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Plane Problems of Cracks in Dissimilar Media[2]

The in-plane extension of two dissimilar materials with cracks or fault lines along their common interface is considered. A method is offered for solving such problems by the application of complex variables integrated with the eigenfunction-expansion technique presented in an earlier paper. The solution to any problem is resolved to finding a single complex potential resulting in a marked economy of effort as contrasted with the more laborious conventional methods which have not yielded satisfactory results. Boundary problems are formulated and solutions are given in closed form. The results of these evaluations also give stress-intensity factors (which determine the onset of rapid fracture in the theory of Griffith-Irwin) for plane problems.

A problem of considerable practical importance is that of two semi-infinite elastic bodies with different elastic properties joined along straight-line segments. The problem represents idealizations of two dissimilar metallic materials welded together with flaws or cracks developed along the original weld line owing to faulty joining techniques. The bonding materials also may be metallic to nonmetallic.

Although a great deal of progress has been made in solving elasticity problems involving lines of discontinuities, mathematical formulation of the problem of cracks between the bonding surfaces of two different materials remains inadequately treated. Recently, several authors have attempted to solve the problem by methods such as the eigenfunction-expansion approach [1, 2],[3] the Hilbert problem [3], and by techniques using integral transforms [4]. However, not one of the foregoing papers has given satisfactory results to the problem. The present investigation therefore is undertaken to give a complete formulation of the "two dissimilar media" crack problem in a manner which is simpler and more thorough than would have been possible by other, hitherto known methods.

In an enlightening paper [1], Williams considered the plane problem of dissimilar materials with a semi-infinite crack. It was discovered for the first time that the stresses possess an oscillatory character of the type $r^{-1/2}$ sin (or cos) of the argument $\epsilon \log r$, where r is the radial distance from the crack tip and ϵ is a function of material constants. This problem was later extended to the case of bending loads by Sih and Rice [2]. While the eigenfunction approach of Williams is an expedient method for determining qualitatively the characteristic behavior of the stress in the vicinity of crack tips, it does not give the solution quantitatively.[4]

Associated with the problem of dissimilar materials having cracks is that of the less complicated one of punches acting on a half-plane. Using the Plemelj formulas and Cauchy integrals, Muskhelishvili [5] has solved the problem of a single punch with straight-line profile pressing on a horizontal base. He showed that the stress changes its sign an infinite number of times underneath the punch. This oscillatory character of the stress is in fact the same as that observed by Williams [1] for the crack problem. Furthermore, upon identifying the width of the punch with the length of the bond line, the Goursat functions for the punch and crack problems take the same form. Similarly, the problem of two collinear punches corresponds to the dissimilar material problem of two semi-infinite planes bonded along two finite line segments. For a detailed account of the similarities between the punch and crack problems, refer to the work of Muskhelishvili [5] in conjunction with that of Erdogan [3]. It should be pointed out that the Hilbert formulation in [3] is based on the condition of free crack surface. A more general application of the problem of linear relationship (or the Hilbert problem) to the problem of straight or circular-arc cracks along the bond line of two different materials will be discussed in a separate paper.

In an effort to obtain a complete solution of the problem, Bahar [4] proposed an alternative method based on integral transforms. He resolved the problem to the solution of simultaneous dual integral equations which in turn were reduced to a system of linear algebraic equations by means of the discontinuous Weber-Sonine-Schafheitlin integrals. In contrast to all the previous results [1, 2, 3, 5], he found that the stresses near the crack

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[3] Numbers in brackets designate References at end of paper.

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[4] Note that in [2] the Goursat functions were expressed independently of uncertainties of both the external loads and the crack dimensions.

Nomenclature

a = half crack length
b = dimension
A, B, C = complex constants
E_j = Young's moduli (j = 1, 2)
$f(z), g(z), F(z)$ = complex function of z
G_j = shear moduli (j = 1, 2)
i = $(-1)^{1/2}$
k_1, k_2 = stress-intensity factors
P, Q = y and x-components of R
R = complex force, $Q + iP$
u_j, v_j = displacements (j = 1, 2)
x, y = rectangular coordinates
z = complex variable, $x + iy$
ϵ = bielastic constant
ϵ_x = strain component
η_j = $3 - 4\nu_j$ for plane strain and $(3 - \nu_j)/(1 + \nu_j)$ for plane stress (j = 1, 2)
μ = $[G_2(\eta_1 + 1)]/[G_1(\eta_2 + 1)]$
ν_j = Poisson's ratio (j = 1, 2)
$(\sigma_x)_j, (\sigma_y)_j, (\tau_{xy})_j$ = stress components (j = 1, 2)
$\Phi_j(z), \Psi_j(z)$ = Goursat functions (j = 1, 2)
ω_j^∞ = rotation at infinity (j = 1, 2)

tip are not oscillatory in character but decay monotonically as r, the radial distance from the crack front, increases. The validity of this result is therefore questioned.

