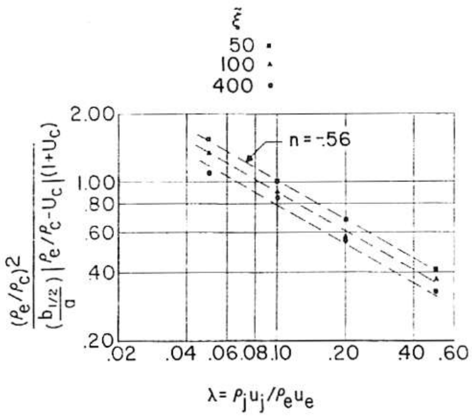

Fig. 5 Power-law dependence of integrand of equation (9) upon λ

Discussion

Within the degree of approximation of the mass-flux scaling rule, considerable freedom of choice of experimental conditions is possible. For example, scaled experiments could be conducted using different gases, at different temperatures and different velocities from the prototype, and provide useful quantitative results. It is interesting that the normal and streamwise coordinates do not scale with λ in the same way as shown by equations (11) and (13). While this does not detract in any important way from the utility of the scaling rule, it is important in that it demonstrates that a purely experimental determination of scaling parameters cannot be based upon centerline concentration measurements alone, as in the case of Zakkay and Krause.[5] However, the results presented herein can be used to determine a streamwise scaling law for the axisymmetric free turbulent jet, again giving a simple power dependence upon λ but, in this case, the exponent is 0.65.

[5] V. Zakkay and E. Krause, "Mixing Problems with Chemical Reaction," AGARD Combustion and Propulsion Colloquium, London, England, 1963, in *Supersonic Flow, Chemical Processes and Radiative Transfer*, Pergamon Press, London, England, 1964.

Starting Transients in the Response of Linear Systems to Stationary Random Excitation

J. R. RICE[1]

THIS NOTE is concerned with the second-order statistics of the nonstationary response, $x(t)$, of a linear time-invariant system to a stationary random excitation, $f(t)$, suddenly applied at time $t = 0$. It is supposed that $x(t)$ satisfies the constant coefficient nth-order differential equation

$$L[x] = a_n x^{(n)}(t) + a_{n-1}x^{(n-1)}(t) + \ldots + a_1 x^{(1)}(t) + a_0 x(t) = \begin{cases} f(t) & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases} \quad (1)$$

with the initial conditions $x(0) = x^{(1)}(0) = \ldots = x^{(n-1)}(0) = 0$. An equation such as (1) would arise in the case of a vibratory mechanical system suddenly subjected to a random force or in the case of a random noise current suddenly switched into an electrical circuit. Such problems have been considered in [1, 2][2] and other works cited therein. However, the relation between the nonstationary response of (1) and the stationary response of the same system, to be discussed here, has not been noted previously.

Let $h(t)$ be the unit impulse response function satisfying $L[h] = \delta(t)$, where $\delta(t)$ is the Dirac delta. Then the solution of (1) is

$$\begin{aligned} x(t) &= \int_0^t h(t-\tau)f(\tau)d\tau \\ &= \int_{-\infty}^t h(t-\tau)f(\tau)d\tau - \int_{-\infty}^0 h(t-\tau)f(\tau)d\tau \end{aligned} \quad (2)$$

The last two integrals in (2) are readily identified. Let $s(t)$ be the stationary response satisfying $L[s] = f(t)$ for all $t > -\infty$. Then the first integral in (2) is $s(t)$ and the second integral represents the response of the system at time $t > 0$ to the excitation $f(\tau)$ applied for only negative time. Thus, the second integral is simply the solution of $L[y] = 0$ under the initial conditions $y(0) = s(0)$, $y^{(1)}(0) = s^{(1)}(0), \ldots, y^{(n-1)}(0) = s^{(n-1)}(0)$. Therefore, (2) becomes

$$\begin{aligned} x(t) &= s(t) - s(0)g_0(t) - s^{(1)}(0)g_1(t) - \ldots - s^{(n-1)}(0)g_{n-1}(t) \\ &= s(t) - \sum_{j=0}^{n-1} s^{(j)}(0)g_j(t) \end{aligned} \quad (3)$$

where $g_j(t)$ is the solution of $L[g_j] = 0$ with the initial conditions that the jth derivative, $g_j^{(j)}(0)$, equal unity and that all other derivatives vanish.

