

First-Occurrence Time of High-Level Crossings in a Continuous Random Process

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This paper deals with the statistical distribution of the first-occurrence and first-recurrence times of the crossing of a given level in a continuous random process. Approximate forms of the first-occurrence and first-recurrence time densities are found by considering the successive crossings to form a renewal process. A relatively simple exponential distribution is found to give an appropriate representation of the limiting case when the crossings of the level under consideration are statistically rare events. Numerical examples are worked out for some stationary Gaussian processes. The method is of use in evaluating survival probabilities for randomly excited mechanical systems subject to failure upon occurrence of a sufficiently high load.

INTRODUCTION

A MECHANICAL system subjected to a random loading may fail when the stress in a critical member reaches a sufficiently high level. This type of failure is generally by fracture or by excessive permanent deformation rendering the system inoperative. If the stress has a finite probability of exceeding the high level, then failure is possible, and an important problem is to find the probability that the system can operate without failure for some given time.

More precisely, the following problem is considered. Given a continuous and differentiable random function $x(t)$, one wishes to find the probability that the value $x = \alpha$ will not be exceeded in the time interval $(0, t)$. This problem is called the first-occurrence time problem and the probability density $p_o(\alpha, t)$ is the first-occurrence density, in the sense that $p_o(\alpha, t)dt$ is the probability, given $x(0) < \alpha$, that $x(t)$ first crosses the level $x = \alpha$ in the time interval $(t, t + dt)$. If failure is defined as the first exceedance of $x = \alpha$, the probability of failure $P_f(\alpha, T)$ in time T may be expressed in terms of the first-occurrence density $p_o(\alpha, t)$. The probability of failure in $(0, T)$ is unity if $x(0) > \alpha$, and the probability of failure in $(0, T)$ is $\int_0^T p_o(\alpha, t)dt$ if $x(0) < \alpha$. Thus, where ϵ_α is the prob-

ability that $x(0) > \alpha$,

$$P_f(\alpha, T) = \epsilon_\alpha + (1 - \epsilon_\alpha) \int_0^T p_o(\alpha, t)dt. \quad (1)$$

For stationary processes, the first-occurrence time density is closely related to the first-recurrence time probability density $p_r(\alpha, \tau)$, where $p_r(\alpha, \tau)d\tau$ is the probability, given $x(t_0) = \alpha$ and $\dot{x}(t_0) < 0$, that the next crossing of $x = \alpha$ occurs in the time interval $(t_0 + \tau, t_0 + \tau + d\tau)$. Thus, $p_r(\alpha, \tau)$ is the density for the time between successive downward and upward crossings of $x = \alpha$. To show the relation between the first-occurrence

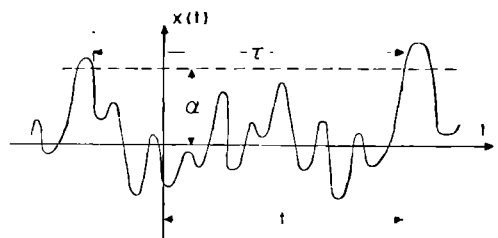


FIG. 1. First-occurrence and first-recurrence times.

and first-recurrence densities, consider a stationary random function $x(t)$ and suppose that $x(0) < \alpha$. With reference to Fig. 1, the time t at which $x = \alpha$ is first crossed is the first-occurrence time. Since it is given that $x(0) < \alpha$, the origin $t = 0$ falls in a time interval between a downward and an upward crossing of $x = \alpha$. Referring again to Fig. 1, this interval has duration τ , where τ is a first-recurrence time. The probability density for the first-occurrence time may be written, by the law of conditional probability, as

$$p_0(\alpha, t) = \int_0^\infty p_0(\alpha, t | \tau) q(\alpha, \tau) d\tau, \quad (2)$$

where $p_0(\alpha, t | \tau)$ is the first-occurrence time density, given that the recurrence time interval including the origin is of duration τ , and where $q(\alpha, \tau) d\tau$ is the probability that the recurrence time interval including the origin has a duration between τ and $\tau + d\tau$.

Changing the point of view slightly, consider the random process of Fig. 1 as a fixed curve and suppose that the time axis is attached to the curve so that the origin $t = 0$ has uniform probability of falling at any point where x is less than α . Then, if it is given that $t = 0$ falls in an interval of length τ , the location of the point $t = 0$ is uniformly distributed on the interval τ , and the time t to the end of the interval τ (that is, the first-occurrence time) has also a uniform distribution. Thus,

$$p_0(\alpha, t | \tau) = 1/\tau \quad \text{if } t < \tau \\ = 0 \quad \text{if } t > \tau. \quad (3)$$

The quantity $q(\alpha, \tau) d\tau$ is the fraction of the time axis (for which $x < \alpha$) taken up by recurrence intervals between τ and $\tau + d\tau$. The fraction of such intervals is $p_r(\alpha, \tau) d\tau$ and, since the duration of each such interval is τ , $q(\alpha, \tau)$ is proportional to $\tau p_r(\alpha, \tau)$. Normalizing,

$$q(\alpha, \tau) = (\tau / \langle \tau \rangle_{av}) p_r(\alpha, \tau), \quad (4)$$

where $\langle \tau \rangle_{av}$ is the mean recurrence time (or the average time between successive downward and upward crossings of $x = \alpha$). Inserting Eqs. 3 and 4 in Eq. 2, the relation between the first-occurrence and first recurrence densities becomes

$$p_0(\alpha, t) = \frac{1}{\langle \tau \rangle_{av}} \int_0^\infty p_r(\alpha, \tau) d\tau = \frac{1}{\langle \tau \rangle_{av}} \left[1 - \int_0^t p_r(\alpha, \tau) d\tau \right]. \quad (5)$$

There is a simple relation between moments of $p_0(\alpha, t)$

and $p_r(\alpha, \tau)$. By direct calculation,

$$\begin{aligned} \langle t^n \rangle_{av} &= \int_0^\infty t^n p_0(\alpha, t) dt = \frac{1}{\langle \tau \rangle_{av}} \int_0^\infty \int_t^\infty t^n p_r(\alpha, \tau) d\tau dt \\ &= \frac{1}{\langle \tau \rangle_{av}} \int_0^\infty \int_0^\tau t^n p_r(\alpha, \tau) dt d\tau \\ &= \frac{1}{(n+1) \langle \tau \rangle_{av}} \int_0^\infty \tau^{n+1} p_r(\alpha, \tau) d\tau \\ &= \frac{\langle \tau^{n+1} \rangle_{av}}{(n+1) \langle \tau \rangle_{av}}. \end{aligned} \quad (6)$$

It is interesting to note the forms of $p_0(\alpha, t)$ and $p_r(\alpha, \tau)$ for a random process in which the probability of an upward crossing of $x = \alpha$ is independent of the past history of the process. In this case, $p_0(\alpha, t) = p_r(\alpha, t)$, since these differ merely by being conditioned on different past events that are here irrelevant. In this case, Eq. 5 becomes a simple integral equation with the readily verified solution $p_0(\alpha, t) = p_r(\alpha, t) = \langle \tau \rangle_{av}^{-1} \exp(-t / \langle \tau \rangle_{av})$.

The expected number of upward crossings N_α^+ of $x = \alpha$ per unit time appears frequently in the work to follow. This is given as¹

$$N_\alpha^+(t) = \int_0^\infty v g_{zz}(\alpha, v; t) dv, \quad (7)$$

where $g_{zz}(u, v; t)$ is the joint probability density of $x(t)$ and $\dot{x}(t)$, represented, respectively, by u and v . Clearly, for stationary processes, $N_\alpha^+(t)$ is independent of t , and in such cases the notation N_α^+ is used.

The solution of first-occurrence and first-recurrence problems is a rather difficult matter, and a tractable exact solution is known² only when $x(t)$ is a Markov process. Apparently, work done to date on occurrence and recurrence problems for non-Markov processes has dealt primarily with the determination of the interval distribution between successive zero (or mean) crossings. Of particular interest is the work by Slepian³ and certain approximations,^{1,4,5} the latter⁵ containing a com-

¹ S. O. Rice, "Mathematical Analysis of Random Noise," in *Selected Papers on Noise and Stochastic Processes*, N. Wax, Ed. (Dover Publications, Inc., New York, 1954), pp. 133-294.

² A. J. F. Siegert, "On the First Passage Time Probability Problem," *Phys. Rev.* **81**, 617-623 (1951).

³ D. Slepian, "The One Sided Barrier Problem for Gaussian Noise," *Bell System Tech. J.* **41**, 463-501 (1962).

⁴ J. A. McFadden, "The Axis Crossing Intervals of Random Functions-II," *IRE Trans. Inform. Theory* **4**, 14-24 (1958).

