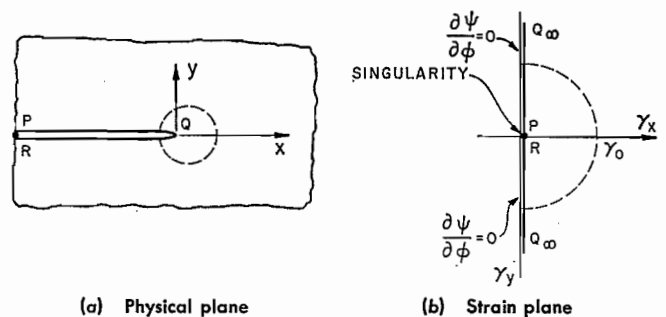


Fig. 7 For small-scale yielding near a crack tip

$$\frac{1}{\gamma} \frac{\partial \psi}{\partial \phi} + i \frac{\partial \psi}{\partial \gamma} \rightarrow -\frac{K_w^2 \gamma_0^2}{2\pi \tau_0^2 \gamma^2} e^{-i\phi} \text{ as } \gamma \rightarrow 0 \quad (78)$$

Integrating (78), it is then required that  $\psi$  have a singularity of the form

$$\psi \rightarrow -\frac{K_w^2 \gamma_0^2 \sin \phi}{2\pi \tau_0^2 \gamma} \text{ as } \gamma \rightarrow 0 \quad (79)$$

A solution for  $\psi$  satisfying the governing equation (24), boundary conditions in Fig. 7(b), and having the required singularity is

$$\psi = -\frac{K_w^2 \gamma_0^2}{2\pi \tau_0^2} \left\{ \frac{1}{\gamma} + \frac{\gamma}{\gamma_0^2} \left[ 2\gamma_0 \tau_0 \int_{\gamma_0}^{\infty} \frac{du}{u^2 \tau(u)} - 1 \right] \right\} \sin \phi, \quad \gamma < \gamma_0 \quad (80)$$

$$\psi = -\frac{\gamma_0}{\pi \tau_0} K_w^2 \left[ \gamma \int_{\gamma}^{\infty} \frac{du}{u^2 \tau(u)} \right] \sin \phi, \quad \gamma > \gamma_0$$

is the solution. When the value of  $K_w$  is that given by (76), the last of equation (80) is identical to (75), and the generality of dependence of small-scale yielding solutions on elastic stress intensity factors is established. This sort of result always has been tacitly assumed in Irwin's [1] approach to fracture mechanics; a similar result for a tensile yielding model has been demonstrated by Rice in [12].

Equations (23) combined with the last of (80) yield, for physical coordinates corresponding to strains in the plastic region,

$$x = \frac{K_w^2}{2\pi \tau_0^2} \left\{ \frac{\gamma_0 \tau_0}{\gamma \tau(\gamma)} \cos 2\phi + \left[ 2\gamma_0 \tau_0 \int_{\gamma}^{\infty} \frac{du}{u^2 \tau(u)} - \frac{\gamma_0 \tau_0}{\gamma \tau(\gamma)} \right] \right\} \quad (81)$$

$$y = \frac{K_w^2}{2\pi \tau_0^2} \frac{\gamma_0 \tau_0}{\gamma \tau(\gamma)} \sin 2\phi$$

Subtracting the term independent of  $\phi$  from the first of (81), squaring both, and adding, the equations of lines in the plastic region along which the strain and stress have the constant magnitudes  $\gamma$  and  $\tau(\gamma)$  are circles

$$[x - X(\gamma)]^2 + y^2 = [R(\gamma)]^2 \quad (82)$$

centered at the distance

$$X(\gamma) = \frac{K_w^2}{2\pi \tau_0^2} \left[ 2\gamma_0 \tau_0 \int_{\gamma}^{\infty} \frac{du}{u^2 \tau(u)} - \frac{\gamma_0 \tau_0}{\gamma \tau(\gamma)} \right] \quad (83)$$

ahead of the crack tip, and with radius

$$R(\gamma) = \frac{K_w^2}{2\pi \tau_0^2} \frac{\gamma_0 \tau_0}{\gamma \tau(\gamma)} \quad (84)$$