With the expression of (3), the statistics of $x(t)$ may be expressed in terms of the statistics of the stationary response, $s(t)$, to the excitation $f(t)$. Let M_s and $R_s(t_2 - t_1)$ be the mean and covariance of $s(t)$:

$$\begin{aligned} M_s &= E\{s(t)\} \\ R_s(t_2 - t_1) &= E\{[s(t_1) - M_s][s(t_2) - M_s]\} \end{aligned} \quad (4)$$

Then the mean of $x(t)$ is

$$\begin{aligned} M_x(t) &= E\{x(t)\} \\ &= E\{s(t)\} - \sum_{j=0}^{n-1} E\{s^{(j)}(0)\}g_j(t) \\ &= M_s[1 - g_0(t)] \end{aligned} \quad (5)$$

since the expected value of any derivative of a stationary process is zero. Since $g_0(0) = 1$ and $g_0(t) \to 0$ as $t \to \infty$, the mean of $x(t)$ is initially zero and approaches the stationary mean M_s for large t. The covariance of $x(t)$ is

$$\begin{aligned} \rho_x(t_1, t_2) &= E\{[x(t_1) - M_x(t_1)][x(t_2) - M_x(t_2)]\} \\ &= E\Bigg\{\Bigg[s(t_2) - M_s + s(0)g_0(t_2) - M_s g_0(t_2) \\ &\quad + \sum_{j=1}^{n-1} s^{(j)}(0)g_j(t_2)\Bigg]\Bigg[s(t_1) - M_s + s(0)g_0(t_1) - M_s g_0(t_1) \\ &\quad + \sum_{j=1}^{n-1} s^{(j)}(0)g_j(t_1)\Bigg]\Bigg\} \end{aligned} \quad (6)$$

Upon performing the foregoing multiplication and taking expectations, noting from (4) the result

$$E\left\{\left(\frac{d^j}{dt_1^{\,j}}[s(t_1) - M_s]\right)\left(\frac{d^k}{dt_2^{\,k}}[s(t_2) - M_s]\right)\right\} = (-1)^j R^{(j+k)}(t_2 - t_1) \quad (7)$$

the covariance of $x(t)$ becomes

$$\begin{aligned} \rho_x(t_1, t_2) &= R_s(t_2 - t_1) \\ &\quad - \sum_{j=0}^{n-1} (-1)^j[R_s^{(j)}(t_1)g_j(t_2) + R_s^{(j)}(t_2)g_j(t_1)] \\ &\quad + \sum_{=0}^{n-1}\sum_{k=0}^{n-1} (-1)^k g_j(t_1)g_k(t_2)R_s^{(j+k)}(0) \end{aligned} \quad (8)$$

[1] Research Associate, Division of Engineering, Brown University, Providence, R. I.

[2] Numbers in brackets designate References at end of Note.

Manuscript received by ASME Applied Mechanics Division, September 9, 1964.

After some algebraic manipulations and noting that $R_s^{(j)}(0) = 0$ if j is an odd integer, equation (8) takes the more compact form

$$\rho_x(t_1, t_2) = R_s(t_2 - t_1) - \sum_{j=0}^{n-1} (-1)^j [R_s^{(j)}(t_1) g_j(t_2) + R_s^{(j)}(t_2) g_j(t_1)] + \sum_{j=0}^{n-1} \left\{ R_s^{(2j)}(0) \sum_{k=\max(0, 2j-n+1)}^{\min(2j, n-1)} (-1)^k g_k(t_1) g_{2j-k}(t_2) \right\} \quad (9)$$

The variance of $x(t)$ is $\sigma_x^2(t) = E\{[x(t) - M_x(t)]^2\} = \rho_x(t, t)$. Since $\{g_j(t)\} \to 0$ as $t \to \infty$, the covariance (9) of the response approaches the stationary covariance $R_s(t_2 - t_1)$ when t_2 and t_1 are large, and terms beyond the first in (9) represent the influence of stochastic starting transients occurring as the system is excited from a state of rest to a state of statistically stationary motion.

The stationary mean, M_s, and covariance, $R_s(\tau)$, may be determined from the mean and covariance of the excitation, $f(t)$, through standard methods as described, for example, in [3]. In the case of some practical importance, when $f(t)$ is Gaussian, $x(t)$ is also Gaussian and probability distribution functions of $x(t)$ may be determined once the mean $M_x(t)$ and covariance $\rho_x(t_1, t_2)$ are known.

References

1 T. K. Caughey and H. J. Stumpf, "Transient Response of a Dynamic System Under Random Excitation," JOURNAL OF APPLIED MECHANICS, vol. 28, TRANS. ASME, vol. 83, Series E, 1961, pp. 563–566.

2 Y. K. Lin, "Nonstationary Response of Continuous Structures to Random Loading," *J. Acoustical Society of America*, vol. 35, February, 1963.