⁵ M. S. Longuet-Higgins, "The Distribution of the Intervals between Zeros of a Stationary Random Function," *Phil. Trans. Roy. Soc. (London)* **A254**, 557-599 (1962).

parison of the results of several investigators. The technique used in Sec. I to solve approximately for the recurrence density $p_r(\alpha, \tau)$ is similar to the technique used elsewhere⁴ for the zero-crossing problem. The method of inclusion and exclusion may be used to write an exact expression for $p_r(\alpha, \tau)$, which, while being untractable, does however serve as a starting point for the approximation of the next sections. Following Bartlett's general development⁶ for first-passage times,

$$p_r(\alpha, \tau) = p_{+|-}(\alpha, \tau) - \int_0^\tau p_{+|+|-}(\alpha, r, \tau) dr + \int_0^\tau \int_r^\tau p_{+|++|-}(\alpha, r, s, \tau) ds dr - \int_0^\tau \int_r^\tau \int_s^\tau p_{+|+++|-}(\alpha, r, s, t, \tau) dt ds dr + \dots \tag{8}$$

where $p_{+|+...+|-}(\alpha, r, s, \dots, w, \tau) dr ds \dots dw d\tau$ is the probability, given a downward crossing of $x=\alpha$ at $t=0$, that upward crossings of $x=\alpha$ occur in the time intervals $(r, r+dr), (s, s+ds), \dots, (w, w+dw),$ and $(\tau, \tau+d\tau)$.

I. RENEWAL-PROCESS APPROXIMATION

The calculation of terms beyond the first in the exact expression Eq. 8 above for the first-recurrence time density $p_r(\alpha, \tau)$ is prohibitively difficult and, indeed, the calculation of the general term is impossible, except for the most trivial of random processes. Thus, an approximation must be constructed that yields a tractable result for $p_r(\alpha, \tau)$. For small values of τ , the first term in the series suffices as all remaining terms are small. But for larger values of τ , this method is quite inadequate, and an approximation valid for all time must be found. The procedure used here consists of considering the crossings of $x=\alpha$ to form a renewal process. That is, we approximate the probability of an upward crossing of $x=\alpha$, given several past upward crossings and the downward crossing at $t=0$, by the probability of an upward crossing of $x=\alpha$, given only the last prior upward crossing. When $x(t)$ is a stationary process (as is assumed throughout this section), the renewal approximation results in a considerable simplification of Eq. 8. Further, it seems intuitively clear that for large α , when the upward crossings of $x=\alpha$ are on the average widely spaced in time as compared to the average time between mean crossings, the probability of an upward crossing should depend almost exclusively on the last prior-given upward crossing.

Upon making the renewal approximation, the various integrands of Eq. 8 become

$$p_{+|+|-}(\alpha, r, \tau) \approx p_{+|+}(\alpha, \tau-r) p_{+|-}(\alpha, r) \\ p_{+|++|-}(\alpha, r, s, \tau) \approx p_{+|+}(\alpha, \tau-s) p_{+|+}(\alpha, s-r) p_{+|-}(\alpha, r) \\ \vdots \tag{9}$$

where $p_{+|-}(\alpha, t) dt$ and $p_{+|+}(\alpha, t) dt$ are probabilities of upward crossings of $x=\alpha$ in $(t, t+dt)$, given, respectively, downward and upward crossings of $x=\alpha$ at $t=0$. In view of Eq. 9, the inclusion-exclusion formula Eq. 8 for $p_r(\alpha, \tau)$ becomes

$$p_r(\alpha, \tau) = p_{+|-}(\alpha, \tau) - \int_0^\tau p_{+|+}(\alpha, \tau-r) p_{+|-}(\alpha, r) dr + \int_0^\tau \int_r^\tau p_{+|+}(\alpha, \tau-s) p_{+|+}(\alpha, s-r) \times p_{+|-}(\alpha, r) ds dr - \dots \tag{10}$$

Owing to the renewal-process approximation, all of the integrals appearing in Eq. 10 are convolution integrals. Thus, the Laplace transform of any multiple integral appearing above is the product of the Laplace transforms of the functions in the integrand. Introducing the transforms through the notation

$$f(\alpha, s) = \int_0^\infty p(\alpha, \tau) e^{-s\tau} d\tau, \tag{11}$$

one has

$$f_r(\alpha, s) = f_{+|-}(\alpha, s) - [f_{+|+}(\alpha, s)] f_{+|-}(\alpha, s) + [f_{+|+}(\alpha, s)]^2 f_{+|-}(\alpha, s) - \dots \tag{12}$$

or

$$f_r(\alpha, s) = [f_{+|-}(\alpha, s)] / [1 + f_{+|+}(\alpha, s)].$$

Multiplying both sides of the above equation by the denominator of the right-hand side, inverting the transform, and remembering that a product of transforms inverts into a convolution integral, one obtains

$$p_r(\alpha, \tau) + \int_0^\tau p_r(\alpha, t) p_{+|+}(\alpha, \tau-t) dt = p_{+|-}(\alpha, \tau). \tag{13}$$

Equation 13 has a very simple interpretation and could be written down immediately. In words, given a downward crossing of $x=\alpha$ at $t=0$, the probability of an upward crossing in $(\tau, \tau+d\tau)$ is the sum of the probability that the first crossing occurs in $(\tau, \tau+d\tau)$ and the probability that the first crossing occurs for $t<\tau$, with a later upward crossing in $(\tau, \tau+d\tau)$.

⁶ M. S. Bartlett, *An Introduction to Stochastic Processes* (Cambridge University Press, London, 1960).

Application of the above equations to particular stationary random processes requires a knowledge of $p_{+|-}(\alpha, \tau)$ and $p_{+|+}(\alpha, \tau)$. These may be expressed in terms of the joint density of $x(t)$ and its first derivative at times $t=0$ and $t=\tau$. Let $g(u, v; u', v'; \tau)$ be the joint-probability density of $x(0), \dot{x}(0), x(\tau)$, and $\dot{x}(\tau)$, which are represented by u, v, u' , and v' , respectively. The probability of a downward (or upward) crossing of $x=\alpha$ in a time interval dt is $N_{\alpha}^+ dt$, where N_{α}^+ is (Eq. 7) the expected number of upward crossings of $x=\alpha$ per unit time. Thus, the joint probability of a downward crossing of $x=\alpha$ in an interval dt at $t=0$ and an upward crossing in an interval $d\tau$ at $t=\tau$ is $N_{\alpha}^+ dt p_{+|-}(\alpha, \tau) d\tau$. But this is also given by the probability that $\alpha < x(0) < \alpha + |\dot{x}(0)| dt$ with $\dot{x}(0) < 0$ and that $\alpha - |\dot{x}(\tau)| d\tau < x(\tau) < \alpha$ with $\dot{x}(\tau) > 0$. Thus,

$$N_{\alpha}^+ dt p_{+|-}(\alpha, \tau) d\tau = \int_0^{\infty} \int_{-\infty}^0 \int_{\alpha-|\dot{x}'|d\tau}^{\alpha} \int_{\alpha}^{\alpha+|\dot{x}|dt} g(u, v; u', v'; \tau) du dv u' dv' d\tau. \tag{14}$$

Carrying out the inner two integrations,

$$p_{+|-}(\alpha, \tau) = \frac{1}{N_{\alpha}^+} \int_0^{\infty} \int_{-\infty}^0 |v v'| g(\alpha, v; \alpha, v'; \tau) dv dv'. \tag{15}$$

Similarly,

$$p_{+|+}(\alpha, \tau) = \frac{1}{N_{\alpha}^+} \int_0^{\infty} \int_0^{\infty} |v v'| g(\alpha, v; \alpha, v'; \tau) dv dv'. \tag{16}$$

For large τ , the values of $x(\tau)$ and $\dot{x}(\tau)$ become independent of $x(0)$ and $\dot{x}(0)$, and $g(\alpha, v; \alpha, v'; \tau)$ approaches $g_{zz}(\alpha, v) g_{zz}(\alpha, v')$. The double integrals in Eqs. 15 and 16 approach $(N_{\alpha}^+)^2$ and, thus,

$$\lim_{\tau \rightarrow \infty} p_{+|-}(\alpha, \tau) = \lim_{\tau \rightarrow \infty} p_{+|+}(\alpha, \tau) = N_{\alpha}^+. \tag{17}$$

Since $p_{+|-}(\alpha, \tau)$ and $p_{+|+}(\alpha, \tau)$ remain finite as $\tau \rightarrow \infty$, their Laplace transforms have singularities of the form N_{α}^+/s . It is convenient to remove this singularity by defining

$$f_{+|\pm}^*(\alpha, s) = f_{+|\pm}(\alpha, s) - N_{\alpha}^+/s = \int_0^{\infty} [p_{+|\pm}(\alpha, \tau) - N_{\alpha}^+] e^{-s\tau} d\tau. \tag{18}$$

