On the Prediction of Some Random Loading Characteristics Relevant to Fatigue[1]

by

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Abstract

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This paper deals with the theoretical prediction of some statistical characteristics of continuous stationary random loadings which are relevant to studies of fatigue. Primary emphasis is given to the determination of the distribution and average height of load rises and falls. A summary is also given of some other statistical information useful in fatigue analysis. Particular examples are worked out for Gaussian processes with idealized spectra and experimental data on fatigue crack propagation under random loadings is cited.

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Introduction

The fluctuating stress responsible for fatigue failure of mechanical systems is often in the form of a continuous stationary random process. It is therefore important to have available techniques for predicting statistical averages and distributions of certain random loading characteristics relevant to fatigue. This is the subject of the work to follow.

We shall devote most of our attention here to statistical aspects of the height of rise and fall in a continuous random loading. By the height of rise we mean the increment h (as depicted in figure 1) in a random function as it passes from a minimum to the next maximum, the height of fall being similarly defined as the decrement in passing from a maximum to the next minimum. Recent work on crack growth rates [1] and fatigue lives [2] under random loading has shown that the primary responsibility for fatigue damage lies with the rise and fall in the loadings rather than other statistical quantities such as the mean load level or distribution of maxima and level crossings. A study of the rise and fall problem in considerably greater detail than to be given here has recently been reported by two of the present writers in [3].

In addition to the material on rises and falls, we shall also summarize some results in [4] relevant to fatigue analysis of random loadings. Numerical examples will be given for the case of Gaussian processes and experimental data cited on crack propagation under random loadings.

Throughout the work to follow we shall deal with stationary processes with zero mean values and with continuous first and second derivatives. Frequent use will be made of the correlation function $R(\tau)$ of such a process, $x(t)$. This function is defined as

$$R(\tau) = E\left\{x(t)\ x(t+\tau)\right\} , \tag{1}$$

where $E\{\ldots\}$ denotes the expected (or average) value of the quantity in $\{\ldots\}$. It is clear that $R(o)$ is the variance (or square of the standard deviation) for the process and that the average value of the product of the j th derivative at time t and k th derivative at time $t+\tau$ is

$$(-1)^j R^{(j+k)}(\tau) = E\left\{x^{(j)}(t)\, x^{(k)}(t+\tau)\right\} . \tag{2}$$

Such processes can alternately be characterized by a power spectral density $F(\omega)$ reflecting the frequency content of the random process and related to $R(\tau)$ by the Weiner-Khinchin relations

$$R(\tau) = \int_0^\infty F(\omega) \cos \omega\tau \, d\omega \tag{3}$$

$$F(\omega) = \frac{2}{\pi} \int_0^\infty R(\tau) \cos \omega\tau \, d\tau . \tag{4}$$

In what follows results will always be given first in general terms and then specialized to the technically important case of Gaussian processes. For a stationary Gaussian process all statistical means and distributions depend only on $R(\tau)$ or alternately $F(\omega)$.

Level Crossings and Extrema

A number of results in the analysis of stationary random processes are reported in [4]. In this section we will summarize some results of [4] pertinent to fatigue analysis dealing with the expected number of level crossings and extrema per unit time in a random loading, our aim being to develop expressions for use in the following sections and to provide a brief summary for readers unaquainted with this area.

Consider first the expected number of crossings of $x(t) = \alpha$ per unit time, N_α. Choosing an infinitessimal time interval dt at some arbitrary point on the time axis, N_α dt may be interpreted as the probability of an α crossing in time dt. Now there will be an α crossing if, (a), $\alpha-|\mathring{x}(t)|dt<x(t)<\alpha$ and $\mathring{x}(t)>0$ or, (b), $\alpha<x(t)<\alpha+|\mathring{x}(t)|dt$ and $\mathring{x}(t)<0$. Letting $g_{x\mathring{x}}(u,v)$ be the joint probability density of $x(t)$ and $\mathring{x}(t)$ (such that du dv $g_{x\mathring{x}}(u,v)$ is the probability that $u<x(t)<u+du$ and $v<\mathring{x}(t)<v+dv$), the probability of an α crossing will be

$$N_\alpha dt = \int_0^\infty \int_{\alpha-|v|dt}^{\alpha} g_{x\mathring{x}}(u,v)du\, dv + \int_{-\infty}^{0}\int_{\alpha}^{\alpha+|v|dt} g_{x\mathring{x}}(u,v)du\, dv \tag{5}$$

Performing the integration in u and recognizing that dt is infinitesimal,

$$N_\alpha = \int_{-\infty}^{+\infty} |v| g_{x\mathring{x}}(\alpha,v)\, dv. \tag{6}$$

Writing $g_{x\dot{x}}(u,v) = g_{\dot{x}|x}(v)|x=u)\ g_x(u)$ where $g_{\dot{x}|x}(v|x=u)$ is the conditional probability density of $\dot{x}(t)$ given that $x(t) = u$ and $g_x(u)$ is the probability density of $x(t)$,

$$N_\alpha = g_x(\alpha)\int_{-\infty}^{+\infty}|v|\,g_{\dot{x}|x}(v|x=\alpha)dv = g_x(\alpha)E\left\{|\dot{x}|\,\middle|\,x=\alpha\right\} . \qquad (7)$$

In the above expression $E\left\{|\dot{x}|\,\middle|\,x = \alpha\right\}$ is the conditional expected value of $|\dot{x}(t)|$ given that $x(t) = \alpha$, or the average of $|\dot{x}(t)|$ at all points where $x(t)$ has the value α. Letting $\alpha=o$, we obtain the expected number N_o of mean crossings per unit time:

$$N_o = \int_{-\infty}^{+\infty} |v|\ g_{x\dot{x}}(o,v)dv = g_x(o)\ E\left\{|\dot{x}|\,\middle|\,x = o\right\} \qquad (8)$$

The expected number of extrema per unit time, N_e, is simply the expected number of zeros per unit time of the process $\dot{x}(t)$. Thus the determination of N_e is identical to that of N_o, except that now $\dot{x}$ replaces x and $\ddot{x}$ replaces $\dot{x}$. Letting $g_{\dot{x}\ddot{x}}(v,w)$ be the joint density of $\dot{x}(t)$ and $\ddot{x}(t)$, we write

$$N_e = \int_{-\infty}^{+\infty}|w|\,g_{\dot{x}\ddot{x}}(o,w)dw = g_{\dot{x}}(o)E\left\{|\ddot{x}|\,\middle|\,\dot{x} = o\right\} \qquad (9)$$

The procedure can be continued to obtain the expected number of inflection points per unit time (zero crossings of $\ddot{x}$) and so on. In general, the expected number of zeros per unit time of the $k^{\underline{th}}$ derivative $x^{(k)}(t)$ is

$$N\left(x^{(k)}=o\right) = g_{x^{(k)}}(o)\ E\left\{|x^{(k+1)}|\,\middle|\,x^{(k)}=o\right\} . \qquad (10)$$

For processes (such as Gaussian processes) where any two successive derivatives $x^{(k)}(t)$ and $x^{(k+1)}(t)$ are independent random variables, (21) and (23) take the simpler forms

$$N_\alpha = g_x(\alpha)\; E\left\{|\dot{x}|\right\} \quad , \; N_e = g_{\dot{x}}(o) E\left\{|\ddot{x}|\right\} \quad . \qquad (11)$$

In the Gaussian case the probability densities of x(t), $\dot{x}(t)$, and $\ddot{x}(t)$ are

$$g_x(u) = \frac{1}{\sqrt{2\pi R(o)}} \quad \exp \left\{- 1/2 \; \frac{u^2}{R(o)} \right\}$$

$$g_{\dot{x}}(v) = \frac{1}{\sqrt{-2\pi R''(o)}} \exp \left\{- 1/2 \; \frac{v^2}{-R''(o)} \right\}$$

$$g_{\ddot{x}}(w) = \frac{1}{\sqrt{2\pi R^{(4)}(o)}} \exp \left\{ - 1/2 \; \frac{w^2}{R^{(4)}(o)} \right\} \qquad (12)$$

The terms R(o), -R"(o), and $R^{(4)}(o)$ are obtained by differentiation of the correlation function defined in (1) and (3), and represent by (2) the variances of x(t), $\dot{x}(t)$, and $\ddot{x}(t)$ respectively. The expected values appearing in (11) are

$$E\left\{|\dot{x}|\right\} = \int_{-\infty}^{+\infty} |v| g_{\dot{x}}(v) dv = \sqrt{- \frac{2}{\pi} R''(o)}$$

$$E\left\{|\ddot{x}|\right\} = \int_{-\infty}^{+\infty} |w| g_{\ddot{x}}(w) dw = \sqrt{\frac{2}{\pi} R^{(4)}(o)} \quad . \qquad (13)$$

Thus for Gaussian process, equations (7), (8), (9), and by an obvious extension, (10) become after noting the simpler form of

(11) appropriate in this case

$$N_\alpha = \frac{1}{\pi} \sqrt{\frac{-R''(o)}{R(o)}} \exp\left\{-1/2 \frac{\alpha^2}{R(o)}\right\} , \tag{14}$$

$$N_o = \frac{1}{\pi} \sqrt{\frac{-R''(o)}{R(o)}} , \tag{15}$$

$$N_e = \frac{1}{\pi} \sqrt{\frac{R^{(4)}(o)}{-R''(o)}} , \tag{16}$$

$$N\left(x^{(k)} = o\right) = \frac{1}{\pi} \sqrt{- \frac{R^{(2k+2)}(o)}{R^{(2k)}(o)}} . \tag{17}$$

These are readily converted to expressions in terms of the spectral density by noting from (3) that

$$R(o) = \int_o^\infty F(\omega)\, d\omega$$

$$-R''(o) = \int_o^\infty \omega^2 F(\omega)\, d\omega$$

$$R^{(4)}(o) = \int_o^\infty \omega^4 F(\omega)\, d\omega \tag{18}$$

The expected number of crossings of $x(t) = \alpha$ per unit time as given by (14) has been used to construct approximate solutions for the probability distribution of the operating time of a randomly loaded system before the occurence of an extreme load level which would cause failure. Such approximate solutions are given in [5] and [6] (which should be read in conjunction with the comment of [7]).