Similarly, the elastic-plastic boundary is also a circle with center and radius

$$X(\gamma_0) = \frac{K_w^2}{2\pi \tau_0^2} \left[ 2\gamma_0 \tau_0 \int_{\gamma_0}^{\infty} \frac{du}{u^2 \tau(u)} - 1 \right], \quad R(\gamma_0) = \frac{K_w^2}{2\pi \tau_0^2} \quad (85)$$

It is of interest to observe that the radius of the plastic zone is independent of the work-hardening stress-strain relation for small-scale yielding. The plastic region extends a distance  $X(\gamma_0) + R(\gamma_0)$  ahead of the crack tip on the  $x$ -axis and a distance  $R(\gamma_0) - X(\gamma_0)$  behind. Division of equations (81) leads to

$$\frac{y}{x - X(\gamma)} = \tan 2\phi \quad (86)$$

so that, if a given point  $(x, y)$  has a strain magnitude  $\gamma$  according to (82), the corresponding angle  $\phi$  between the strain vector and the  $y$ -axis is one half of the angle made with the  $x$ -axis by the line from  $(X(\gamma), 0)$  to  $(x, y)$ . These geometrical properties of the plastic region strain field are shown in Fig. 8, where the elastic-plastic boundary appears as a dashed-line circle and a line of constant strain magnitude  $\gamma (> \gamma_0)$  as a solid-line nonconcentric circle. Physical coordinates corresponding to strains in the elastic region may be computed from the solution for  $\psi$  in the first equation of (81). Omitting details of calculation, one may show that the resulting relation between stresses and coordinates is

$$\tau_y - i\tau_x = \frac{K_w}{\{2\pi[x - X(\gamma_0) - iy]\}^{1/2}} \quad (87)$$

Comparing with (77), it is seen that stresses in the elastic region for the elastic-plastic case are given by the original linear elastic solution, but for a longer crack having its tip at the center of the plastic zone  $(X(\gamma_0), 0)$  instead of  $(0, 0)$ . Irwin and Koskinen [13] and Rice [8] have noted a similar result for the perfectly plastic case. It follows from (87) that lines of constant strain magnitude  $\gamma (< \gamma_0)$  are circles concentric with the elastic-plastic