Thus, Eq. 12 for the Laplace transform of the recurrence density becomes, upon multiplying numerator and denominator by s ,

$$f_r(\alpha, s) = [N_{\alpha}^+ + s f_{+|-}^*(\alpha, s)] / [N_{\alpha}^+ + s + s f_{+|+}^*(\alpha, s)]. \tag{19}$$

The mean recurrence time $\langle \tau \rangle_{av}$ (that is, the average time between successive downward and upward crossings of $x=\alpha$) is

$$\langle \tau \rangle_{av} = \int_0^{\infty} \tau p_r(\alpha, \tau) d\tau = -\frac{\partial}{\partial s} [f_r(\alpha, s)]_{s=0}. \tag{20}$$

Computing the derivative of the transform from Eq. 19, yields

$$\begin{aligned} (\partial/\partial s)[f_r(\alpha, s)]_{s=0} &= -(1/N_{\alpha}^+) \\ &\quad \times [1 + f_{+|+}^*(\alpha, 0) - f_{+|-}^*(\alpha, 0)] \end{aligned} \tag{21}$$

or

$$\langle \tau \rangle_{av} = \frac{1}{N_{\alpha}^+} \left\{ 1 - \int_0^{\infty} [p_{+|-}(\alpha, t) - p_{+|+}(\alpha, t)] dt \right\}.$$

Higher moments of the recurrence time are related to moments of the first-occurrence time by Eq. 6, and may be calculated from

$$\langle \tau^n \rangle_{av} = \int_0^{\infty} \tau^n p_r(\alpha, \tau) d\tau = (-1)^n \left[\frac{\partial^n}{\partial s^n} f_r(\alpha, s) \right]_{s=0}. \tag{22}$$

A general expression for the n th moment, valid for any n , seems difficult to obtain. However, particular moments may be found and, after some algebraic manipulations, results for $\langle \tau^2 \rangle_{av}$ and $\langle \tau^3 \rangle_{av}$ are

$$\begin{aligned} \langle \tau^2 \rangle_{av} &= \frac{2}{N_{\alpha}^+} \langle \tau \rangle_{av} \left\{ 1 + \int_0^{\infty} [p_{+|+}(\alpha, \tau) - N_{\alpha}^+] d\tau \right\} \\ &\quad + \frac{2}{N_{\alpha}^+} \int_0^{\infty} \tau [p_{+|+}(\alpha, \tau) - p_{+|-}(\alpha, \tau)] d\tau, \end{aligned} \tag{23}$$

$$\begin{aligned} \langle \tau^3 \rangle_{av} &= \frac{3}{N_{\alpha}^+} \langle \tau^2 \rangle_{av} \left\{ 1 + \int_0^{\infty} [p_{+|+}(\alpha, \tau) - N_{\alpha}^+] d\tau \right\} \\ &\quad + \frac{6}{N_{\alpha}^+} \langle \tau \rangle_{av} \int_0^{\infty} \tau [p_{+|+}(\alpha, \tau) - N_{\alpha}^+] d\tau \\ &\quad + \frac{3}{N_{\alpha}^+} \int_0^{\infty} \tau^2 [p_{+|+}(\alpha, \tau) - p_{+|-}(\alpha, \tau)] d\tau. \end{aligned} \tag{24}$$

The integral

$$\int_0^{\infty} p_r(\alpha, \tau) d\tau = f_r(\alpha, 0) \tag{25}$$

should yield unity. It is readily verified from Eq. 19 that $f_r(\alpha, 0) = 1$, and, thus, the renewal process approximation gives a result for $p_r(\alpha, \tau)$ that satisfies this basic restriction on a probability density. It is perhaps surprising that the renewal-process approximation also gives the correct mean recurrence time; this result is proved in Appendix A.

The renewal-process approximation is applied, in a later section, to provide numerical examples of the first-occurrence density for some special cases of Gaussian processes. Summarizing briefly, the method of calculation is first to compute $p_{+|-(\alpha, t)}$ and $p_{+|+(\alpha, t)}$ from the joint-probability densities of $x(t)$ and $\dot{x}(t)$ as in Eqs. 15 and 16. Then, the mean recurrence time $\langle \tau \rangle_{av}$ is computed from Eq. 21, and the recurrence density $p_r(\alpha, \tau)$ from the renewal integral Eq. 13. The first-occurrence density $p_o(\alpha, t)$, necessary for the computation of the probability of failure given in Eq. 1, is determined from Eq. 5 in terms of $\langle \tau \rangle_{av}$ and $p_r(\alpha, \tau)$.

II. LIMITING FIRST-OCCURRENCE DENSITY

Since, in applications, one is generally concerned with the statistically rare crossings of a high level α , it is of considerable interest to investigate the limiting form of the first-occurrence density as α approaches infinity. Some simple arguments, given below, suggest an exponential distribution of first-occurrence times. However, a rigorous proof has not been obtained and some of the difficulties encountered in this connection are pointed out.

Results for the limiting distribution are derived in a form valid for both stationary and some nonstationary random processes. It is convenient first to redefine the first-occurrence time density so that $p_o(\alpha, t)dt$ is the probability that the first upward crossing of $x = \alpha$ occurs in the time interval $(t, t + dt)$. This differs from the previous definition in that it is no longer given that $x(0) < \alpha$; the difference is unimportant, since for large α there is a negligible probability that $x(0) > \alpha$. The probability of the first upward crossing of $x = \alpha$ in $(t, t + dt)$ is the product of the probability of an upward crossing in $(t, t + dt)$, given no prior upward crossing in $(0, t)$, and the probability of no prior upward crossing. Thus,

$$p_o(\alpha, t) = \mu[\alpha, t | (0, t)] \left\{ 1 - \int_0^t p_o(\alpha, \tau) d\tau \right\}, \quad (26)$$

where $\mu[\alpha, t | (0, t)]dt$ is the probability of an upward crossing in $(t, t + dt)$, given no prior upward crossing in $(0, t)$. Solving Eq. 26 for the first-occurrence density,

$$p_o(\alpha, t) = \mu[\alpha, t | (0, t)] \exp \left\{ - \int_0^t \mu[\alpha, \tau | (0, \tau)] d\tau \right\}. \quad (27)$$

The definition of $\mu[\alpha, t | (0, t)]$ suggests that, for large α , $\mu[\alpha, t | (0, t)]$ approaches $N_{\alpha^+}(t)$. Consider first the case when t is small. Here, one has, regardless of the value of α , $\mu[\alpha, t | (0, t)] \simeq N_{\alpha^+}(t)$ (with an equality holding as $t \rightarrow 0$), since the probability of a crossing in $(0, t)$ prior to the crossing at t is correspondingly small. When t is not small, the same approximation is suggested for large values of α , since the crossings of $x = \alpha$ will then be statistically rare events and prior crossings may be expected to have a negligible influence on the probability of a crossing in $(t, t + dt)$. Thus, for large α , $\mu[\alpha, t | (0, t)] \simeq N_{\alpha^+}(t)$, and Eq. 27 yields, for the limiting first-occurrence time density,

$$p_o(\alpha, t) \simeq N_{\alpha^+}(t) \exp \left\{ - \int_0^t N_{\alpha^+}(\tau) d\tau \right\}. \quad (28)$$

For stationary processes, $N_{\alpha^+}(t) = N_{\alpha^+}$, a constant, and Eq. 28 becomes

$$p_o(\alpha, t) \simeq N_{\alpha^+} e^{-N_{\alpha^+} t}. \quad (29)$$

The corresponding recurrence time density may be found directly from Eq. 5. Noting that the mean recurrence time $\langle \tau \rangle_{av}$ approaches $1/N_{\alpha^+}$ for large α , one has

$$p_r(\alpha, \tau) \simeq N_{\alpha^+} e^{-N_{\alpha^+} \tau}. \quad (30)$$

Another way of viewing Eq. 28 is as follows. Owing to the very large average time interval between excursions above $x = \alpha$ and the comparatively short duration of the excursions, one may view the excursions above $x = \alpha$ as a random process of point events in time occurring independently at a mean rate $N_{\alpha^+}(t)$. It is well-known⁷ that such a process leads to an exponential distribution identical to Eq. 28 for the waiting time before occurrence of an event or, in present terminology, the first-occurrence time.