3 J. S. Bendat, *Principles and Applications of Random Noise Theory*, John Wiley & Sons, Inc., New York, N. Y., 1958.

Apparatus for Measuring Thrust Produced on Rotating Body Immersed in a Liquid

W. H. HOPPMANN, II,[1] **and C. N. BARONET**[2]

Introduction

IT IS important to be able to measure the torque and thrust on a solid body rotating about a fixed axis and partially immersed in a liquid. In the past, various attempts have been made to measure these important physical variables. If the rotating solid body is symmetrical in shape and the liquid is held in a symmetrical container whose axis of symmetry coincides with the axis of rotation, the vectors representing the resultant torque and force are directed along the axis of rotation. These physical variables are of considerable interest in programs of research using rotational flow generators.

Within the last two years, a fluid flow generator for the purpose of making general studies of flow in liquids induced by rotating symmetrical bodies was designed and built.[3] As part of that apparatus, a transducer to measure the torque was satisfactorily developed.

Recently, effort has been devoted to the development of a suitable transducer to measure the force which hereinafter is referred to as the thrust. It is the purpose of the present paper to fully describe that instrument and to present interesting experimental data which have been obtained with its aid.

[1] Professor of Mechanics, Rensselaer Polytechnic Institute, Troy, N. Y. Mem. ASME.

[2] Graduate Student, Rensselaer Polytechnic Institute, Troy, N. Y.

[3] W. H. Hoppmann, II, and C. E. Miller, "Rotational Fluid Flow Generator for Studies in Rheology," *Transactions of the Society of Rheology*, vol. 7, 1963, pp. 181–193.

Manuscript received by ASME Applied Mechanics Division, April 17, 1964.

Nature and Design of the Thrust Transducer

The thrust transducer was conceived and designed for use with the rotational fluid flow generator mentioned previously. A sufficiently informative sketch of that apparatus with the present transducer in place is shown in Fig. 1. The flow-inducing body shown is a cone. The cylindrical plexiglass pot symmetrically located is used to hold the liquid.

A detailed view of the thrust transducer is shown in Fig. 2. It consists of a short sleeve which attaches to the drive shaft of the flow generator and a solid circular plate which attaches to any experimental rotor under study. The sleeve and plate are connected to each other by a spider consisting of a set of three equispaced flat steel strips. The relative displacement of the sleeve and ring produced by an axial thrust flexes the steel strips, producing measurable strains. In this manner, the desired thrust can be measured. Silicon-type strain gages are cemented to the flexure strips in the form of a Wheatstone bridge for the purpose of measuring the strain. The electrical output from the gages is taken from contacts on the rotating shaft through

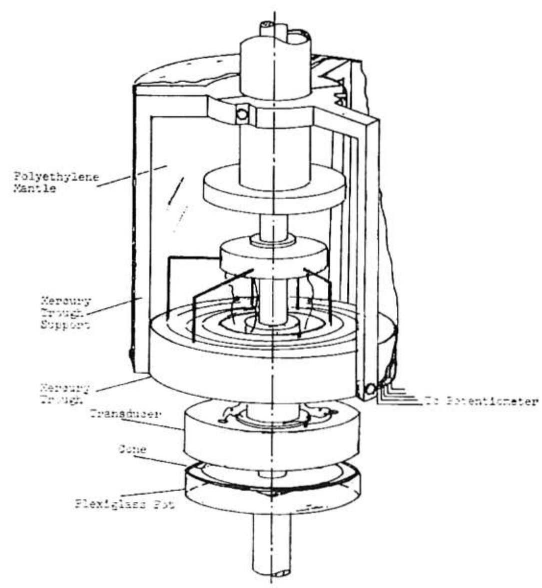

Fig. 1 Assembly drawing

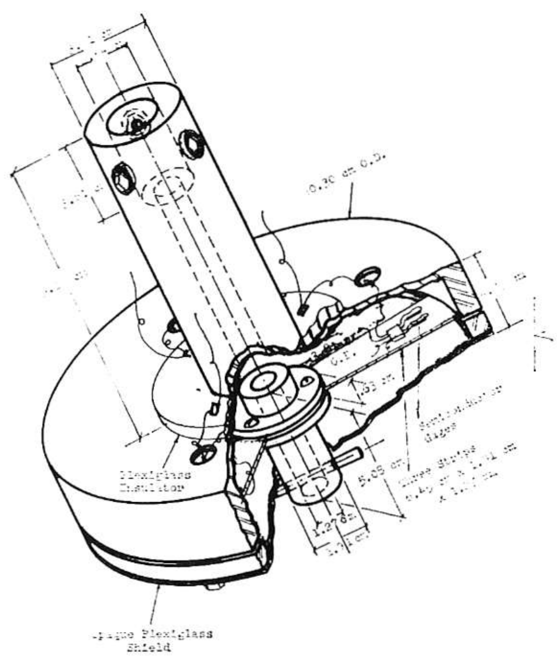

Fig. 2 Thrust transducer