In what follows, it is shown how the complex-variable method combined with eigenfunction expansion in [1, 2] can be applied to formulate the problem of bonded dissimilar elastic planes containing cracks along the bond. Solutions are given in closed form for a number of extensional problems of fundamental interest. In particular, the problem of an isolated complex force, i.e., a force vector having components in the x and y-directions, applied at an arbitrary location on each side of the crack surface is solved. Aside from its application to such problems as wedge loading at an arbitrary angle, the isolated-force solution may be used as Green's functions to obtain the stresses in welded dissimilar plates owing to any arbitrary distribution of tractions on the crack surface.

The results in this paper are also discussed in connection with the Griffith-Irwin theory of fracture. In their theory, the critical length of a crack may be predicted from the crack-tip stress-intensity factors. It is shown that these factors can be evaluated readily from a complex potential function $\Phi(z)$.

Statement of Problem

Let a material with elastic properties E_1 and ν_1 occupy the upper half-plane, $y > 0$, and a material with elastic properties E_2 and ν_2 occupy the lower half-plane, $y < 0$. The two materials are bonded along straight-line segments of the x-axis. In the following, all quantities such as the elastic constants, stresses, and so on, pertaining to the region $y > 0$ and $y < 0$ will be marked with subscripts 1 and 2, respectively.

Muskhelishvili [5] and others have shown that the solution to an individual problem in the plane theory of elasticity can be reduced to finding two complex functions, which satisfy the boundary conditions of that problem. In the case of two different materials, however, the elastic properties are discontinuous across the bond line, and a complete solution to the problem requires the knowledge of four complex functions $\Phi_j(z)$, $\Psi_j(z)$, $j = 1, 2$, of the complex variable $z = x + iy$. The basic equations for two-dimensional isotropic elasticity in the form used by Kolosov-Muskhelishvili are

$$\begin{aligned}(\sigma_x)_j + (\sigma_y)_j &= 4\,\mathrm{Re}[\Phi_j(z)]\\ (\sigma_y)_j - (\sigma_x)_j + 2i(\tau_{xy})_j &= 2[\bar{z}\Phi_j'(z) + \Psi_j(z)]\end{aligned} \tag{1}$$

and

$$2G_j(u_j + iv_j) = \eta_j \int \Phi_j(z)dz - z\bar{\Phi}_j(\bar{z}) - \int \bar{\Psi}_j(\bar{z})d\bar{z} \tag{2}$$

where u_j, v_j are components of displacement, $(\sigma_x)_j$, $(\sigma_y)_j$, $(\tau_{xy})_j$ are components of stress, and G_j is the shear modulus. Also $\eta_j = 3 - 4\nu_j$ for plane strain and $\eta_j = (3 - \nu_j)/(1 + \nu_j)$ for generalized plane stress, ν_j being Poisson's ratio.

Isolated Forces on Surface of a Semi-Infinite Crack

Let the semi-infinite planes of different materials be joined along the positive x-axis, Fig. 1. A line crack is situated along the negative x-axis extending from $x = 0$ to $x = -\infty$ and is opened by a complex force $R = Q + iP$ at $z = -a$ on each side of the crack.

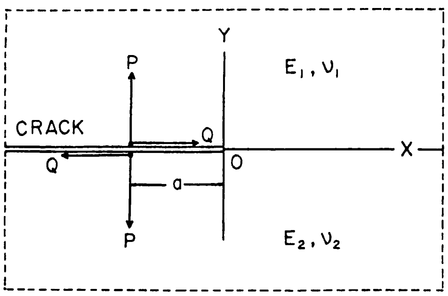

Fig. 1 Isolated forces on a semi-infinite crack

For this problem, the general forms of the Goursat functions are given by equation (41) in [2]. These were derived by expressing the Airy stress function, obtained by the Williams method as a power series in terms of polar coordinates r and θ, in the form

$$\mathrm{Re}\,[\bar{z}\phi_j(z) + \chi_j(z)]$$

The functions $\phi_j'(z)$ and $\chi_j''(z)$ are $\Phi_j(z)$ and $\Psi_j(z)$ in this paper, respectively. Upon defining

$$f(z) = 2\sum_{n=1}^{\infty} [(n - \tfrac{1}{2}) - i\epsilon][(n + \tfrac{1}{2}) - i\epsilon]\bar{B}^{(n)}z^{n-1} \tag{3}$$

it is possible to express the functions $\Phi_j(z)$ and $\Psi_j(z)$ in terms of $f(z)$ alone. The results are