The approximate first-occurrence time density for large α given by Eq. 28 requires only a knowledge of $N_{\alpha^+}(t)$, which is readily computed from Eq. 7 once the second-order joint-probability density $g_{x\dot{x}}(u, v; t)$ of $x(t)$ and $\dot{x}(t)$ is known. This results in a considerable simplification when compared to the renewal-process approximation of Sec. I or when compared to a procedure based on retaining only, say, the first two terms in the inclusion-exclusion series. Both of the latter approximations require a knowledge of the fourth-order joint density of $x(t_1), \dot{x}(t_1), x(t_2)$, and $\dot{x}(t_2)$. Aside from computational difficulties that may arise even if this density is known, the information available on a particular stochastic process may not be sufficient to determine the fourth-order density. For example, in the case of

⁷ E. Parzen, *Stochastic Processes* (Holden-Day, San Francisco, 1962).

stationary Gaussian processes, the second-order density requires only a knowledge of the mean of x and variance of x and \dot{x} ; the fourth-order density requires in addition that the correlation function of the process and its first two derivatives be known for all time. Furthermore, the second-order density $g_{zz}(u,v)$ may be found⁸ as the stationary solution to the Fokker-Planck Equation for a general class of nonlinear dynamical systems subjected to white excitation, leading⁹ to expressions for N_{α}^{+} in terms of the system potential energy at the level $x=\alpha$. Corresponding results are unknown for the fourth-order densities of such systems.

Equations in some respects similar to Eqs. 29 and 30 have been given,^{10,11} where,¹⁰ in the present notation, the relation $p_r(\alpha,t) = 2N_{\alpha}^{+}e^{-2N_{\alpha}^{+}t}$ is obtained and where¹¹ the relation $p_o(\alpha,t) = 2N_{\alpha}^{+}e^{-2N_{\alpha}^{+}t}$ is obtained. That the first result¹⁰ is inappropriate is readily seen by noting that it gives a mean recurrence time $\langle \tau \rangle_{av} = 1/2N_{\alpha}^{+}$, instead of the correct $1/N_{\alpha}^{+}$ for large α . The second result¹¹ is similarly inappropriate, since it yields $p_o(\alpha,0) = 2N_{\alpha}^{+}$. But, from Eq. 5, it is clear that $p_o(\alpha,0) = \langle \tau \rangle_{av}^{-1}$, which approaches N_{α}^{+} for large α .

Defining failure as the first exceedance of $x=\alpha$, the probability $P_f(\alpha,T)$ of failure in time T is from Eq. 1, after using Eq. 28 for $p_o(\alpha,t)$,

$$P_f(\alpha,T) = \epsilon_{\alpha} + (1 - \epsilon_{\alpha}) \left\{ 1 - \exp \left[- \int_0^T N_{\alpha}^{+}(\tau) d\tau \right] \right\}. \quad (31)$$

Methods for determining upper and lower bounds on the failure probability $P_f(\alpha,T)$ for processes starting at $x(0)=0$ are known.¹² Generalizing these results to account for processes that do not necessarily start at zero, one obtains an upper bound by noting that the probability of failure in dT is

$$\begin{aligned} dP_f(\alpha,T) &= P_f(\alpha, T+dT) - P_f(\alpha,T) \\ &= \text{prob} \left\{ \max_{0 < t < T} x(t) < \alpha \text{ and } \max_{T < t < T+dT} x(t) > \alpha \right\} \\ &\leq \text{prob} \left\{ x(T) < \alpha \text{ and } \max_{T < t < T+dT} x(t) > \alpha \right\} \end{aligned} \quad (32)$$

$$dP_f(\alpha,T) \leq N_{\alpha}^{+}(T)dT,$$

since the probability that $x(t) < \alpha$ for all points of $(0,T)$ is less than the probability that $x(t) < \alpha$ for any one point of $(0,T)$. Integrating subject to the initial condition $P_f(\alpha,0) = \epsilon_{\alpha}'$,

$$P_f(\alpha,T) < \epsilon_{\alpha} + \int_0^T N_{\alpha}^{+}(t) dt. \quad (33)$$

Comparing with Eq. 31 and noting that $1 - e^{-x} < x$ for any positive x , it is seen that the first-occurrence density approximation of Eq. 28 yields through Eq. 31 a failure probability always below the upper bound of Eq. 33. A lower bound to the failure probability is found by writing

$$\begin{aligned} P_f(\alpha,T) &= \text{prob} \left\{ \max_{0 < t < T} x(t) > \alpha \right\} \\ &\geq \text{prob} \{ x(t) > \alpha \text{ for any } t \text{ in } (0,T) \}; \\ P_f(\alpha,T) &\geq \epsilon_{\alpha}(t), \quad \text{for any } t \text{ in } (0,T). \end{aligned} \quad (34)$$

Here, $\epsilon_{\alpha}(t)$ is the probability that $x(t) > \alpha$. For a stationary process $\epsilon_{\alpha}(t) = \epsilon_{\alpha}(0) = \epsilon_{\alpha}$ for all t , and thus the expression for $P_f(\alpha,T)$ given by equation Eq. 31 is always above the lower bound of Eq. 34. There is no obvious reason why this expression should satisfy the bound of Eq. 34 in the general case of nonstationary processes, and apparently each case must be checked separately.

In spite of the plausibility of the result, a convincing proof that $p_{\alpha}(\alpha,t)$ tends to the exponential distribution of Eq. 28 for large α has not been obtained. Sufficient conditions under which Eqs. 28-30 result from both the densities as given by the renewal-process approximation of Sec. I and as given by exact inclusion-exclusion series have been discussed.¹³ Essentially, the type of conditions required are, in the stationary case,

$$\lim_{\alpha \rightarrow \infty} \int_0^{\infty} [\dot{p}_{+|-}(\alpha,t) - N_{\alpha}^{+}] dt = 0, \quad (35a)$$

$$\lim_{\alpha \rightarrow \infty} \int_0^{\infty} \int_0^{\infty} [\dot{p}_{++|-}(\alpha,t_1,t_2) - (N_{\alpha}^{+})^2] dt_1 dt_2 = 0. \quad (35b)$$

The difficulty in verifying these expressions is due to the extreme complexity (as may be noted from expressions of Sec. III) of the functions $\dot{p}_{+|-}(\alpha,t)$, $\dot{p}_{++|-}(\alpha,t_1,t_2)$, \dots . The meaning of Eqs. 35 is made clear by discussing the first. The integral

$$\int_0^T \dot{p}_{+|-}(\alpha,t) dt \quad (36)$$

¹³ J. R. Rice, "Theoretical Prediction of Some Statistical Characteristics of Random Loadings Relevant to Fatigue and Fracture," Ph.D. thesis, Lehigh Univ. (1964).

⁸ T. K. Caughey, "Derivation and Application of the Fokker-Planck Equation to Discrete Nonlinear Dynamic Systems Subjected to White Random Excitation," J. Acoust. Soc. Am. 35, 1683-1692 (1963).

⁹ S. H. Crandall, "Zero Crossings, Peaks, and Other Statistical Measures of Random Responses," J. Acoust. Soc. Am. 35, 1693-1699 (1963).

¹⁰ M. Kac and D. Slepian, "Large Excursions of Gaussian Processes," Ann. Math. Statist. 30, 1215-1228 (1959).

¹¹ J. J. Coleman, "Reliability of Aircraft Structures in Resisting Chance Failure," Operations Res. 7, 639-645 (1959).

¹² M. Shinozuka, "On Upper and Lower Bounds of the Probability of Failure of Simple Structures under Random Excitation," Columbia Univ. Inst. Study Fatigue Reliability Tech. Rept. 01 (Dec. 1963).

is the conditional expected number of upward crossings of $x=\alpha$ in $(0, T)$ given a downward crossing of $x=\alpha$ at $t=0$, and

$$\int_0^T N_{\alpha^+} dt \tag{37}$$

is the unconditional expected number of upward crossings of $x=\alpha$ in $(0, T)$. Equation 35a then requires that, as $\alpha \rightarrow \infty$, the difference between the conditional and unconditional expected numbers of upward crossings in $(0, \infty)$ approaches zero. There is little difficulty in justifying $p_{+|-(\alpha, t)} \rightarrow 0$ and $N_{\alpha^+} \rightarrow 0$ as $\alpha \rightarrow \infty$. Further, from Eq. 17, $p_{+|-(\alpha, t)} \rightarrow N_{\alpha^+}$ as $t \rightarrow \infty$ as is required for the integral to exist. Thus, if the integral converges uniformly¹⁴ so that the limit on α may be taken inside the integral sign, Eq. 35a is satisfied. Uniform convergence is assured if

$$\int_T^\infty [p_{+|-(\alpha, t)} - N_{\alpha^+}] dt \tag{38}$$

[which represents the difference between the conditional and unconditional expected numbers of upward crossings in (T, ∞)] can be made arbitrarily small by choosing, independently of α , a correspondingly large T . Essentially, then, it is required that the dependence of $p_{+|-(\alpha, t)}$ on its conditioning at $t=0$ dies out sufficiently fast in time for all α . One expects the conditioning influence to dominate $p_{+|-(\alpha, t)}$ only for times comparable to some characteristics of the process, such as $1/2N_{0^+}$, the mean time between crossings of $x=0$, so that the time of conditioning influence is negligible in comparison to times of the order of the mean recurrence time $1/N_{\alpha^+}$ for large α .