Average Rise and Fall Height

The average height of rise and fall is easily determined (as will be seen later, the determination of the distribution of rise and fall heights is a very difficult problem) and is a simple parameter describing the fatigue-damaging effect of a random loading. The average rise and fall per unit time is the expected value of the aboslute value of $\dot{x}(t)$, $E\{|\dot{x}|\}$; dividing by the expected number of extrema per unit time, N_e, yields the following expression for the average height:

$$\bar{h} = \frac{E\{|\dot{x}|\}}{N_e} \qquad (19)$$

It was shown in the last section that N_e is related to the conditional expectation of $|\ddot{x}|$ given that $\dot{x} = o$ by

$$N_e = g_{\dot{x}}(o)E\left\{|\ddot{x}| \;\middle|\; \dot{x} = o\right\} , \qquad (8)$$

where $g_{\dot{x}}(v)$ is the probability density of $\dot{x}(t)$.

The average height of rise and fall may be related to the ratio of N_e and the expected number of zero crossings per unit time, N_o, in the special case of processes for which $x(t)$ and $\dot{x}(t)$ are independent random variables. This is the case for at least one type of processes, namely Gaussian processes. The expected number of zero crossings (or mean level crossings) per unit time is

$$N_o = g_x(o)\ E\left\{|\dot{x}| \;\middle|\; x = o\right\} \qquad (9)$$

where $g_x(u)$ is the probability density of $x(t)$. If $x(t)$ and $\dot{x}(t)$ are independent, $E\left\{|\dot{x}|\,\middle|\,x=o\right\} = E\left\{|\dot{x}|\right\}$ and, using (8), (19) becomes

$$\bar{h} = \frac{1}{g_x(o)} \quad \frac{N_o}{N_e} \tag{20}$$

This expression provides a very convenient experimental method for determing $\bar{h}$; one need only count the number of zero crossings and extrema in a sufficiently large time interval and an actual measurement of rise and fall heights is unnecessary. Since N_e must always exceed N_o, equation (8) gives as an upper bound on $\bar{h}$

$$\bar{h} \leq \frac{1}{g_x(o)} \tag{21}$$

For the case of Gaussian processes the values of $E\left\{|\dot{x}|\right\}$, N_e, N_o, and $g_x(o)$ are given by (13), (16), (15), and (12) respectively. Thus equations (19), (20), and (21) take the form

$$\bar{h} = -R''(o)\sqrt{\frac{2\pi}{R^{(4)}(o)}} \tag{22}$$

$$\bar{h} = \sqrt{2\pi R(o)}\,\frac{N_o}{N_e} \quad \text{or} \quad \frac{\bar{h}}{\sigma} = \sqrt{2\pi}\,\frac{N_o}{N_e} \tag{23}$$

$$\bar{h} \leq \sqrt{2\pi R(o)} \quad \text{or} \quad \frac{\bar{h}}{\sigma} \leq \sqrt{2\pi} \tag{24}$$

The alternate forms of the last two expressions follow from noting that $R(o) = \sigma^2$, where σ is the root mean square value of the process $x(t)$. The average rise and fall height of (22) above may be expressed in terms of the second and fourth moments of the power spectrum through equations (18).

Distribution of Load Maxima and Minima

A statistical distribution of considerable interest in fatigue and fracture analysis is the distribution of load levels at which maxima or minima occur. We give below a slight modification of results in [4] pertinent to this problem. Let $f_M(\alpha)\,d\alpha$ be the probability that, given a maximum at t, its value falls in the interval $\alpha<x(t)<\alpha+d\alpha$. This is expressible in terms of the triple joint probability density $g_{x\dot{x}\ddot{x}}(u,v,w)$ for $x(t)$, $\dot{x}(t)$, and $\ddot{x}(t)$. The probability of having a maximum in an infinitesimal time dt with a value between α and $\alpha+d\alpha$ is by the law of conditional probabilities

$$f_M(\alpha)d\alpha\,\frac{N_e}{2}\,dt = \int_{-\infty}^{o}\int_{o}^{|w|dt}\int_{\alpha}^{\alpha+d\alpha} g_{x\dot{x}\ddot{x}}(u,v,w)du\;dv\;dw, \qquad (25)$$

since such an event will occur if $\alpha<x(t)<\alpha+d\alpha$, $o<\dot{x}(t)<|\ddot{x}(t)|dt$, and $\ddot{x}(t)<o$. Thus

$$f_M(\alpha) = \frac{2}{N_e}\int_{-\infty}^{o}|w|\;g_{x\dot{x}\ddot{x}}(\alpha,o,w)dw \quad . \qquad (26)$$

Similarly, the probability density $f_m(\alpha)$ that, given a minimum at t, its value falls in the interval $\alpha<x(t)<\alpha+d\alpha$ is

$$f_m(\alpha) = \frac{2}{N_e}\int_{o}^{\infty}|w|\;g_{x\dot{x}\ddot{x}}(\alpha,o,w)dw \quad . \qquad (27)$$

The average number of maxima and minima per unit time with values between α and $\alpha+d\alpha$ are respectively

$$\frac{N_e}{2} \; f_M(\alpha)d\alpha \qquad , \frac{N_e}{2} f_m(\alpha)d\alpha \quad . \tag{28}$$

In the Gaussian case $\dot{x}(t)$ is independent of $x(t)$ and $\ddot{x}(t)$. Thus $g_{x\dot{x}\ddot{x}}(u,v,w) = g_{\dot{x}}(v) \; g_{x\ddot{x}}(u,w)$, where $g_{\dot{x}}(v)$ is given in (12) and

$$g_{x\ddot{x}}(u,w) = \frac{1}{2\pi\sqrt{k}} \exp[- \frac{1}{2k} \left\{R^{(4)}(o)u^2 - 2R''(o)uw + R(o)w^2\right\}] \; , \tag{29}$$

where

$$k = R(o) \; R^{(4)}(o) - \left\{R''(o)\right\}^2 . \tag{30}$$

Carrying out the integration in (26),

$$f_M(\alpha) = \frac{-R''(o)}{2R(o)\sqrt{R^{(4)}(o)R(o)}} \exp\left(- \frac{\alpha^2}{2R(o)}\right) \left[\alpha\left\{1+\mathrm{erf}\left(\frac{-R''(o)\alpha}{\sqrt{2kR(o)}}\right)\right\} - \frac{\sqrt{2kR(o)}}{\sqrt{\pi}\, R''(o)} \exp\left(- \frac{\{R''(o)\alpha\}^2}{2kR(o)}\right) \right] . \tag{31}$$

For large values of α, the error function approaches unity ($R''(o)$ being negative) and dominates the exponential term. Thus, when α is large

$$f_M(\alpha) \approx \frac{-R''(o)\alpha}{R(o)\sqrt{R(o)R^{(4)}(o)}} \exp\left\{- \frac{\alpha^2}{2R(o)}\right\} . \tag{32}$$

The density of minimum values is given, in the Gaussian case as in the case of any symmetrical process, by

$$f_m(\alpha) = f_M(-\alpha) \quad . \tag{33}$$

The distribution of maxima in a randomly varying stress at the tip of a growing fatigue crack has been employed in [8] for a study of structural failure through crack propagation.

In deriving expressions for N_α and N_e in an earlier section we have interpreted $N_\alpha dt$ and $N_e dt$ as representing, respectively, the probability of an α crossing and the probability of an extremum of x(t) in the time interval t to t+dt. The expression for $f_M(\alpha)$ derived in this section was interpreted as the probability density of x(t), given that a maximum occured at time t. With these interpretations, the results of (6), (9), and (26) for N_α, N_e, and $f_M(\alpha)$, respectively, clearly hold true even when x(t) is a non-stationary process.

Rise and Fall Distribution

The exact determination of the distribution of rises and falls in a continuous random process is a very difficult problem, although the average rise and fall height is relatively easy to compute. We shall give a brief summary of an approximate technique below, referring the reader to [3] for a more general formulation with considerably more attention paid to details of derivations and analysis of the approximations introduced.

Assume the random process $x(t)$ under consideration to have continuous second derivatives and let $P(h,\tau)dh\,d\tau$ be the probability that, given a minimum in $x(t)$ at $t=o$, the next maximum occurs in the time interval $\tau<t<\tau+d\tau$ and that $h<x(\tau)-x(o)<h+dh$ (see fig. 1). Then the rise and fall density is

$$P(h) = \int_o^\infty P(h,\tau)d\tau \quad . \tag{34}$$

As a first step in the determination of $P(h,\tau)$ we write

$$P(h,\tau) = f_o(h|\tau)p_o(\tau) \quad , \tag{35}$$

where $f_o(h|\tau)$ is the conditional density of h given that the next maximum occurs at $t = \tau$ and $p_o(\tau)$ is the density of the time interval between successive extrema. These two functions will be approximated in the work to follow, our aim being to replace them by functions which can be calculated.