$$\begin{aligned}\Phi_1(z) &= z^{-\frac{1}{2}-i\epsilon}f(z)\\ \Psi_1(z) &= e^{2\pi\epsilon}z^{-\frac{1}{2}+i\epsilon}\bar{f}(z) - z^{-\frac{1}{2}-i\epsilon}[(\tfrac{1}{2} - i\epsilon)f(z) + zf'(z)]\end{aligned} \tag{4}$$

for the region $y > 0$ and

$$\begin{aligned}\Phi_2(z) &= e^{2\pi\epsilon}z^{-\frac{1}{2}-i\epsilon}f(z)\\ \Psi_2(z) &= z^{-\frac{1}{2}+i\epsilon}\bar{f}(z) - e^{2\pi\epsilon}z^{-\frac{1}{2}-i\epsilon}[(\tfrac{1}{2} - i\epsilon)f(z) + zf'(z)]\end{aligned} \tag{5}$$

for the region $y < 0$. In equations (3) through (5), ϵ is defined as the bielastic constant given by (see equation (39) in [2])

$$\epsilon = \frac{1}{2\pi}\log\left[\left(\frac{\eta_1}{G_1} + \frac{1}{G_2}\right)\Big/\left(\frac{\eta_2}{G_2} + \frac{1}{G_1}\right)\right] \tag{6}$$

The problem is to find the function $f(z)$ such that it is holomorphic in a region close to the crack tip. Outside of this region, $f(z)$ may have poles of the order $1/z$. In effect, this permits loading on the crack surface except for isolated loads near the tip of the crack. Hence, in the proximity of the requisite force R at $z = -a$ in Fig. 1, the Goursat functions, say for $y > 0$, must take the form

$$\begin{aligned}\Phi_1(z) &= -\frac{R}{2\pi}\frac{1}{z + a}\\ \Psi_1(z) &= \frac{\bar{R}}{2\pi}\frac{1}{z + a} + \frac{R}{2\pi}\frac{a}{(z + a)^2}\end{aligned} \tag{7}$$

Equation (7) represents Boussinesq's solution [6] of an isolated force R acting on the boundary of a half-plane, but now expressed in terms of complex potentials. Upon comparing both equations (4) and (7) for the stresses in the neighborhood of the pole at $z = -a$, it is found that

$$f(z) = \frac{P - iQ}{2\pi e^{\pi\epsilon}}\frac{a^{\frac{1}{2}+i\epsilon}}{z + a} \tag{8}$$

Inserting equation (8) into (4) and (5) gives the Goursat functions from which the stresses and displacements can be computed without difficulty.

Goursat Functions for Finite-Crack Problems

The Goursat functions, equations (4) and (5), originally derived for a semi-infinite crack, may be modified to solve the problem of a finite crack, Fig. 2. Since the branch points are now located at $z = \pm a$ (the crack tips), the singular terms $(z - a)^{-\frac{1}{2}-i\epsilon}$ and $(z + a)^{-\frac{1}{2}+i\epsilon}$ must be introduced into the complex potentials $\Phi_j(z)$ and $\Psi_j(z)$. This is accomplished by defining $f(z)$ in equations (4) and (5) as

$$f(z) = (z + a)^{-\frac{1}{2}+i\epsilon}g(z) \tag{9}$$

in which $g(z)$ is well behaved at $z = \pm a$ and it may have poles sufficiently far away from the crack tip when isolated forces are present.

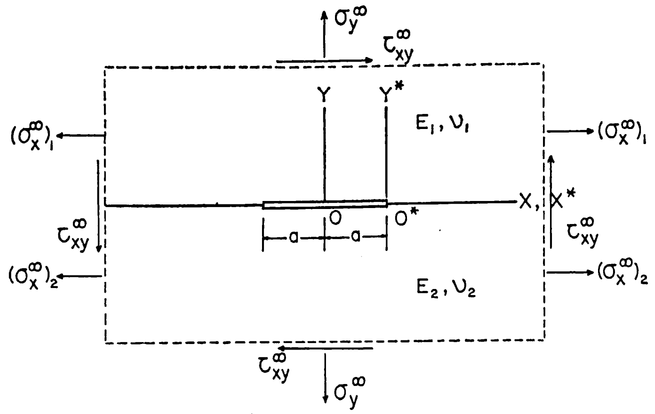

Fig. 2 Infinite plate with a crack subjected to stresses at infinity

In addition, the effect of the translation of coordinate axes on the Goursat functions must be considered. Denote by $\Phi_j(z)$, $\Psi_j(z)$ the functions referred to the axes x, y and $\Phi_j^*(z)$, $\Psi_j^*(z)$ to x^*, y^* (see Fig. 2). Since the stress components, equation (1), are not altered by the translation $z = z^* + a$, it is observed that

$$\Phi_j(z) = \Phi_j^*(z - a)$$
$$\Psi_j(z) = \Psi_j^*(z - a) - a\Phi_j^{*\prime}(z - a) \tag{10}$$