III. APPLICATION TO STATIONARY GAUSSIAN PROCESSES

Formulas required for the determination of the first-occurrence and first-recurrence time densities are given in this section for the technically important case of stationary Gaussian processes. Expressions are given for N_{α^+} as required in the limiting distributions for large α of Eq. 29 and 30 and for $p_{+|-(\alpha, t)}$ and $p_{+|+(\alpha, t)}$, defined, respectively, by Eqs. 15 and 16, as required in the renewal-process approximation. Numerical examples are given for processes with idealized spectra.

For a Gaussian process, N_{α^+} as defined by Eq. 7 is¹

$$N_{\alpha^+} = (1/2\pi)[-R''(0)/R(0)]^{1/2} \exp[-\frac{1}{2}\alpha^2/R(0)], \tag{39}$$

where $R(\tau) = (x(t)x(t+\tau))_{av}$ is the correlation function of the process. The fourth-order joint-density function

$g(u, v; u', v'; \tau)$ of $x(0), \dot{x}(0), x(\tau)$, and $\dot{x}(\tau)$ is required in the determination of $p_{+|-(\alpha, \tau)}$ and $p_{+|+(\alpha, \tau)}$. In the Gaussian case, this is¹⁵

$$g(u, v; u', v'; \tau) = [(2\pi)^2 \sqrt{|M|}]^{-1} \exp\{-\frac{1}{2}[s_{11}u^2 + s_{22}v^2 + s_{33}u'^2 + s_{44}v'^2 - s_{12}uv - s_{23}vu' - s_{34}u'v' - s_{14}uv' - s_{13}uu' - s_{24}vv']\}, \tag{40}$$

where s_{ij} is the element of the i th row and j th column of the inverse of matrix $[M]$ and where $|M|$ is the determinant of matrix $[M]$ —the matrix $[M]$ being defined as

$$[M] = \begin{bmatrix} R(0) & 0 & R(\tau) & R'(\tau) \\ 0 & -R'(0) & -R'(\tau) & -R''(\tau) \\ R(\tau) & -R'(\tau) & R(0) & 0 \\ R'(\tau) & -R''(\tau) & 0 & -R''(0) \end{bmatrix}. \tag{41}$$

The considerable amount of algebra required to express Eq. 40 in terms of the correlation function $R(\tau)$ and its first two derivatives is omitted here. After making the change of variables $x = -v(s_{22}/2)$, $y = v'(s_{44}/2)$ in Eq. 15 and $x = v(s_{22}/2)$, $y = v'(s_{44}/2)$ in Eq. 16, one obtains

$$p_{+|-(\alpha, \tau)} = A e^{-B\alpha^2} \int_0^\infty \int_0^\infty xy e^{-(x^2 + 2cxv + v^2) + 2D\alpha(x+v)} dx dy, \tag{42}$$

$$p_{+|+(\alpha, \tau)} = A e^{-B\alpha^2} \int_0^\infty \int_0^\infty xy e^{-(x^2 - 2cxv + v^2) + 2D\alpha(v-x)} dx dy. \tag{43}$$

Here, A, B, c , and D are functions of τ expressible in terms of the correlation function $R(\tau) = R_\tau$ and its derivatives by

$$\begin{aligned} A &= A(\tau) = (2/\pi)(-R_0/R''_0)(|M|^{1/2}/\beta^2), \\ B &= B(\tau) = [(R''_0 - R''_\tau)/|M|] \gamma - (1/2R_0), \\ c &= c(\tau) = (1/\beta)[R''_\tau(R_0^2 - R_\tau^2) + R_\tau R'_\tau{}^2], \end{aligned} \tag{44}$$

and

$$D = D(\tau) = \frac{-R'_\tau \gamma}{[(2|M|\beta)]^{1/2}},$$

where

$$\begin{aligned} |M| &= (R_0 R_0'' - R_\tau R_\tau'' + R_\tau'^2)^2 - (R_0 R_\tau'' - R_\tau R_0'')^2, \\ \beta &= -R_0''(R_0^2 - R_\tau^2) - R_0 R_\tau'^2, \\ \gamma &= (R_0 - R_\tau)(R_0'' + R_\tau'') + R_\tau'^2. \end{aligned} \tag{45}$$

The double integrals in Eqs. 42 and 43 cannot be evaluated in closed form, but, by changing to polar

¹⁴ E. B. Wilson, *Advanced Calculus* (Ginn & Co., Boston, Mass., 1912).

¹⁵ J. S. Bendat, *Principles and Applications of Random Noises: Theory* (John Wiley & Sons, Inc., New York, 1958).

coordinates, both can be reduced to single finite integrals. Performing the change of variables $x=r \cos\theta$, $y=r \sin\theta$, Eq. 42 becomes

$$p_{+|-}(\alpha, \tau) = \frac{1}{2} A e^{-B\alpha^2} \int_0^{\pi/2} \int_0^\infty r^3 \sin 2\theta \times e^{-r^2(1+c \sin 2\theta)+2D\alpha r(\sin\theta+\cos\theta)} dr d\theta. \quad (46)$$

Carrying out the integration in r ,

$$p_{+|-}(\alpha, \tau) = \frac{1}{4} A e^{-B\alpha^2} \int_0^{\pi/2} \frac{\sin 2\theta}{(1+c \sin 2\theta)^2} \times [\sqrt{\pi z_1} (\frac{3}{2} + z_1^2) e^{z_1^2} (1 + \operatorname{erf} z_1) + 1 + z_1^2] d\theta, \quad (47)$$

where

$$z_1 = z_1(\alpha, \tau, \theta) = D\alpha(\sin\theta + \cos\theta) / (1+c \sin 2\theta)^{1/2}. \quad (48)$$

The equation for $p_{+|+}(\alpha, \tau)$ is identical to Eq. 47, except that c is replaced by $-c$ and z_1 is replaced by z_2 , where

$$z_2 = z_2(\alpha, \tau, \theta) = D\alpha(\sin\theta - \cos\theta) / (1-c \sin 2\theta)^{1/2}. \quad (49)$$

Noting the symmetry of z_1 about $\theta = \pi/4$ and the antisymmetry of z_2 about $\theta = \pi/4$, the equations for $p_{+|-}(\alpha, \tau)$ and $p_{+|+}(\alpha, \tau)$ become

$$p_{+|-}(\alpha, \tau) = \frac{1}{2} A e^{-B\alpha^2} \int_0^{\pi/4} \frac{\sin 2\theta}{(1+c \sin 2\theta)^2} \times [\sqrt{\pi z_1} (\frac{3}{2} + z_1^2) e^{z_1^2} (1 + \operatorname{erf} z_1) + 1 + z_1^2] d\theta, \quad (50)$$

$$p_{+|+}(\alpha, \tau) = \frac{1}{2} A e^{-B\alpha^2} \int_0^{\pi/4} \frac{\sin 2\theta}{(1-c \sin 2\theta)^2} \times [\sqrt{\pi z_2} (\frac{3}{2} + z_2^2) e^{z_2^2} (\operatorname{erf} z_2) + 1 + z_2^2] d\theta. \quad (51)$$

The above integrals are expressed in a form convenient for numerical evaluation.