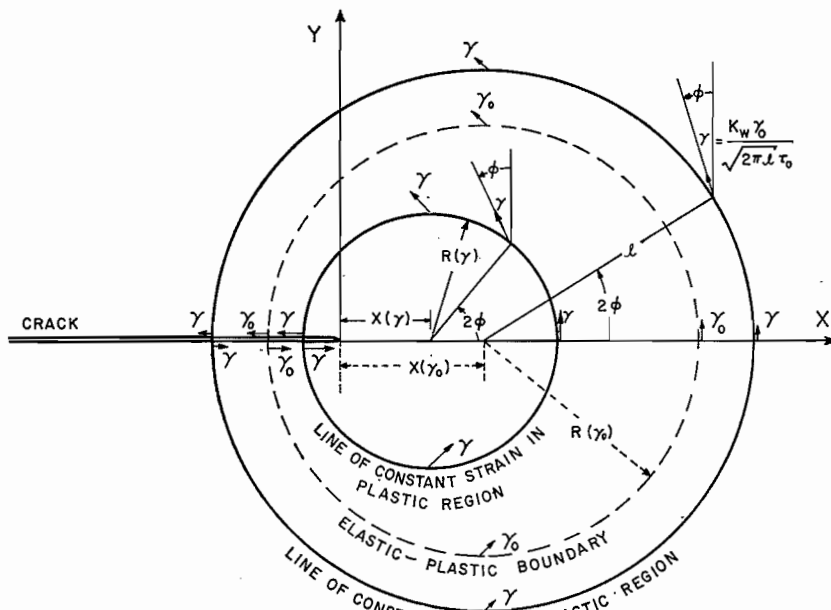


Fig. 8 Geometry of small-scale yielding near a crack tip

**Table 1** Tabulated values of  $D_i$  for crack in power-law-hardening materials

$\tau_w/\tau_0$	$-D_1/\gamma_0^a$	$-D_2/\gamma_0^a$	$-D_3/\gamma_0^a$	$-D_4/\gamma_0^a$	$-D_5/\gamma_0^a$	$-D_6/\gamma_0^a$
<b>N = 0</b>						
.2	.0408	.0004	0	0	0	0
.4	.1740	.0070	.0006	.0001	0	0
.6	.4414	.0398	.0071	.0016	.0004	.0001
.8	.9735	.1553	.0489	.0186	.0078	.0028
<b>N = .1</b>						
.2	.0370	.0004	0	0	0	0
.4	.1557	.0057	.0004	0	0	0
.6	.3851	.0318	.0055	.0012	.0003	.0001
.8	.8059	.1182	.0363	.0139	.0058	.0022
<b>N = .3</b>						
.2	.0311	.0003	0	0	0	0
.4	.1286	.0046	.0004	0	0	0
.6	.3072	.0249	.0044	.0010	.0002	.0001
.8	.6022	.0869	.0270	.0105	.0046	.0019
<b>N = .5</b>						
.2	.0268	.0002	0	0	0	0
.4	.1096	.0040	.0003	0	0	0
.6	.2556	.0212	.0037	.0008	.0002	.0001
.8	.4810	.0709	.0223	.0088	.0039	.0017

**Table 2** Tabulated values of  $D_i$  for 45 deg V-notch in power-law hardening materials

$\tau_w/\tau_0$	$-D_1/\gamma_0^a$	$-D_2/\gamma_0^a$	$-D_3/\gamma_0^a$	$-D_4/\gamma_0^a$	$-D_5/\gamma_0^a$	$-D_6/\gamma_0^a$
<b>N = 0</b>						
.2	.0248	.0001	0	0	0	0
.4	.1297	.0026	.0001	0	0	0
.6	.3670	.0221	.0028	.0004	.0001	0
.8	.8804	.1156	.0311	.0103	.0037	.0012
<b>N = .1</b>						
.2	.0220	.0001	0	0	0	0
.4	.1141	.0021	.0001	0	0	0
.6	.3157	.0175	.0021	.0003	.0001	0
.8	.7160	.0862	.0227	.0075	.0027	.0009
<b>N = .3</b>						
.2	.0184	.0001	0	0	0	0
.4	.0942	.0018	.0001	0	0	0
.6	.2535	.0139	.0017	.0003	0	0
.8	.5405	.0645	.0171	.0057	.0021	.0008
<b>N = .5</b>						
.2	.0160	0	0	0	0	0
.4	.0814	.0015	.0001	0	0	0
.6	.2151	.0121	.0015	.0002	0	0
.8	.4423	.0537	.0143	.0048	.0018	.0007

boundary, such that, if  $l = \{[x - X(\gamma_0)]^2 + y^2\}^{1/2}$  is the distance from the center of the plastic zone to a given point  $(x, y)$ , the corresponding strain magnitude is

$$\gamma = \frac{\gamma_0}{\tau_0} \frac{K}{(2\pi l)^{1/2}} \quad (88)$$

The angle  $\phi$  between the strain vector and  $y$ -axis is one half the angle made with the  $x$ -axis by the line drawn from the center  $(X(\gamma_0), 0)$  of the plastic zone to the point  $(x, y)$ . These geometrical properties of the small-scale yielding solution in the elastic region are also shown in Fig. 8.