Taking $f_o(h|\tau)$ first, we approximate it by

$$f_o(h|\tau) \approx f(h|\tau) \quad , \tag{36}$$

where $f(h|\tau)$ dh is the probability, given a minimum in $x(t)$ at $t=o$ and a maximum (not necessarily the first) at $t=\tau$, that $h<x(\tau)- x(o)<h+dh$. The approximation becomes exact for small τ since the probability of having another maximum in $o<t<\tau$ is correspondingly small. For large values of τ there will be a discrepancy, but this discrepancy will be unimportant since $p_o(\tau)$ is small for large τ. The approximation for $p_o(\tau)$ (which represents the zero crossing density for $\mathring{x}(t)$) is obtained after expressing it first in the form derived in [3]

$$p_o(\tau)=p(\tau|o<t<\tau) \exp\left\{-\int_o^\tau p(s|o<t<s)\, ds\right\}, \tag{37}$$

where $p(\tau|o<t<\tau)\, d\tau$ is the probability of a maximum in $\tau<t<\tau+d\tau$, given a minimum at $t=o$ and no other maxima in the internal $o<t<\tau$. We now approximate $p(\tau|o<t<\tau)$ by

$$p(\tau|o<t<\tau) \approx p(\tau), \tag{38}$$

where $p(\tau)d\tau$ is the probability, given a minimum at $t=o$, that a maximum occurs in $\tau<t<\tau+d\tau$. Thus (37) becomes

$$p_o(\tau) \approx p(\tau) \exp\left\{- \int_o^\tau p(s)\, ds\right\} . \tag{39}$$

For small τ the exponential term is approximately unity and (39) yields the type of approximation given in [4] for the zero crossing density. This is a good approximation for small τ, since the probability of other maxima in $o<t<\tau$ is also small. As $\tau \to \infty$, the right-hand member in (39) approaches zero exponentially, since $p(\tau)$ approaches a constant representing the expected number of maxima per unit time. Little can be said about the closeness of the approximation when τ is not small, but we note that integrating the right-hand member of (39) in τ from zero to infinity yields unity.

We note that the approximation obtained for $P(h,\tau)$ through the use of equations (35), (36), and (39) involves the expression $f(h|\tau)\, p(\tau)$. By the law of conditional probability this expression is equal to $f(h,\tau)$ where $f(h,\tau)\, dh\, d\tau$ is the probability, given a minimum at $t=o$, that a maximum (not necessarily the first) occurs in $\tau<t<\tau+d\tau$ and that $h<x(\tau)-x(o)<h+dh$. Thus, equation (34) for the rise and fall density becomes

$$P(h) \approx \int_o^\infty f(h,\tau)\, \exp\left\{- \int_o^\tau p(s)ds\right\} d\tau \quad . \tag{40}$$

The functions $f(h,\tau)$ and $p(\tau)$ appearing in (51) are expressible in terms of joint density functions for the process $x(t)$ and its derivatives. The results follow from essentially the same sort of approach which led to (6) and (26). Letting N_e be the expected number of extrema per unit time,

$$p(\tau) = \frac{2}{N_e} \int_{-\infty}^{o} \int_o^\infty |ww'|\, g(o,w;o,w';\tau)dw\, dw' \tag{41}$$

where $g(v,w;\ v',w';\tau)$ is the joint density of $\mathring{x}(o)$, $\ddot{x}(o)$, $\mathring{x}(\tau)$, and $\ddot{x}(\tau)$ represented by v,w,v', and w' respectively, and

$$f(h,\tau) = \frac{2}{N_e}\int_{-\infty}^{o}\int_{o}^{\infty}\int_{-\infty}^{+\infty}|ww'|g(u,o,w;u+h,o,w';\tau)du\ dw\ dw' \qquad (42)$$

where $g(u,v,w;u',v',w';\tau)$ is the joint density of $x(o)$, $\mathring{x}(o)$, $\ddot{x}(o)$, $x(\tau)$, $\dot{x}(\tau)$, and $\ddot{x}(\tau)$.

For applications to the case of stationary Gaussian processes the multi-dimensional Gaussian distribution [4] must be used. The expression (41) for $p(\tau)$ becomes

$$p(\tau) = \frac{1}{2\pi}\left\{\frac{-R''(o)}{R^{(4)}(o)}\right\}^{1/2}\left\{J^2-K^2\right\}^{1/2}$$

$$\left[\left\{R''(o)\right\}^2 - \left\{R''(\tau)\right\}^2\right]^{-3/2}\left\{1+H\cot^{-1}(-H)\right\}\ , \qquad (43)$$

where

$$J = R^{(4)}(o)\left[\left\{R''(o)\right\}^2 - \left\{R''(\tau)\right\}^2\right] + R''(o)\left\{R^{(3)}(\tau)\right\}^2$$

$$K = -R^{(4)}(\tau)\left[\left\{R''(o)\right\}^2 - \left\{R''(\tau)\right\}^2\right] - R''(\tau)\left\{R^{(3)}(\tau)\right\}^2$$

$$H = K\left\{J^2 - K^2\right\}^{-1/2} \qquad (44)$$

and $o \leq \cot^{-1}(-H) \leq \pi$. The expression for $f(h,\tau)$ cannot be expressed in closed form, but may be reduced (see [3]) to a single integral in a form convenient for numerical evaluation:

$$f(h,\tau) = 1/2\ A \exp\left(-B\ h^2\right) \int_0^{\pi/4} \frac{\sin 2\theta}{(1+c \sin 2\theta)^2} \Big[1+z^2 + \sqrt{\pi}\, z(\tfrac{3}{2}+z^2) \exp(z^2) \left\{1+ \mathrm{erf}(z)\right\} \Big] \, d\theta \quad , \tag{45}$$

where

$$z = z(h,\tau,\theta) = \frac{Dh(\sin\theta + \cos\theta)}{\sqrt{1+C \sin 2\theta}}$$

$$A = \left\{\frac{-R''(o)}{R^{(4)}(o)}\right\}^{1/2} \frac{1}{\pi^{3/2}\, q^2 \sqrt{|M|(s_{11}+s_{14})}} \quad ,$$

$$B = 1/4\ (s_{11}-s_{14}) \quad ,$$

$$C = -\ \frac{s_{36}-\ s_{33}+q}{q} \quad ,$$

$$D = \frac{s_{13}-s_{16}}{2\sqrt{2q}} \quad ,$$

$$q = s_{33} - \frac{(s_{13}+s_{16})^2}{2(s_{11}+s_{14})} \quad , \tag{46}$$

and where $|M|$ is the determinant of the matrix M and s_{ij} is the $i^{\underline{th}}$ row and $j^{\underline{th}}$ column member of the inverse matrix of M, M being the six by six correlation matrix

$$
M = \begin{bmatrix}
R(o) & o & R''(o) & R(\tau) & R'(\tau) & R''(\tau) \\
o & -R''(o) & o & -R'(\tau) & -R''(\tau) & -R^{(3)}(\tau) \\
R''(o) & o & R^{(4)}(o) & R''(\tau) & R^{(3)}(\tau) & R^{(4)}(\tau) \\
R(\tau) & -R'(\tau) & R''(\tau) & R(o) & o & R''(o) \\
R'(\tau) & -R''(\tau) & R^{(3)}(\tau) & o & -R''(o) & o \\
R''(\tau) & -R^{(3)}(\tau) & R^{(4)}(\tau) & R''(o) & o & R^{(4)}(o)
\end{bmatrix} \quad (47)
$$

It is seen that, for Gaussian processes, the evaluation of $p(\tau)$ and $f(h,\tau)$ required in the approximate expression (40) of the rise and fall density presupposes a knowledge of the correlation function $R(\tau)$ and of its first four derivatives. According to [9] we are assured that these exist for any process for which second derivatives exist. In terms of the spectral density this means that $F(\omega)$ must be bounded by an expression of the form $c/\omega^{5+\varepsilon}$ for large ω, where c is a constant and ε is any positive number.

More particular information on the rise and fall distribution, such as the conditional density of rises given a minimum at some fixed level, can also be obtained through modifications (see [3]) of the techniques given above.

It is appropriate to mention two other attempts at the determination of the rise and fall distribution in references [10] and [11]. Schjelderup [10] gives a treatment of the rise and fall problem for some special limiting cases of Gaussian processes with power spectra composed of sharp peaks at very widely separated frequencies. His method fails to take account of the dependence of a maximum on the last minimum and is in general not capable of extension to other cases. Kowalewski [11] gives, without derivation or reference, an equation for the rise and fall distribution in a Gaussian process. That the equation is incorrect can easily be seen by computing the average rise and fall height; the result differs considerably from $\bar{h}$ as given by (22) or (23), which has been checked for validity several times by measuring and averaging rise and fall heights in experimental records of random processes.