Results of numerical calculations of the first-occurrence time density $p_0(\alpha, t)$ are given below for some stationary Gaussian processes with idealized power spectral densities. The spectra considered are constant over a certain frequency range and zero for all other frequencies, having the mathematical representation

$$S(\omega) = \begin{cases} \sigma^2 / (1-\beta)\omega_c & \text{FOR } \beta\omega_c < \omega < \omega_c, \\ 0 & \text{OTHERWISE,} \end{cases} \quad (52)$$

where $\beta\omega_c$ is a lower cutoff frequency, ω_c is an upper cutoff frequency, and σ^2 is the variance of $x(t)$. The

correlation function for the process $x(t)$ is¹

$$R(\tau) = \langle x(t)x(t+\tau) \rangle_{av} = \int_0^\infty S(\omega) \cos \omega \tau d\omega, \\ R(\tau) = [\sigma^2 / (1-\beta)\omega_c \tau] [\sin \omega_c \tau - \beta \sin \beta \omega_c \tau]. \quad (53)$$

In performing numerical calculations, it is convenient to suppress explicit dependence of the results on the variance σ^2 and cutoff frequency ω_c . To this end, the dimensionless times $\nu = \omega_c t$ and $\Psi = \omega_c \tau$ are introduced, and the normalized dimensionless process $y(\nu) = x(t)/\sigma$ is considered. The correlation function $R_y(\Psi)$ of the process $y(\nu)$ is

$$R_y(\Psi) = \langle y(\nu)y(\nu+\Psi) \rangle_{av} = (1/\sigma^2) \langle x(t)x(t+\Psi/\omega_c) \rangle_{av} = (1/\sigma^2) R(\Psi/\omega_c). \quad (54)$$

Substituting from Eq. 53 for $R(\tau)$,

$$R_y(\Psi) = [(1-\beta)\Psi]^{-1} [\sin \Psi - \beta \sin \beta \Psi]. \quad (55)$$

Calculations are made of the first-occurrence time density $p_0(\alpha/\sigma, \Psi)$, where $p_0(\alpha/\sigma, \Psi)d\Psi$ is the probability, given $y(0) < \alpha/\sigma$, that the normalized process $y(\nu)$ first crosses the level $y = \alpha/\sigma$ in the dimensionless time interval $(\Psi, \Psi + d\Psi)$. Clearly, this is also the probability, given $x(0) < \alpha$, that $x(t)$ first crosses the level $x = \alpha$ when $\omega_c t$ is in the interval $(\Psi, \Psi + d\Psi)$.

Equation 29 gives an approximation to the first-occurrence density for the statistically rare crossings of high levels in terms of the expected number of upward crossings of the level per unit time. Using Eq. 39, the expected number of upward crossings of $y = \alpha/\sigma$ per unit dimensionless time (that is, the expected number of upward crossings of $x = \alpha$ per unit of $\omega_c t$) becomes

$$N_{\alpha/\sigma^+} = \frac{1}{2\pi} [-R_y''(0)/R_y(0)]^{1/2} \exp[-\frac{1}{2}(\alpha/\sigma)^2/R_y(0)] \\ = \frac{1}{2\pi} [(1-\beta^3)/3(1-\beta)]^{1/2} \exp[-\frac{1}{2}\alpha^2/\sigma^2], \quad (56)$$

since $R_y(0) = 1$ and $R_y''(0) = -(1-\beta^3)/3(1-\beta)$. Thus, Eq. 29 for the first-occurrence time density for large α —namely, $p_0(\alpha/\sigma, \Psi) \simeq N_{\alpha/\sigma^+} \exp(-N_{\alpha/\sigma^+}\Psi)$ —becomes

$$p_0(\alpha/\sigma, \Psi) \simeq \frac{1}{2\pi} [(1-\beta^3)/3(1-\beta)]^{1/2} \exp(-\frac{1}{2}\alpha^2/\sigma^2) \\ \exp\left\{-\frac{1}{2\pi} [(1-\beta^3)/3(1-\beta)]^{1/2} \exp(-\frac{1}{2}\alpha^2/\sigma^2) \Psi\right\}. \quad (57)$$

Numerical results were obtained from the renewal-process approximation for values of α equal to σ , 2σ , and 3σ for each of two random processes—one process having an ideal wide-band spectrum with $\beta=0$ and the other process having an ideal narrow-band spectrum with $\beta=\frac{1}{2}$. The calculation is started by finding the functions $p_{+|-}(\alpha/\sigma, \Psi)$ and $p_{+|+}(\alpha/\sigma, \Psi)$ defined for Gaussian processes by Eqs. 50 and 51, for $\Psi=\omega_c t=0, 0.25, 0.50, 0.75, \dots, 50.00$. The value $\Psi=50$ is approximately ten times the average distance between zero crossings. The integrations from 0 to $\pi/4$ on θ required in Eqs. 50 and 51 were carried out by computing the integrand for $\theta=0, \pi/64, \pi/32, \dots, \pi/4$ and summing. Once $p_{+|-}(\alpha/\sigma, \Psi)$ and $p_{+|+}(\alpha/\sigma, \Psi)$ are determined, the first-recurrence time density $p_r(\alpha/\sigma, \Psi)$ is found from the renewal integral Eq. 13, which was obtained by making the renewal-process approximation in the exact inclusion-exclusion expression for the recurrence density. In terms of the present dimensionless notation, Eq. 13 becomes

$$p_{+|-}(\alpha/\sigma, \Psi) = p_r(\alpha/\sigma, \Psi) + \int_0^\Psi p_r(\alpha/\sigma, \nu) p_{+|+}(\alpha/\sigma, \Psi - \nu) d\nu. \quad (58)$$

The Equation was solved for $\Psi=0, 0.25, 0.50, \dots, 125.00$ by replacing the integral by a summation. Clearly, the solution for $p_r(\alpha/\sigma, \Psi)$ depends only on the known functions $p_{+|-}$ and $p_{+|+}$ and the past values of p_r ; thus, the solution of Eq. 58 is readily obtained recursively. The limiting value $N_{\alpha/\sigma}^+$ was used for $p_{+|-}$ and $p_{+|+}$ when Ψ was greater than 50. The mean recurrence time is obtained from Eq. 21, an expression that, although derived through the renewal-process approximation, yields the exact value of the mean recurrence time, as shown in Appendix A. In terms of the present notation, the mean dimensionless recurrence time $\langle \Psi \rangle_{av}$ is

$$\langle \Psi \rangle_{av} = \frac{1}{N_{\alpha/\sigma}^+} \left\{ 1 - \int_0^\infty [p_{+|-}(\alpha/\sigma, \Psi) - p_{+|+}(\alpha/\sigma, \Psi)] d\Psi \right\}. \quad (59)$$

The integration was carried out numerically by replacing the infinite integral by a summation in Ψ from 0 to 50, at intervals of 0.25. Finally, Eq. 5 yields the following expression for the first-occurrence time density in terms of the first-recurrence time density:

$$p_0(\alpha/\sigma, \Psi) = \frac{1}{\langle \Psi \rangle_{av}} \left[1 - \int_0^\Psi p_r(\alpha/\sigma, \nu) d\nu \right]. \quad (60)$$

Again, the integration was replaced by a summation

giving the first occurrence density for $\Psi=\omega_c t=0, 0.25, 0.50, \dots, 125.00$.

Results of the computations are shown by the solid lines in Fig. 2(a-c) for the wide-band spectrum ($\beta=0$) with $\alpha=\sigma, 2\sigma$, and 3σ , respectively. The dashed lines are plots of the limiting exponential distribution for the first-occurrence density as given by Eq. 57. It is seen that, as α increases, the agreement between the renewal-process approximation (solid lines) and the exponential distribution (dashed lines) becomes increasingly good. When $\alpha=2\sigma$, the difference between the two curves as shown in Fig. 2(b) is less than 7% for small values of $\Psi=\omega_c t$, and, for larger values of Ψ , the difference becomes negligible. When $\alpha=3\sigma$ as shown in Fig. 2(c), the difference is completely negligible, having a value of less than $\frac{1}{2}$ %.

The results verify the validity of the exponential distribution for large α in the case of processes with wide spectra, and show a very rapid approach to the limiting distribution as α is increased. A similar verification is obtained in the case of processes with narrow spectra, but here the approach to the limiting distribution is considerably slower.

Figure 3(a-c) contain results of the computations for the narrow-band spectrum ($\beta=\frac{1}{2}$) with $\alpha=\sigma, 2\sigma$, and 3σ , respectively. When $\alpha=2\sigma$, the difference between the renewal-process approximation and the exponential distribution, as shown in Fig. 3(b) has a maximum of about 11% for small values of Ψ and the difference persists, in contrast to the wide-band case, for larger values of Ψ . When $\alpha=3\sigma$, as shown in Fig. 3(c), the difference decreases to a value of about 5%, which persists over the entire portion of the time axis shown. It is clear that, ultimately, the two curves of Fig. 3(c) meet, since it can be shown that the area under each is equal to unity.

The results indicate a considerable difference between processes with wide- and narrow-band spectra, with regard to the rapidity of approach to the exponential first-occurrence time distribution, and indicate that, in situations requiring great accuracy, some caution is necessary in applying the exponential distribution to narrow-band processes when the crossings of the level under consideration are not statistically rare.

Unfortunately, the renewal-process approximation seems least appropriate in the case of narrow-band processes, for these have correlation functions that approach zero rather slowly with time, indicating a high degree of dependence on past values. Basic to the renewal approximation is the assumption that the probability of an upward α crossing, given several past upward crossings, depends approximately only on the last prior crossing. Clearly, such an approximation is best for processes with little memory. In fact, for all three cases of narrow-band processes considered here, the renewal approximation yielded some negative values of the recurrence time density $p_r(\alpha/\sigma, \Psi)$. This

FIG. 2. First-occurrence density. Wide spectrum. (a) $\alpha=1\sigma$. (b) $\alpha=2\sigma$. (c) $\alpha=3\sigma$.