Of some interest for fracture mechanics is the relation between stress or strain at a point directly ahead of the crack tip on the  $x$ -axis and distance,  $x$ , from the crack tip to the point under consideration. Noting that  $\phi = 0$  ahead of the crack, from (81) and (88), the relation is

$$\begin{aligned} x &= \frac{K_w^2}{\pi\tau_0^2} \gamma_0 \int_{\gamma_0}^{\infty} \frac{du}{u^2\tau(u)} = X(\gamma) + R(\gamma) \\ &= \frac{K_w^2}{\pi\tau_0^2} \left\{ \frac{\tau_0\gamma_0}{\tau\gamma(\tau)} - \tau_0\gamma_0 \int_{\tau}^{\infty} \frac{dt}{t^2\gamma(t)} \right\}, \quad \gamma > \gamma_0 \\ x &= \frac{K_w^2}{\pi\tau_0^2} \left\{ \gamma_0\tau_0 \int_{\gamma_0}^{\infty} \frac{du}{u^2\tau(u)} + \frac{1}{2} \left[ \left( \frac{\gamma_0}{\gamma} \right)^2 - 1 \right] \right\} \\ &= \frac{K_w^2}{\pi\tau_0^2} \left\{ 1 - \tau_0\gamma_0 \int_{\tau_0}^{\infty} \frac{dt}{t^2\gamma(t)} + \frac{1}{2} \left[ \left( \frac{\tau_0}{\tau} \right)^2 - 1 \right] \right\}, \quad \gamma < \gamma_0 \end{aligned} \quad (89)$$

with the latter forms appropriate if one prefers to work in terms of stress, as permissible except when a bounded stress accompanies infinite strain.

Employing (26) and arbitrarily setting the displacement  $w = 0$  at the crack tip, the displacement at  $x = -[R(\gamma_0) - X(\gamma_0)]$ ,

where the elastic-plastic boundary intersects the upper crack surface, is

$$w_0 = \frac{\gamma_0 K_w^2}{\pi\tau_0^2} \quad (90)$$

and is independent of the hardening stress-strain relation employed.

### Some Particular Examples

If one assumes a power law

$$\tau(\gamma) = \tau_0 \left( \frac{\gamma}{\gamma_0} \right)^N \quad (91)$$

in the work-hardening range, the small-scale yielding solution is a correct incremental solution. As  $N$  varies from zero to unity, (91) passes from a description of perfect plasticity to a description of perfect elasticity. For  $N$  greater than unity, (90) describes a material of the locking type. The parameters  $X(\gamma)$ ,  $R(\gamma)$  describing the geometry of the plastic region in accord with the foregoing discussion and Fig. 8 are

$$\begin{aligned} X(\gamma) &= \frac{1 - N}{1 + N} \frac{K_w^2}{2\pi\tau_0^2} \left( \frac{\gamma_0}{\gamma} \right)^{N+1}, \\ R(\gamma) &= \frac{K_w^2}{2\pi\tau_0^2} \left( \frac{\gamma_0}{\gamma} \right)^{N+1} \end{aligned} \quad (92)$$

and the circular plastic zone extends a distance  $R(\gamma_0) + X(\gamma_0) = K_w^2[(1 + N)\pi\tau_0^2]$  ahead of and  $R(\gamma_0) - X(\gamma_0) = NK_w^2/[(1 + N)\pi\tau_0^2]$  behind the crack tip. From (89), distance ahead of the crack tip and strains are related, in the plastic zone, by

$$x = \frac{1}{N + 1} \frac{K_w^2}{\pi\tau_0^2} \left( \frac{\gamma_0}{\gamma} \right)^{N+1} \quad (93)$$

so that, for  $x > 0$  and in the plastic region,

$$\gamma_w(x, 0) = \gamma_0 \left[ \frac{K_w^2}{(N + 1)\pi\tau_0^2 x} \right]^{1/(N+1)} \quad (94)$$