Particular Examples - Gaussian Processes With Idealized Spectra

Examples are given in this section for stationary Gaussian processes with idealized power spectra of the form

$$F(\omega) = \begin{cases} \dfrac{\sigma^2}{(1-\beta)\omega_c} & \text{for } \beta\omega_c<\omega<\omega_c \\ o & \text{otherwise} \end{cases} \qquad (48)$$

where $o \leqslant \beta < 1$. Here ω_c is an upper cut-off frequency and $\beta\omega_c$ a lower cut-off frequency. When $\beta = o$ the spectrum is that of an ideal 'low-pass' filter and when β is close to unity the spectrum is 'narrow band'. Since $\int_o^\infty F(\omega)d\omega = \sigma^2$, σ is the root mean square value of $x(t)$. It is convenient to use a dimensionless time $\phi = \omega_c t$ and to deal with the process $y(\phi) = {}^{x(t)}/_{\sigma}$ which is dimensionless with unity root mean square. Then the correlation function for $y(\phi)$ is, after an appropriate modification of (3),

$$R(\phi) = E\left\{y(\psi)y(\psi+\phi)\right\} = \frac{1}{\sigma^2}\int_o^\infty F(\omega)\cos\left(\frac{\omega}{\omega_c}\phi\right)d\omega \qquad (49)$$

or

$$R(\phi) = \frac{1}{(1-\beta)\phi}\left\{\sin\phi - \sin\beta\phi\right\}. \qquad (50)$$

For several applications discussed earlier one needs only know certain derivatives at $\phi = o$:

$$R(o) = 1 \quad , \quad R''(o) = -1/3\,\frac{1-\beta^3}{1-\beta} \; , \; R^{(4)}(o) = 1/5\,\frac{1-\beta^5}{1-\beta} \quad . \qquad (51)$$

Denoting by h a rise or fall in x(t), we find from (22) the average rise and fall height of the process $y(\phi)$ is

$$\bar{h}/\sigma = \frac{\sqrt{10\pi}}{3} \frac{1-\beta^3}{\sqrt{(1-\beta)(1-\beta^5)}} . \tag{52}$$

It is easily seen that for $\beta = o$ (ideal low pass filter), $\bar{h} = \sqrt{10\pi}/3\ \sigma \approx 1.87\sigma$, whereas when $\beta \to 1$ (narrow band filter) the average approaches $\bar{h} = \sqrt{2\pi}\sigma \approx 2.51\sigma$ which is the upper bound given by (24). The expected number of crossings of $y(\phi) = \alpha/\sigma$ (that is, of $x(t) = \alpha$), zero crossings, and extrema per unit of time ϕ (that is, per unit of $\omega_c t$) are from (14), (15), and (16) respectively

$$N_\alpha = \frac{1}{\pi} \sqrt{\frac{1-\beta^3}{3(1-\beta)}} \exp\left(-\frac{1}{2} \frac{\alpha^2}{\sigma^2}\right)$$

$$N_o = \frac{1}{\pi} \sqrt{\frac{1-\beta^3}{3(1-\beta)}}$$

$$N_e = \frac{1}{\pi} \sqrt{\frac{3(1-\beta^5)}{5(1-\beta^3)}} . \tag{53}$$

Equations (52) and (53) illustrate a point of some interest in the analysis of stationary processes. Suppose there are two processes $x_1(t)$ and $x_2(t)$ with variances σ_1^2 and σ_2^2 , and that the spectra $F_1(\omega)$ and $F_2(\omega)$ are similar in the sense that they can be made to coincide by appropriate scaleing of the ω and F axes. If ω_1 and ω_2 are characteristic frequencies such as a cut-off frequency or a center frequency, then the processes $y_1(\phi) = {}^{x_1(t)}/\sigma_1$ (where $\phi=\omega_1 t$) and $y_2(\phi) = {}^{x_2(t)}/\sigma_2$ (where $\phi=\omega_2 t$) have identical statistical distributions both in dimensionless amplitude $y_i={}^{x_i}/\sigma_i$ and in dimensionless time $\phi=\omega_i t$.

The rise and fall density for $y(\phi)$ (that is, $P(h/\sigma)$) may be computed from the approximate formulation given in the last section. Considerable difficulties were met in performing numerical computations on a digital computer due to the extreme accuracy required in inverting the correlation matrix of (47) near its singular point at $\tau=o$ and due to the extensive amount of numerical integration. These are discussed at greater length in [3]. Computations of the rise and fall density were made for values of β equal to 0,.25,.50,&.75. Results for $\beta= o$ are shown in figure 2 and for $\beta=0.75$ in figure 3. Figure 4 is a combined plot of computed results for all four cases. The dashed line in figure 2 is a plot of experimental data collected by Leybold [12] from measurements of approximately 53,000 rises and falls in a digitally generated random function with an ideal low pass filter power spectrum. The dashed line in figure 3 is a plot of the Rayleigh distributed rise and fall density which would occur in an extremely narrow band process consisting of a sine wave with an amplitude varying negligibly as the process passes from a minimum to the next maximum. The amplitude R of such a process has the density [4]

$$q(R) = \frac{R}{\sigma_2} \exp\left(-1/2\, \frac{R^2}{\sigma_2}\right) . \qquad (54)$$

Noting that $h/\sigma = {}^{2R}/\sigma$ and using the usual rules for tranformation of stochastic variables, the limiting form of the rise and fall density as the bandwidth is narrowed toward zero is

$$P(h/\sigma) = 1/4(h/\sigma)\exp\left\{-\, 1/8\ (h/\sigma)^2\right\} . \qquad (55)$$

This is the density plotted as a dashed line in fig. 3; it is easily checked that it yields for an average rise and fall height $\bar{h} = \sqrt{2\pi}\sigma$, which is the upper bound of (24). The comparison with the case β=.75 seems appropriate since the value of $\bar{h}$ is only 1.5% less than $\sqrt{2\pi}$. Computed results for small h/σ could not be accurately obtained in this case and thus are not shown.

The agreement of predicted results with experimental data for β=o and the limiting Rayleigh distribution for β=.75 indicates that the approximate solution to the rise and fall problem is satisfactory at least for large h. Since one is generally interested in higher moments of h (see the next section), the results are sufficient. The table below gives the first four moments of the computed rise and fall density where the $n^{\underline{th}}$ moment is defined as

$$\overline{h^n}/\sigma^n = \int_0^\infty (h/\sigma)^n\ P(h/\sigma)\ dh/\sigma \quad . \qquad (56)$$

The first line gives exact values of $\bar{h}/\sigma$ as found from equation (52).

moment	$\beta = 0$	$\beta = 0.25$	$\beta = 0.50$	$\beta = 0.75$
$(\bar{h}/\sigma)$ exact	1.868	2.111	2.351	2.478
$(\bar{h}/\sigma)$ computed	1.810	2.026	2.244	2.625
$(\overline{h^2}/\sigma^2)$ computed	4.469	5.473	6.415	8.227
$(\overline{h^3}/\sigma^3)$ computed	13.112	17.484	21.491	31.139
$(\overline{h^4}/\sigma^4)$ computed	43.573	63.110	81.297	132.856

Agreement between exact values of the average height and the average as computed from the approximate rise and fall density is generally good. The moments for $\beta=0.75$ can be compared with those of the limiting form of the distribution as β approaches one. The first four moments of the limiting expression given by (55) are, respectively, 2.51, 8.00, 30.70, and 128.00. The lower moments of h as computed for $\beta=0.75$ seem too large, the reason being the inaccurate results mentioned earlier for small values of h. This apparently has less effect on the higher moments.

Fatigue Crack Propagation Under Random Loadings

Experimental data on crack propagation under random loadings are cited here to show the relation of rise and fall statistics to the prediction of fatigue life. Recent work in [13,14,15,16] and particularly in [1] and [17] has shown that the rate of propagation of a fatigue crack in a plane sheet under cyclic loading depends only on the variation of the crack tip stress intensity factor and is otherwise independent of the method of loading, crack length, and specimen goemetry. This is true when the scale of plastic yielding at the crack tip is small compared with other geometric dimensions and is a very reasonable result since, when yielding does not alter the stress state far from the crack, the stress intensity factor is a single parameter describing the stress distribution near the crack tip. It has been observed experimentally that the crack growth per load cycle depends primarily on the amplitude of the intensity factor variation and is relatively insensitive to the mean load level. Further, if one insists on fitting a power law type of relationship to the data, the crack extension per load cycle is roughly proportional the fourth power of the amplitude of stress intensity factor variation. Some theoretical justification of these results is given in [1] and [18].

On the basis of evidence on crack propagation under cyclic loading it is not unreasonable to expect that, under random loading, the average crack extension per load peak should be approximately proportional to the average value of the fourth power of the rise and fall height in the stress intensity factor variation. This ignores the effect of rise and fall height sequence, but nevertheless provides a useful starting point. For a crack of length 2a in a large plate loaded with a stress s acting perpendicularly to the crack line, the stress intensity factor k is [19] $k = s\sqrt{a}$. Thus, if the stress s = s(t) is a stationary random process with an average fourth power of the rise and fall height $\overline{h^4}_s$, the corresponding fourth power rise and fall of the stress intensity factor is

$$\overline{h^4}_k = \overline{h^4}_s \quad a^2 \quad . \tag{57}$$

Our hypothesis then is that the crack extension per load peak should depend only on the combination shown in equation (57) and, thus, should be otherwise relatively independent of the crack length a, the mean value of s(t), and any other statistical properties of the process s(t). Citing experimental results given in [1], we verify that the preceding statement is true.

Figures 5 and 6 show respectively samples of five different stationary random stress processes and the corresponding power spectra. Processes A,B, and C were generated by Fuller [20] and E by Leybold [12] (process E was used for the comparison of predicted and experimental rise and fall densities in figure 2).

The densities of rise and fall heights, made dimensionless through division by the standard deviation σ_s, were determined by measurements of rises and falls and are shown in figure 7. The average fourth power rise and fall heights of the stresses divided by the standard deviations, $\overline{h_s^4}$ over σ_s^4, are 128 for A, 88 for B, 101 for C, 64 for D, and 40 for E. The stress processes A,B, and C were applied to cracked plates of 7075 T6 aluminum alloy by S. Smith of the Boeing Company who kindly reported his results to the writers. The results are reduced to a plot of $(\overline{h_k^4})^{1/4}$ from (57) against the average crack extension per load peak d(2a)/dn in figure 8.

The fact that data from three very distinctly different random loadings fall essentially into the same curve in figure 8 is a verification of the usefulness of rise and fall statistics and of the stress intensity factor approach to crack propagation. It should be cautioned that it appears that the crack propagation rate under random loadings does not depend on $\overline{h_k^4}$ in exactly the same way as it does for cyclic loads. This point is illustrated by figure 9 where growth rates under both cyclic and random loadings are compared.