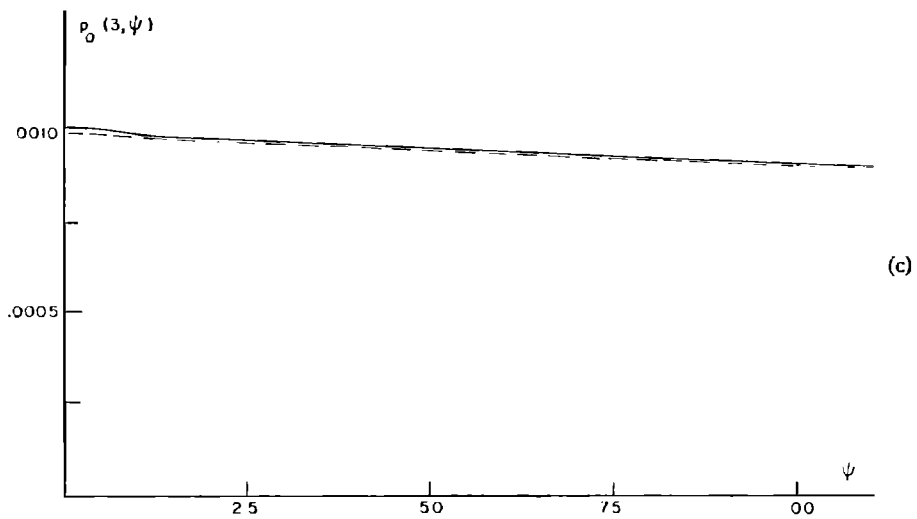
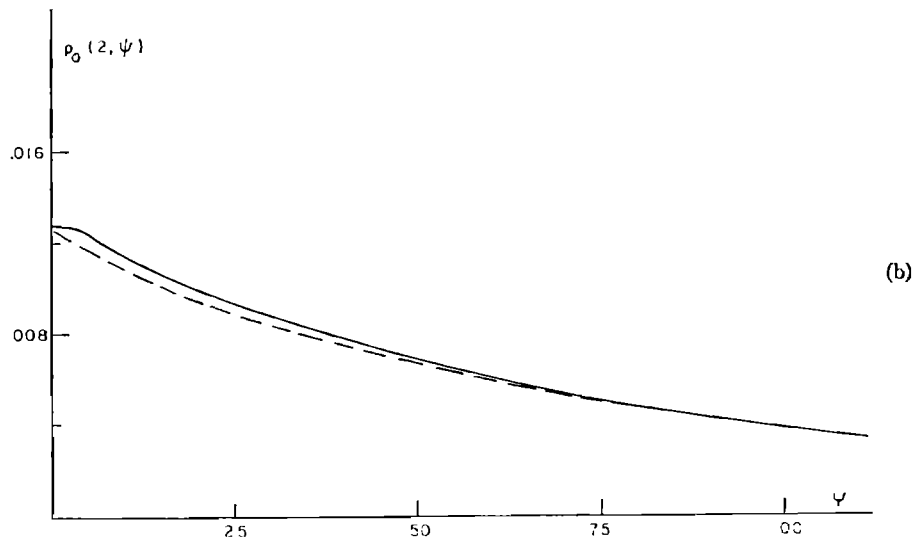
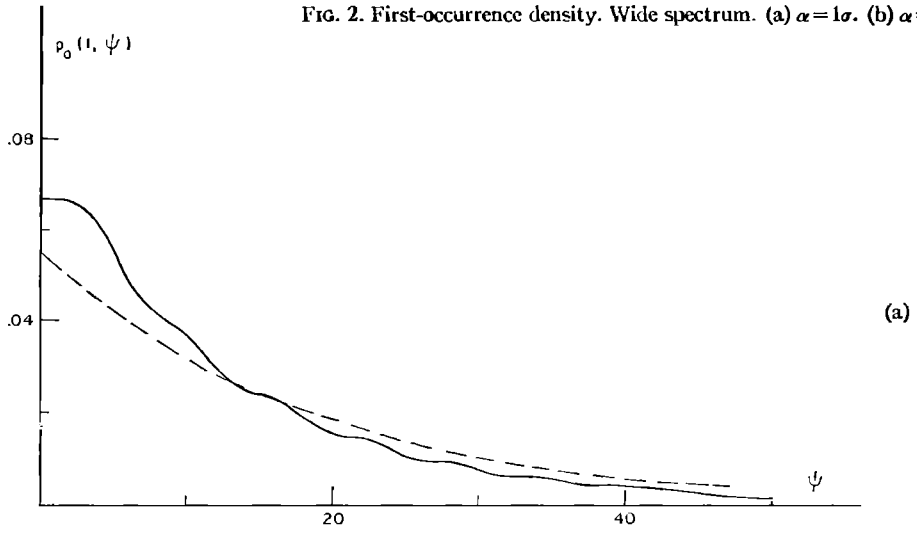
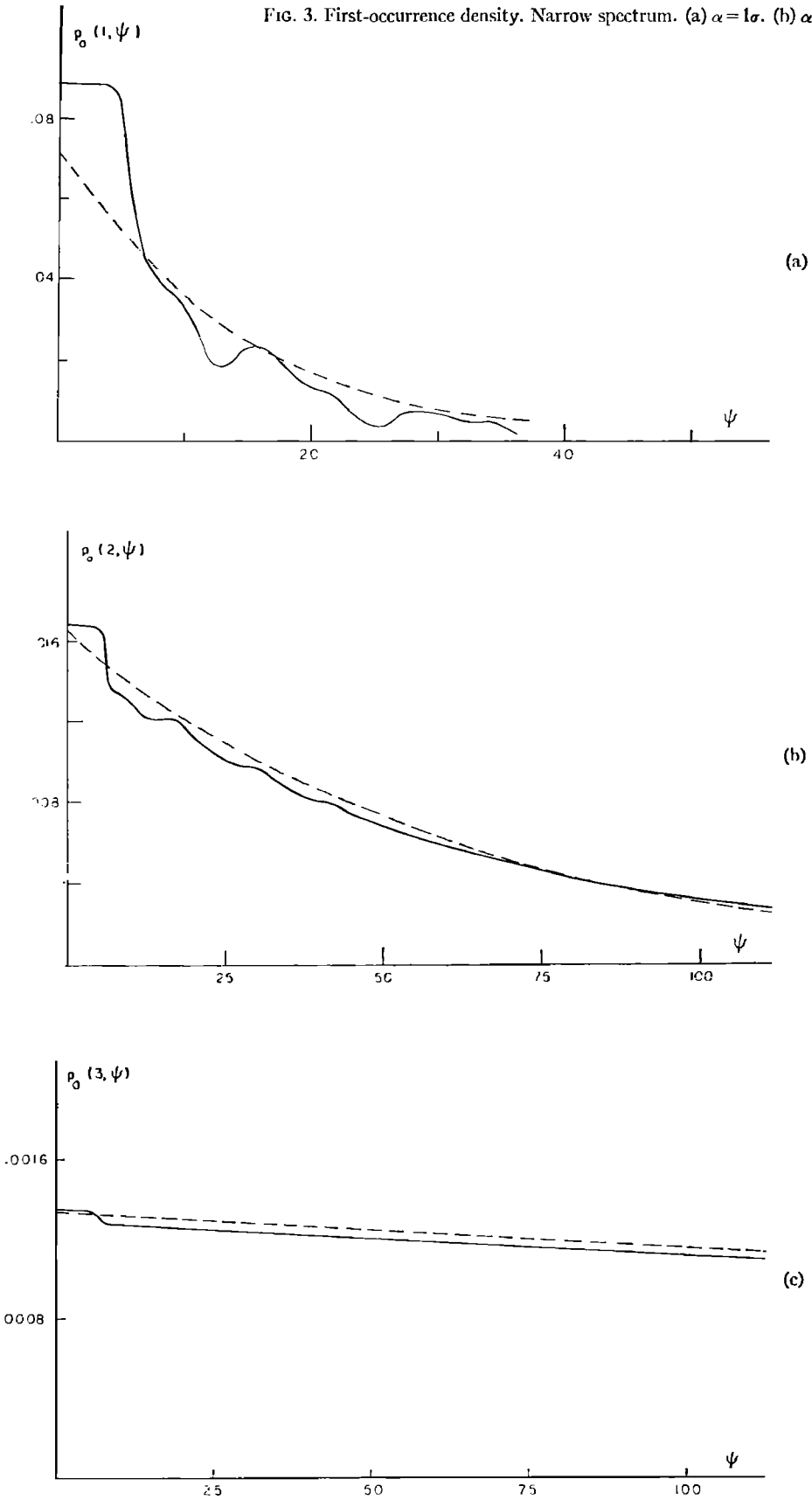


FIG. 3. First-occurrence density. Narrow spectrum. (a) $\alpha=1\sigma$. (b) $\alpha=2\sigma$. (c) $\alpha=3\sigma$.



may be inferred by noting the existence of relative minima in the graphs of $p_o(\alpha/\sigma, \Psi)$ in Fig. 3(a-c). From Eq. 60, the derivative with respect to Ψ of $p_o(\alpha/\sigma, \Psi)$ is proportional to $-p_r(\alpha/\sigma, \Psi)$, indicating that $p_o(\alpha/\sigma, \Psi)$ should have no minima. The fact that the calculated values of $p_r(\alpha/\sigma, \Psi)$ took small negative values over some short time intervals is reflected in the Figures by the small positive slope of the curves at certain intervals on the time axes. A similar behavior was noted⁴ in

connection with the application of a renewal-process approximation to the zero-crossing problem.