**Table 3** Tabulated values of  $D_i$  for 90 deg V-notch in power-law-hardening materials

$\tau_w/\tau_0$	$-D_1/\gamma_0^a$	$-D_2/\gamma_0^a$	$-D_3/\gamma_0^a$	$-D_4/\gamma_0^a$	$-D_5/\gamma_0^a$	$-D_6/\gamma_0^a$
<b>N = 0</b>						
.2	.0089	0	0	0	0	0
.4	.0721	.0004	0	0	0	0
.6	.2571	.0072	.0004	0	0	0
.8	.7216	.0656	.0129	.0032	.0009	.0002
<b>N = .1</b>						
.2	.0077	0	0	0	0	0
.4	.0618	.0003	0	0	0	0
.6	.2168	.0056	.0003	0	0	0
.8	.5769	.0483	.0093	.0023	.0006	.0002
<b>N = .3</b>						
.2	.0064	0	0	0	0	0
.4	.0511	.0003	0	0	0	0
.6	.1766	.0046	.0003	0	0	0
.8	.4471	.0374	.0072	.0018	.0005	.0001
<b>N = .5</b>						
.2	.0056	0	0	0	0	0
.4	.0449	.0002	0	0	0	0
.6	.1534	.0041	.0003	0	0	0
.8	.3378	.0320	.0062	.0015	.0004	.0001

$$\tau_y(x, 0) = \tau_0 \left[ \frac{K_w^2}{(N+1)\pi\tau_w^2 x} \right]^{N/(N+1)} \quad (94) \quad (\text{Cont.})$$

For the assumption of a bilinear relation between stress and strain, in the work-hardening range

$$\tau(\gamma) = \tau_0 + \epsilon \frac{\tau_0}{\gamma_0} (\gamma - \gamma_0) \quad (95)$$

where  $\epsilon$  is the ratio of the work-hardening shear modulus to elastic shear modulus. Here one has

$$X(\gamma) = \frac{K_w^2}{2\pi\tau_0^2} \left\{ \frac{2\gamma_0}{(1-\epsilon)\gamma} - \frac{\gamma_0^2}{\gamma[(1-\epsilon)\gamma_0 + \epsilon\gamma]} + \frac{2\epsilon}{(1-\epsilon)^2} \ln \left[ \frac{\epsilon\gamma}{(1-\epsilon)\gamma_0 + \epsilon\gamma} \right] \right\} \quad (96)$$

$$R(\gamma) = \frac{K_w^2}{2\pi\tau_0^2} \frac{\gamma_0^2}{\gamma[(1-\epsilon)\gamma_0 + \epsilon\gamma]}$$

with the plastic zone extending a distance  $R(\gamma_0) + X(\gamma_0) = [1 + \epsilon(1-\epsilon)^{-1} \ln \epsilon] K_w^2 / [(1-\epsilon)\pi\tau_0^2]$  ahead of the crack tip. Considerable difficulties accompany an attempt to solve for  $\gamma$  in terms of  $x$  from (89); however, one may determine the nature of the crack tip singularity. When  $\epsilon \neq 0$ , a calculation shows

$$\frac{2\pi\epsilon\tau_0^2\gamma^2}{K_w^2\gamma_0^2} [X(\gamma) + R(\gamma)] \rightarrow 1 \quad \text{as } \gamma \rightarrow \infty,$$

so that for  $x = X(\gamma) + R(\gamma)$  very small [that is, for large  $\gamma$  or more precisely,  $\gamma \gg (1-\epsilon)\gamma_0/\epsilon$ ], one has

$$\begin{aligned} \gamma_y(x, 0) &\rightarrow \frac{\gamma_0 K_w}{\tau_0 \sqrt{2\pi \epsilon x}} \\ \tau_y(x, 0) &\rightarrow \frac{\sqrt{\epsilon} K_w}{\sqrt{2\pi x}} \end{aligned} \quad \text{as } x \rightarrow 0 \quad (\text{if } \epsilon \neq 0) \quad (97)$$