Some Practical Considerations in Applying Rise and Fall Statistics

Basing predictions of fatigue life on averages of the rise and fall height can, in some cases, lead to substantially incorrect results. As an exmaple, consider a random loading x(t) for which a particular sample has the appearance of one of the samples shown in figure 5. If a very high frequency oscillation with extremely small amplitude is superimposed on the loading x(t), one would expect the resultant loading to give essentially the same fatigue life. But the distribution of rises and falls would be markedly changed, for in place of a relatively large rise and fall in x(t) one would now have several smaller rises and falls.

A random process generated by applying white noise to a lightly damped linear spring-mass-damper system affords an example of this phenomena and provides a useful means of illustrating what can be done to eliminate this high frequency effect from the analysis.

Let x(t) be the response satisfying

$$\ddot{x}(t) + \varepsilon\omega_0 \dot{x}(t) + \omega_0^2 \; x(t) = w(t) \quad , \qquad (59)$$

where w(t) is white noise with a power spectral density equal to a constant, K, for all frequencies. Then using standard methods in the stochastic analysis of linear systems [9,21], the power spectral density of the response x(t) is

$$F(\omega) = \frac{K}{\omega_o^4} \frac{1}{[1-(\frac{\omega}{\omega_o})^2]^2+\varepsilon^2(\frac{\omega}{\omega_o})^2} \quad . \qquad (60)$$

When ε is small (light damping) the spectrum has a 'narrow band' appearance with a large peak centered near $\omega = \omega_o$, and thus a sample of x(t) might be expected to resemble a sinusoidal wave with slowly varying amplitude and phase. The average rise and fall height is, however, equal to zero. This is readily seen by noting that from (22) $\bar{h}$ varies inversely with the square root of $R^{(4)}(o)$. By (18), $R^{(4)}(o)$ is the fourth moment of the spectrum $F(\omega)$. But in the case under consideration $F(\omega)$ approaches K/ω^4 for large ω, implying that the fourth moment is infinite and thus the average height of rise is zero. The mathematical idealization of white noise cannot be realized physically; rather than having a vanishing average height in the actual case $\bar{h}$ would be very small due to the high frequency content of $F(\omega)$ which causes rapid oscillations of small amplitude in x(t), even though upon casual observation x(t) would appear to be quite regular and typically 'narrow band' in its variation.

This difficulty is not insurmountable, and the use of some judgement can lead to meaningful results for the rise and fall distribution. Suppose we consider a process $x_*(t)$ with a spectrum defined as $F(\omega)$ (as given in (60)) for $\omega<\omega_*$, and defined as zero for $\omega>\omega_*$. Then $x_*(t)$ has a non-zero $\bar{h}$ if ω_* is finite.

The process $x_*(t)$ is simply the process $x(t)$ with a high frequency component, having variance $\sigma_o^2 = \int_{\omega_*}^{\infty} F(\omega)d\omega$, substracted out. If we choose a value of ω_* slightly greater than ω_o but at the same time sufficiently large such that σ_o^2 is negligible in comparison to the variance of $x(t)$, we have effectively eliminated the troublesome part of the process $x(t)$ but retained its important features with regard to fatigue analysis through consideration of $x_*(t)$.

A similar method may be employed in other cases. Taking the ratio N_o/N_e (or equivalently from (23), $\overline{h}/ 2\pi\sigma$) as a measure of the degree of irregularity in a process, when this ratio is not an appreciable fraction of unity the exact distribution of rises and falls may have little relevance to the load variations important in fatigue, and an alteration of the type performed above may be necessary to yield useful results.

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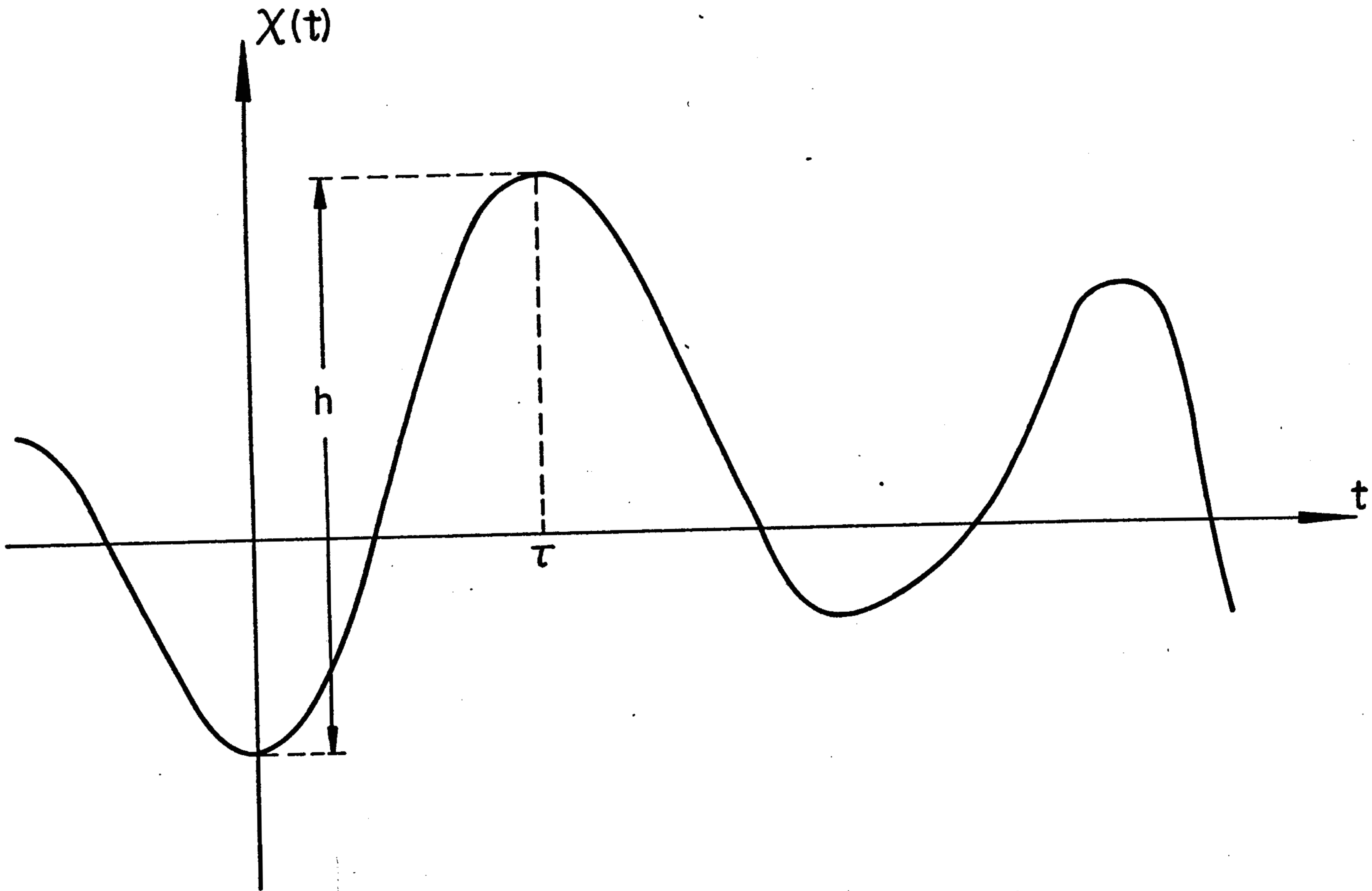

fig. 1 Definition of height of rise, h.