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Appendix A

It was indicated that the expression obtained for the first-recurrence-time probability density $p_r(\alpha, \tau)$ by the renewal-process approximation not only satisfies the condition

$$\int_0^\infty p_r(\alpha, \tau) d\tau = 1, \tag{A1}$$

but also yields the correct value of the mean recurrence time $\langle \tau \rangle_{av}$. This last point is now proved as follows.

Let $\langle \tau' \rangle_{av}$ be the average time between successive upward and downward crossings of $x = \alpha$ (Fig. A-1). Then, $\langle \tau \rangle_{av} + \langle \tau' \rangle_{av} = 1/\lambda_{\alpha^+}$. Solving for $\langle \tau \rangle_{av}$, yields

$$\langle \tau \rangle_{av} = (1/\lambda_{\alpha^+})(1 - N_{\alpha^+} \langle \tau' \rangle_{av}). \tag{A2}$$

It is shown that this expression is identical with the expression Eq. 21 obtained by the renewal-process approximation.

Let $\lambda(\tau')$ be the probability density for the time τ' , shown in Fig. A-1, between successive upward and downward crossings of $x = \alpha$, and let $p_{+|-}(\alpha, \tau | \tau')$ be the probability of an upward crossing of $x = \alpha$ in $(t, t + dt)$ given, as in Fig. A-1, a downward crossing at $t = 0$, and that the last upward crossing prior to $t = 0$ occurred at $t = -\tau'$. Then, by the law of conditional probability

$$p_{+|-}(\alpha, t) = \int_0^\infty p_{+|-}(\alpha, t | \tau') \lambda(\tau') d\tau'. \tag{A3}$$

Consider

$$\int_0^T p_{+|+}(\alpha, t) dt, \tag{A4}$$

which is the expected number of upward crossings of $x = \alpha$ in $(0, T)$, given an upward crossing at $t = 0$. Referring to Fig. A-2, let τ' be the time of the first downward crossing after the upward crossing at $t = 0$. If τ' is given and $\tau' < T$, the expected number of upward crossings in $(0, T)$ is

$$\int_{\tau'}^T p_{+|-}(\alpha, t - \tau' | \tau') dt. \tag{A5}$$

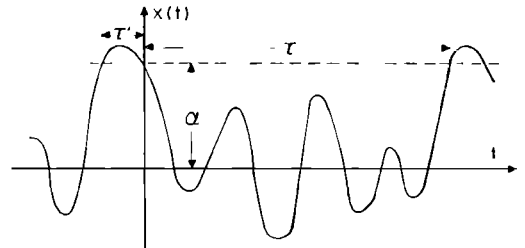


FIG. A-1. Random process with downward crossing of $x = \alpha$ at $t = 0$ and last prior upward crossing at $t = -\tau'$.

If $\tau' > T$, then the expected number of upward crossings in $(0, T)$ is zero. Thus, the difference between the conditional expected number of upward crossings of $x = \alpha$ in $(0, T)$, given a downward crossing at $t = 0$, and the conditional expected number of upward crossings of $x = \alpha$ in $(0, T)$, given an upward crossing at $t = 0$, is

$$\begin{aligned} & \int_0^T [p_{+|-}(\alpha, t) - p_{+|+}(\alpha, t)] dt \\ &= \int_0^T p_{+|-}(\alpha, t) dt - \int_0^T \lambda(\tau') \int_{\tau'}^T p_{+|-}(\alpha, t - \tau' | \tau') dt d\tau' \\ &= \int_0^T p_{+|-}(\alpha, t) dt - \int_0^T \lambda(\tau') \int_0^{T-\tau'} p_{+|-}(\alpha, t | \tau') dt d\tau' \\ &= \int_0^T \left[p_{+|-}(\alpha, t) - \int_0^T p_{+|-}(\alpha, t | \tau') \lambda(\tau') d\tau' \right] dt \\ & \quad + \int_0^T \lambda(\tau') \int_{T-\tau'}^T p_{+|-}(\alpha, t | \tau') dt d\tau' \\ &= \int_0^T \left[p_{+|-}(\alpha, t) - \int_0^\infty p_{+|-}(\alpha, t | \tau') \lambda(\tau') d\tau' \right] dt \\ & \quad + \int_0^T \int_T^\infty \lambda(\tau') p_{+|-}(\alpha, t | \tau') d\tau' dt \\ & \quad + \int_0^T \lambda(\tau') \int_{T-\tau'}^T p_{+|-}(\alpha, t | \tau') dt d\tau'. \tag{A6} \end{aligned}$$

The first of the three integrals in the last member of Eq. A6 is identically zero by Eq. A3. The second integral is now shown to approach zero as T approaches infinity. We assume that, regardless of the value of τ' shown in Fig. A-1, the average number of upward crossings of $x=\alpha$ per unit in any interval $(0, T)$ following the upward crossing at $t=-\tau'$ and the downward crossing at $t=0$ shown in Fig. A-1 is bounded by some number M . Thus,

$$\frac{1}{T} \int_0^T p_{+|-}(\alpha, t | \tau') dt < M \quad \text{FOR ALL } \tau', \quad (A7)$$

a mathematical statement of the plausible assumption that an arbitrarily large τ' does not induce an infinite number of crossings of $x=\alpha$ in any finite time interval. Then,

$$\begin{aligned} & \int_0^T \int_T^\infty \lambda(\tau') p_{+|-}(\alpha, t | \tau') d\tau' dt \\ &= \int_T^\infty \lambda(\tau') \int_0^T p_{+|-}(\alpha, t | \tau') dt d\tau' \\ &< \int_T^\infty \lambda(\tau') M T d\tau' < M \int_T^\infty \tau' \lambda(\tau') d\tau'. \quad (A8) \end{aligned}$$

The last term of Eq. A8 clearly approaches zero as T approaches infinity, since $\langle \tau' \rangle_{av}$ exists, and thus the second integral in the last member Eq. A6 approaches zero. Thus, in computing the limit of Eq. A6, one needs only to find the limit of the third integral in Eq. A6.

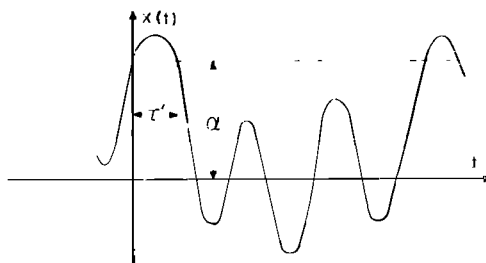


FIG. A-2. Random process with upward crossing of $x=\alpha$ at $t=0$ and first later downward crossing at $t=\tau'$.

Therefore,

$$\begin{aligned} & \int_0^\infty [p_{+|-}(\alpha, t) - p_{+|+}(\alpha, t)] dt \\ &= \lim_{T \rightarrow \infty} \int_0^T [p_{+|-}(\alpha, t) - p_{+|+}(\alpha, t)] dt \\ &= \lim_{T \rightarrow \infty} \int_0^T \lambda(\tau') \int_{T-\tau'}^T p_{+|-}(\alpha, t | \tau') dt d\tau' \\ &= \int_0^\infty \lambda(\tau') \lim_{T \rightarrow \infty} \int_{T-\tau'}^T p_{+|-}(\alpha, t | \tau') dt d\tau' \\ &= \int_0^\infty \lambda(\tau') N_{\alpha^+} \tau' d\tau' = N_{\alpha^+} \langle \tau' \rangle_{av}, \quad (A9) \end{aligned}$$

since

$$\lim_{t \rightarrow \infty} p_{+|-}(\alpha, t | \tau') = N_{\alpha^+}. \quad (A10)$$

Thus, Eqs. 21 and A2 are identical, proving that the renewal-process approximation yields the exact value of the mean recurrence time.