### Power-Law-Hardening Materials—Numerical Results

When a power law,  $\tau(\gamma) = \tau_0(\gamma/\gamma_0)^N$ , relates stress to strain in the work-hardening range, the solution of equations (45, 46) for  $f_k(\gamma)$  is

$$f_k(\gamma) = \left( \frac{\gamma}{\gamma_0} \right)^{-\mu_k} \quad (98)$$

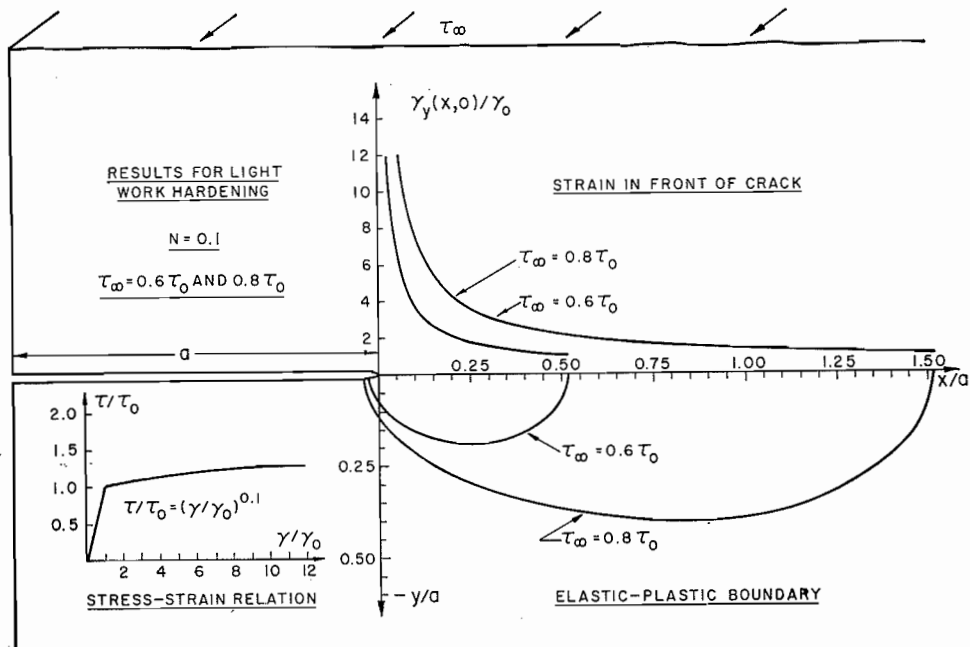
where

$$\mu_k = \left( \frac{(1-N)^2}{4} + (2k-1)^2 \lambda^2 N \right)^{1/2} - \frac{(1-N)}{2} \quad (99)$$

The  $C_k$  defined by (52) are

$$C_k = \frac{(2k-1)\lambda - \mu_k}{(2k-1)\lambda + \mu_k} \quad (100)$$

Substituting (100) into equations (68), the constants  $D_1, D_2, \dots, D_6$  were numerically determined in the dimensionless form