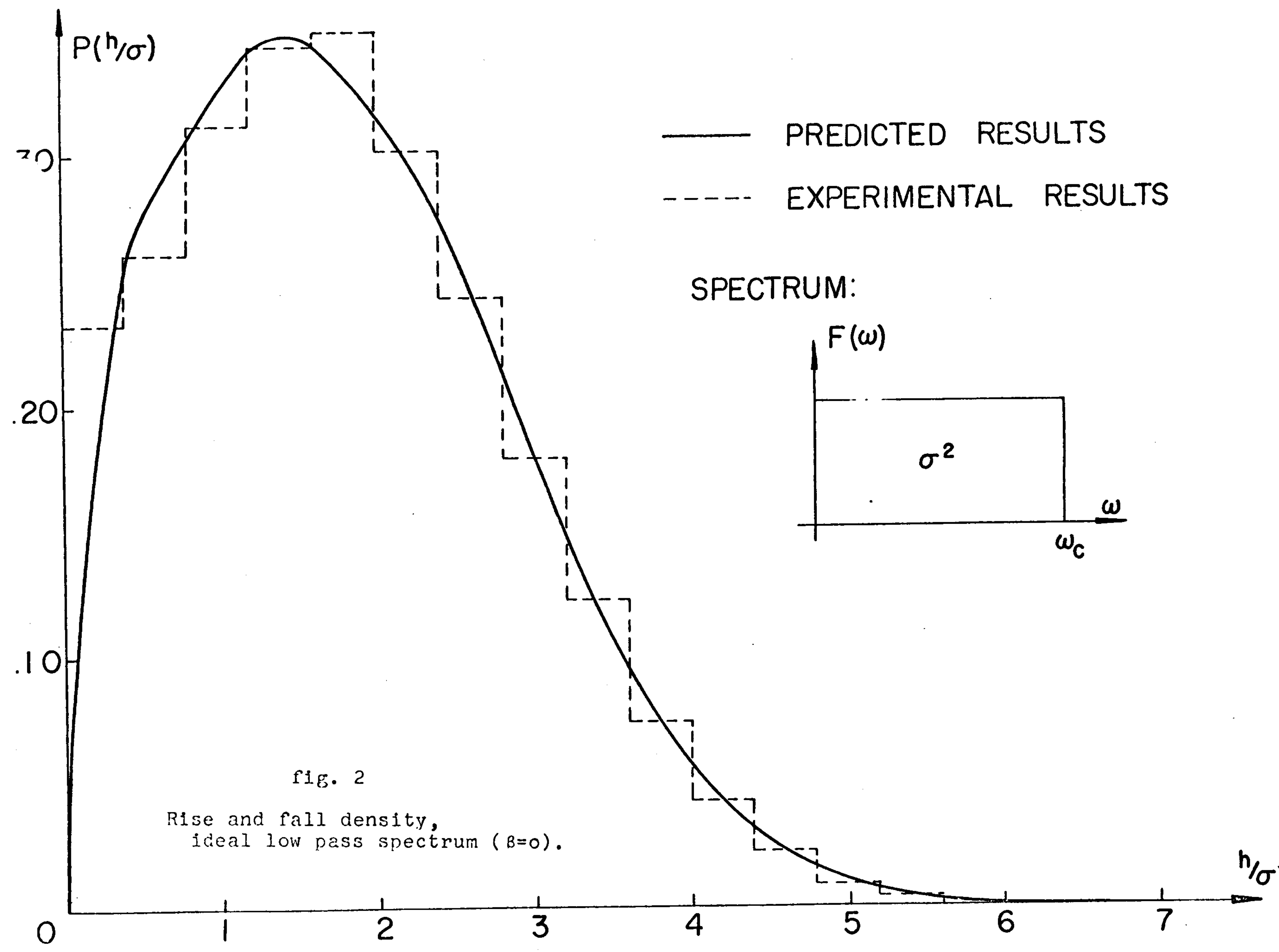

fig. 2

Rise and fall density,
ideal low pass spectrum (β=o).

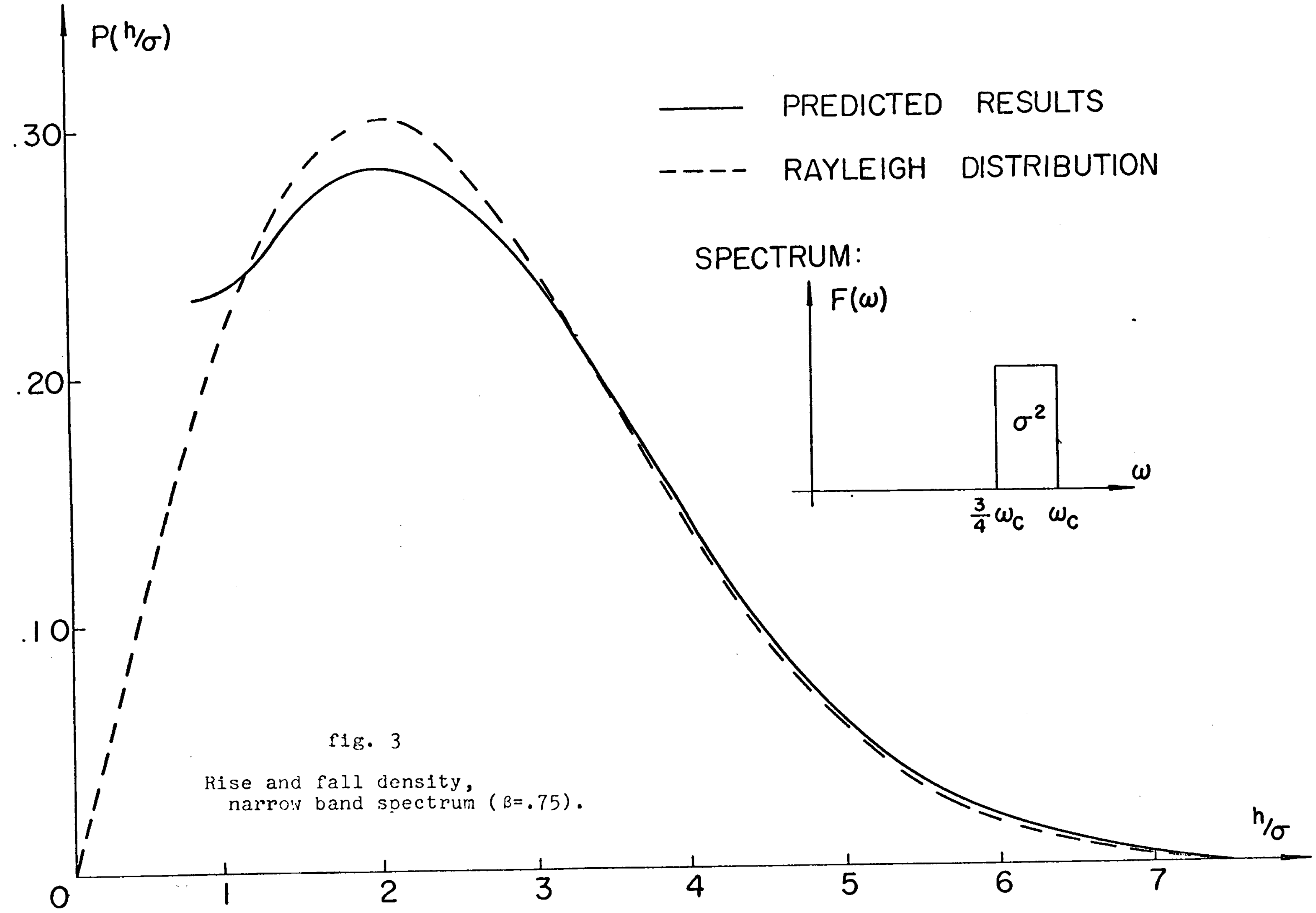

fig. 3

Rise and fall density,
narrow band spectrum (β=.75).

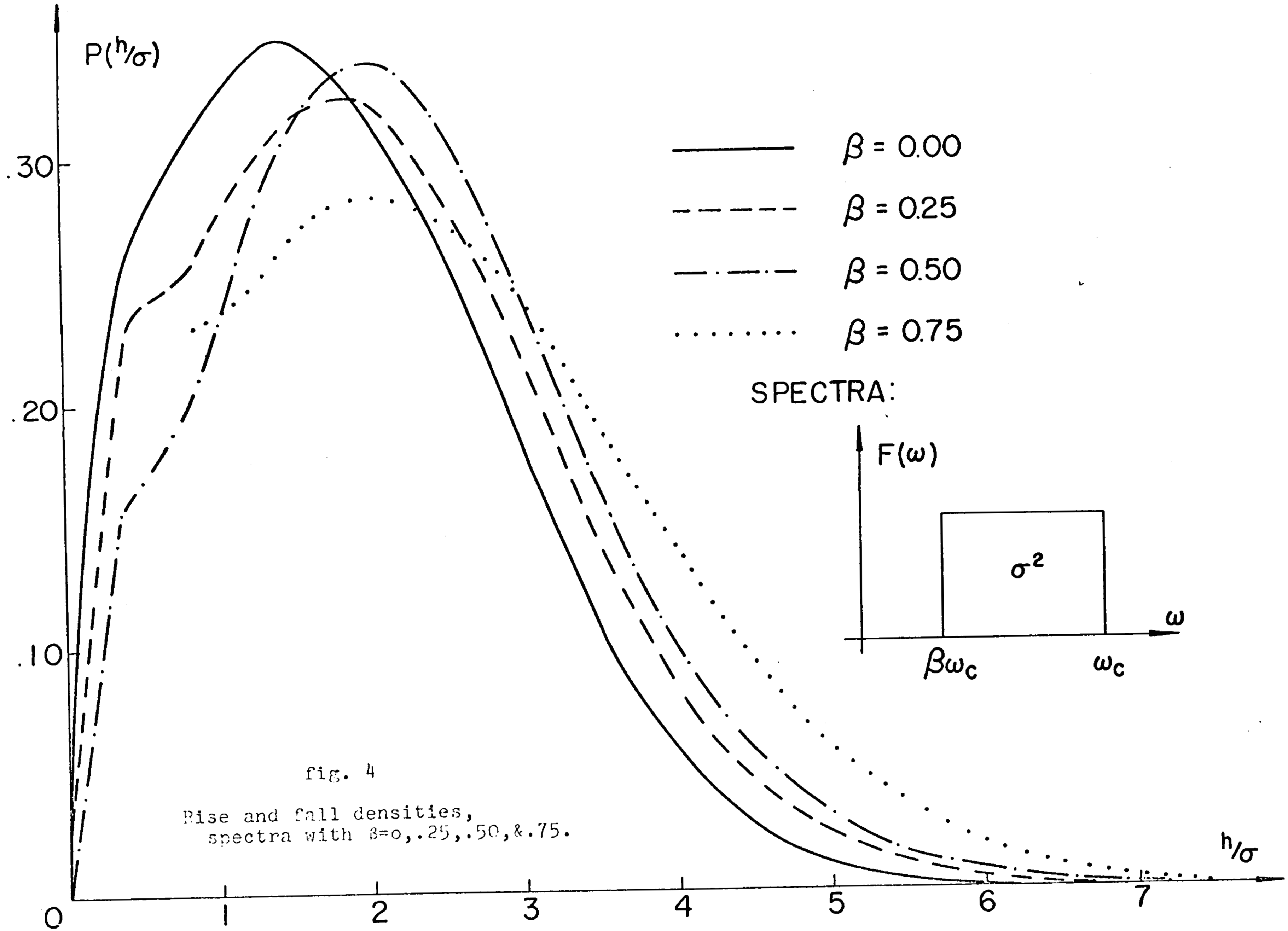

fig. 4

Rise and fall densities,
spectra with β=o,.25,.50,&.75.

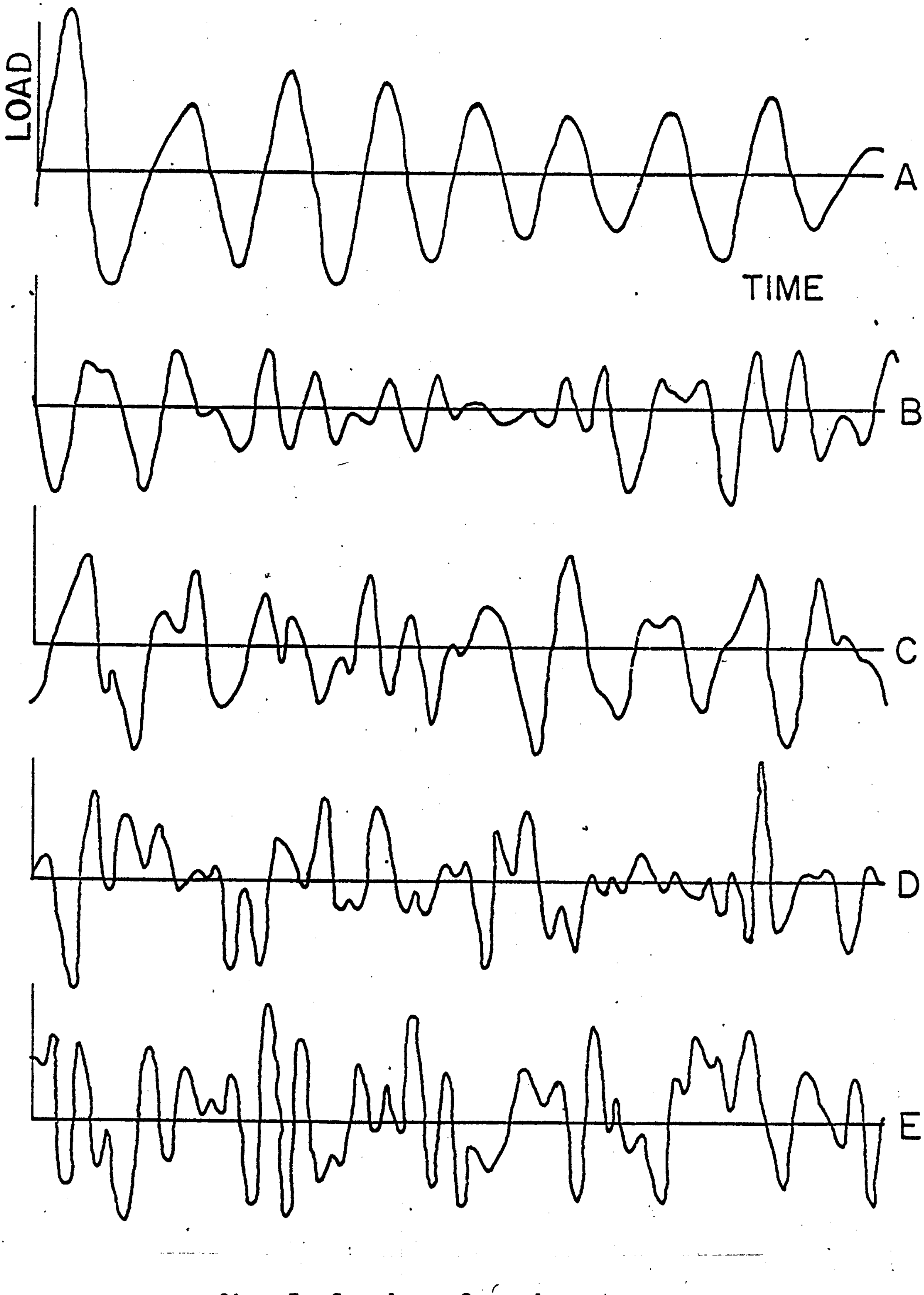

fig. 5 Samples of random stresses.

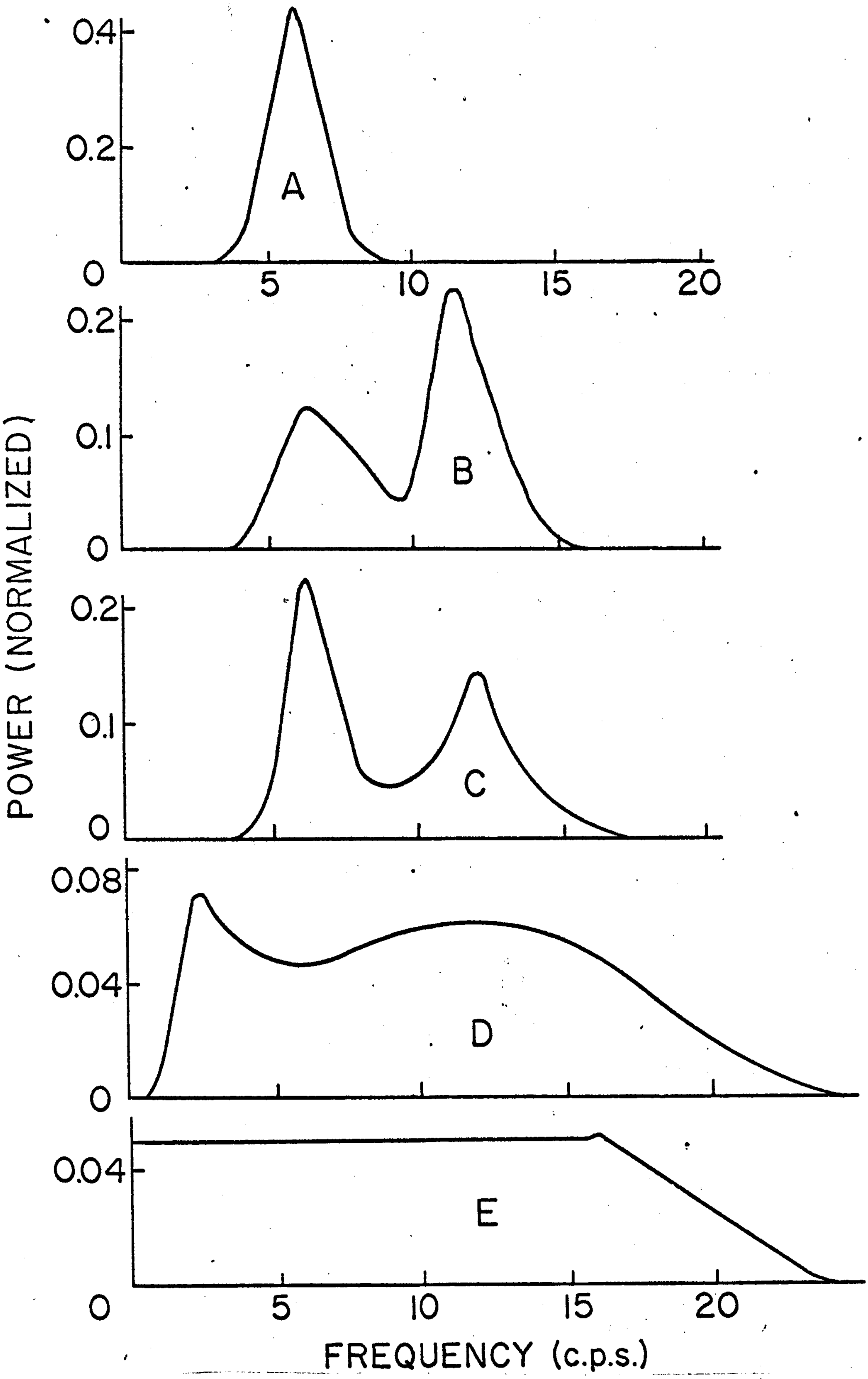

fig. 6 Power spectra.