**Fig. 9**

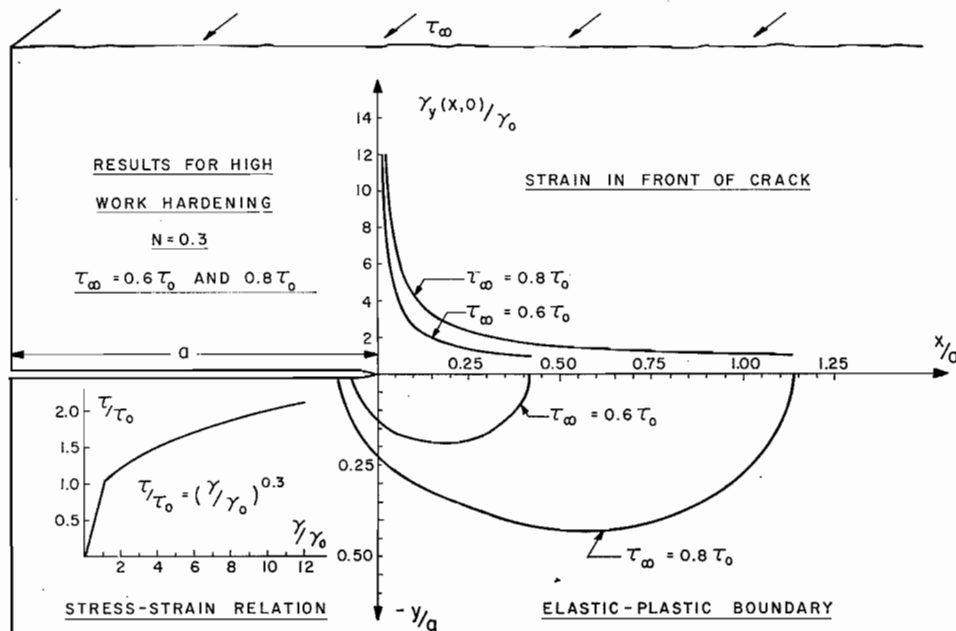


Fig. 10

$-D_k/\gamma_0 a$ , as a function of  $\tau_\infty/\tau_0$ , for various values of notch angle  $2\alpha$  and hardening exponent  $N$  [recall, to the order of accuracy of (68),  $D_7 = D_8 = \dots = 0$ ]. Tables 1, 2, and 3, respectively, tabulate results for: (a) A crack ( $\alpha = 0$ ); (b) a 45 deg V-notch ( $\alpha = 22.5$  deg); and (c) a 90 deg V-notch ( $\alpha = 45$  deg). With a bit of interpolation within and between tables, the reader may determine the  $D_k$  for a wide variety of problems and, through (44), (23), and (98), fill in all details of interest for the plastic region solution.

Values from Table 1 were employed in preparing Figs. 9 and 10 which show the solution for a crack in a lightly hardening ( $N = 0.1$ ) and highly hardening ( $N = 0.3$ ) material, respectively. Two stress levels,  $\tau_\infty = 0.6\tau_0$  and  $0.8\tau_0$ , are considered in each figure as these nicely demonstrate the transition from the circular plastic zones of small-scale yielding to the highly elongated zones of large-scale yielding. Stress-strain curves appear at the lower left, positions of the elastic-plastic boundary in dimensionless coordinates  $x/a$  and  $y/a$  at the lower right, and graphs of dimensionless principal shear strain  $\gamma_y(x, 0)/\gamma_0$  along the  $x$ -axis ahead of the crack tip at the upper right.

The general plastic region, expression (44), for the potential function  $\psi$ , with strain plane gradient equal (19) to physical coordinates, may be put in the form

$$\psi = D_1 \left(\frac{\gamma}{\gamma_0}\right)^{-\mu_1} \left\{ \sin \lambda \phi + \sum_{k=2}^{\infty} \frac{D_k}{D_1} \left(\frac{\gamma}{\gamma_0}\right)^{-(\mu_k - \mu_1)} \sin (2k - 1)\lambda \phi \right\} \quad (101)$$

for power-law-hardening, after employing (98). Since  $\mu_k - \mu_1 > 0$  from (99) for  $k = 2, 3, 4, \dots$ , when  $\gamma$  is extremely large compared to  $\gamma_0$  (and thus corresponding physical coordinates very near the notch tip), the terms inside the summation of (101) are negligible in comparison to unity, and the bracketed term is essentially equal to  $\sin \lambda \phi$ . Thus, very near the crack tip,

$$\psi \rightarrow D_1 \left(\frac{\gamma}{\gamma_0}\right)^{-\mu_1} \sin \lambda \phi \quad (102)$$

But this limiting form is equation (70) for  $\psi$  as appropriate for small-scale yielding, except that now  $D_1$  is given by (68) instead of (69), and the crack tip strain singularity has the same functional form as for small-scale yielding. Since power-law-hardening resulted in radial loading in the small-scale yielding plastic region, we conclude that the dominant (singular term) crack tip

strains always correspond to radial loading, and thus the deformation theory is expected to closely approximate an incremental solution even at high stress levels for a power-law-hardening material.

A generalization of the foregoing argument on the dependence of the notch tip singularity only on the first term of (44) is of some interest in developing a fracture criterion.

### Acknowledgment

Work reported here was performed through the financial support of a National Science Foundation Engineering Research Initiation Grant. This support is gratefully acknowledged.

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