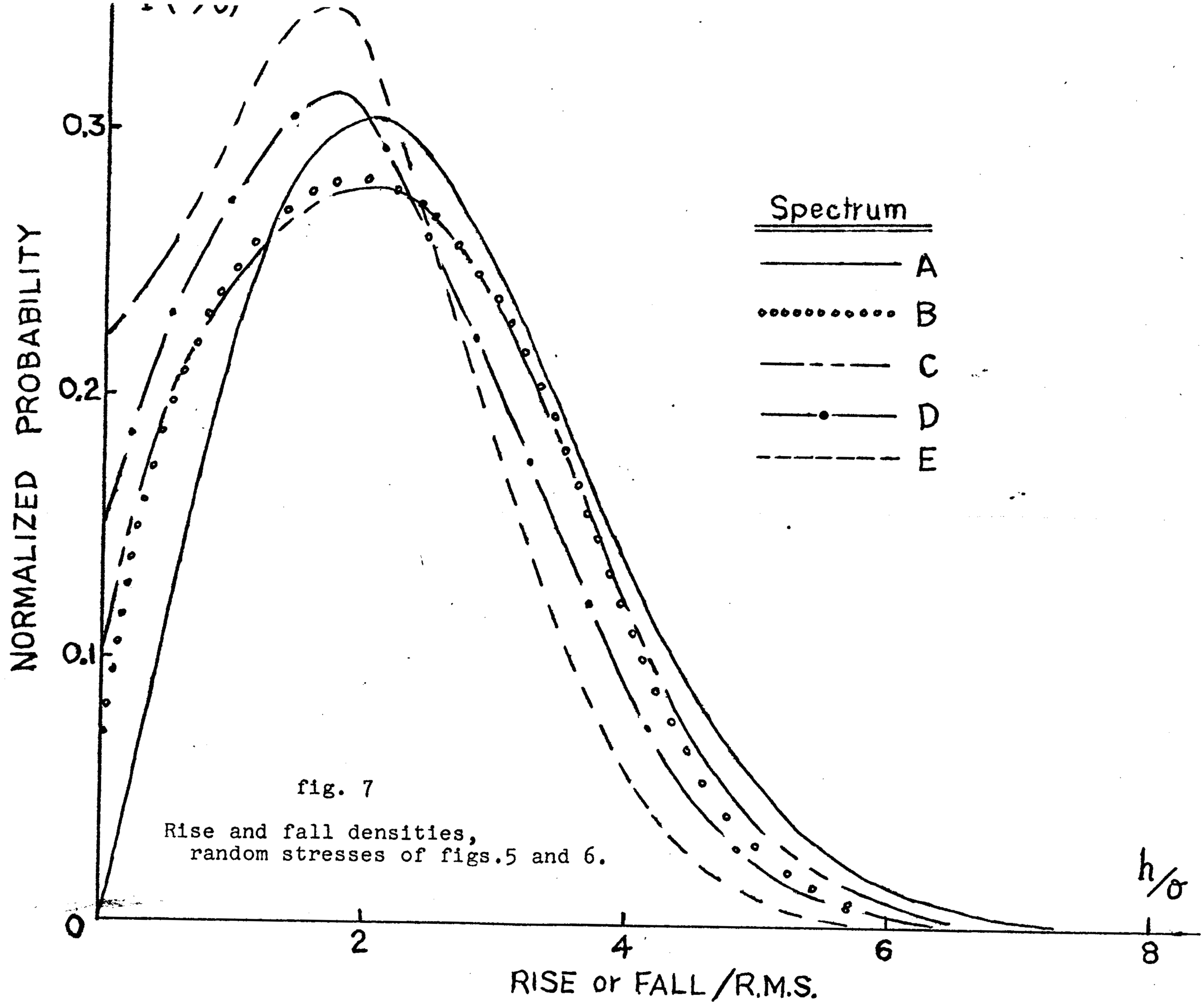

fig. 7

Rise and fall densities,
random stresses of figs.5 and 6.

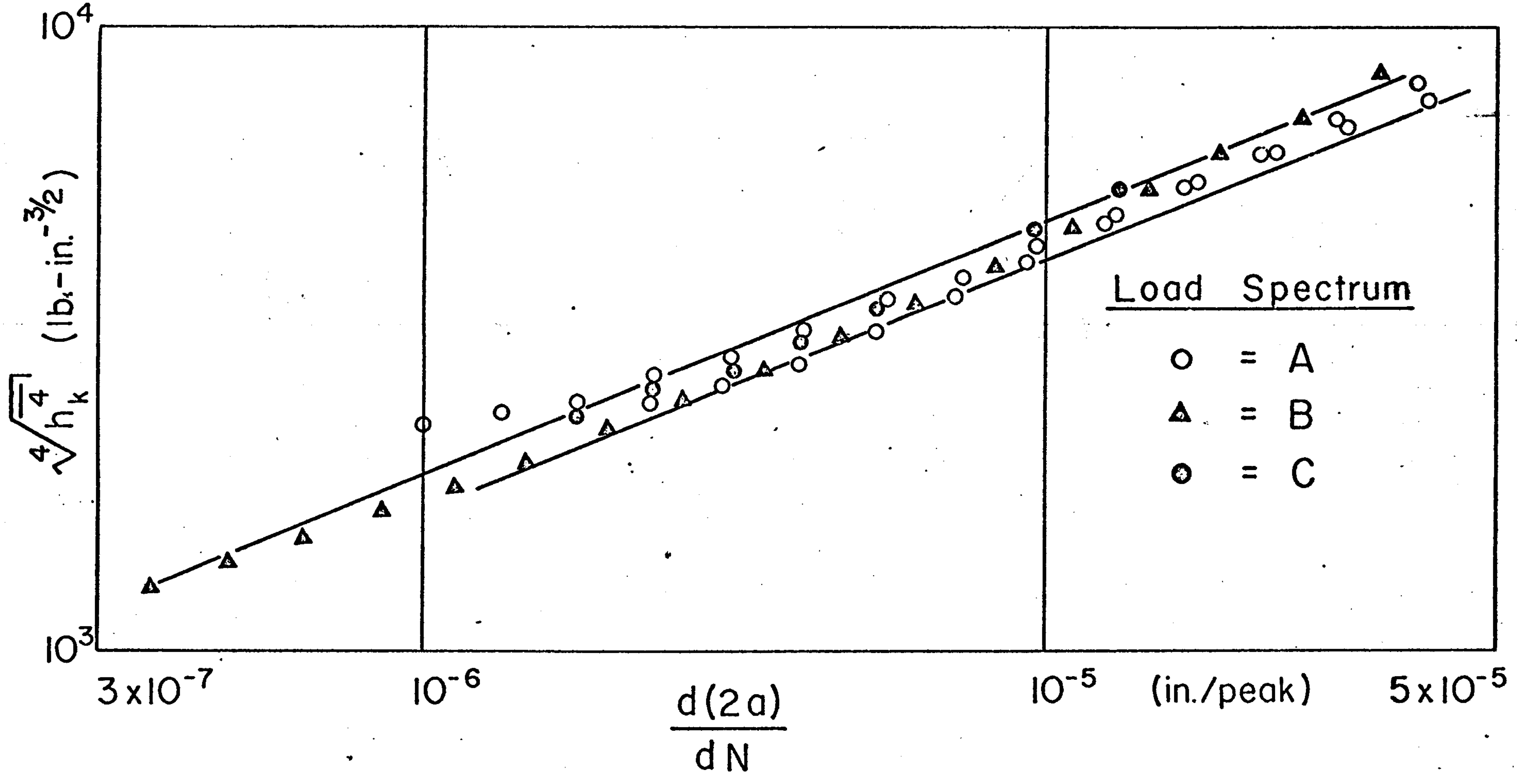

fig. 8 Crack growth under random loadings.

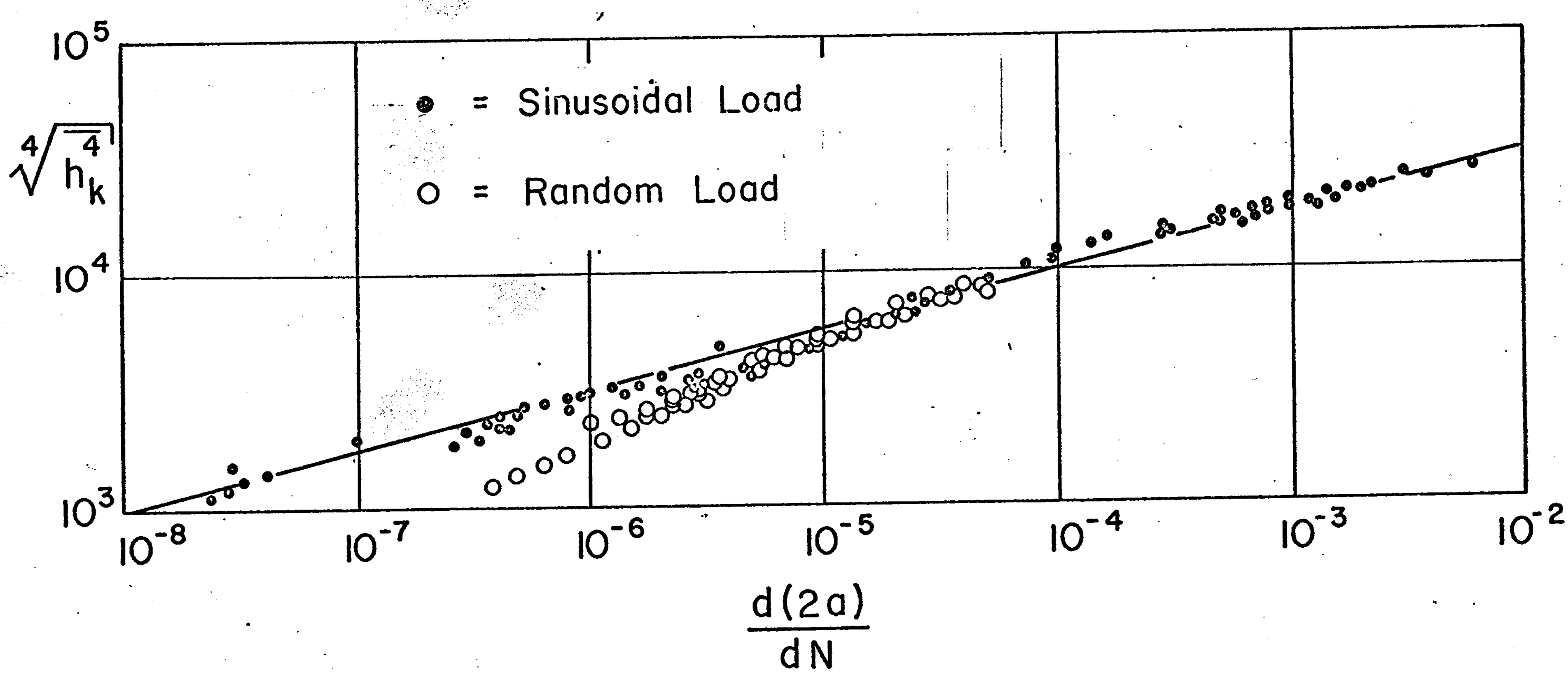

fig. 9 Comparison of growth rates under sinusoidal and random loadings.