Making use of equations (9) and (10), equations (4) and (5) may be rearranged to read as

$$\Phi_1(z) = g(z)F(z) + A$$
$$\Psi_1(z) = e^{2\pi\epsilon}\bar{g}(z)\bar{F}(z) + \left[\frac{a^2 + 2i\epsilon az}{z^2 - a^2}\, g(z) - zg'(z)\right] F(z) - (A + \bar{A}) \tag{11}$$

and

$$\Phi_2(z) = e^{2\pi\epsilon}g(z)F(z) + \mu A$$
$$\Psi_2(z) = \bar{g}(z)\bar{F}(z) + e^{2\pi\epsilon}\left[\frac{a^2 + 2i\epsilon az}{z^2 - a^2}\, g(z) - zg'(z)\right] F(z) - \mu(A + \bar{A}) \tag{12}$$

where

$$\mu = [G_2(\eta_1 + 1)]/[G_1(\eta_2 + 1)]$$

and

$$F(z) = (z^2 - a^2)^{-1/2}\left(\frac{z + a}{z - a}\right)^{i\epsilon} \tag{13}$$

Those terms containing the complex constant A represent the degenerate case of $\Phi_j(z)$ and $\Psi_j(z)$. A detailed derivation is given in Appendix 1.

It can be shown that equations (11) and (12) give zero stresses on the crack surface and continuous stresses across the bond line, $|z| > a$ for $z = \bar{z}$. The continuity of displacements may be verified by first computing for the complex displacements

$$2G_1(u_1 + iv_1) = \eta_1\int g(z)F(z)dz - e^{2\pi\epsilon}\int g(\bar{z})F(\bar{z})d\bar{z} + (\bar{z} - z)\bar{g}(\bar{z})\bar{F}(\bar{z}) \tag{14}$$

in the upper half-plane and

$$2G_2(u_2 + iv_2) = e^{2\pi\epsilon}\eta_2\int g(z)F(z)dz - \int g(\bar{z})F(\bar{z})d\bar{z} - e^{2\pi\epsilon}(\bar{z} - z)\bar{g}(\bar{z})\bar{F}(\bar{z}) \tag{15}$$

in the lower half-plane. From equations (14) and (15), the difference between the displacements for $y > 0$ and $y < 0$ is found to be

$$2[(u_1 + iv_1) - (u_2 + iv_2)] = \left(\frac{\eta_1}{G_1} - e^{2\pi\epsilon}\frac{\eta_2}{G_2}\right)\int g(z)F(z)dz + \left(\frac{1}{G_2} - e^{2\pi\epsilon}\frac{1}{G_1}\right)\int g(\bar{z})F(\bar{z})d\bar{z} \tag{16}$$

on the bond line $z = \bar{z}$. In view of equation (6), this difference is indeed zero when

$$\int g(z)F(z)dz$$

is single-valued on the bond line.

Hence, the Goursat functions given by equations (11) and (12) satisfy all the conditions of the problem of a finite crack between two dissimilar materials. The foregoing analysis may be extended easily to a finite or infinite number of collinear cracks. An example is given in Appendix 2. It is now more pertinent to illustrate the use of this method by finding the constant A and the function $g(z)$ for specific problems.

Extension of Infinite Plate With a Crack

From the point of view of application, the consideration of infinite region is of interest when the crack length is small in comparison with plate dimensions. The geometry of the present problem is shown in Fig. 2, where the plate composed of two different materials is subjected to normal and shear stresses at infinity.

In order for the stresses to be bounded as $z \to \infty$, the function $g(z)$ can at most be linear in z; i.e.,

$$g(z) = Bz + C \tag{17}$$

where B and C are complex constants yet to be found. The physical interpretation of the constant $A = A_1 + iA_2$ in equations (11), (12) and $B = B_1 + iB_2$ in equation (17) is considerably more complicated than in the case of similar material, $\epsilon = 0$. Putting equation (17) into (11) and (12) and letting $z \to \infty$, then by way of equation (1), the stresses at infinity lead to

$$A_1 = \frac{(\sigma_x^\infty)_1 + \sigma_y^\infty}{4} - \frac{\sigma_y^\infty}{1 + e^{2\pi\epsilon}}$$
$$B = B_1 + iB_2 = \frac{\sigma_y^\infty - i\tau_{xy}^\infty}{1 + e^{2\pi\epsilon}} \tag{18}$$

It should be mentioned that the normal stress, σ_x, in the x-direction is discontinuous across the bond line. Thus, it is necessary to distinguish $(\sigma_x^\infty)_1$ in the region $y > 0$ from $(\sigma_x^\infty)_2$ in the region $y < 0$. In fact, it follows directly that they are related to each other by

$$(\sigma_x^\infty)_2 = \mu(\sigma_x^\infty)_1 + \frac{(3 + \mu)e^{2\pi\epsilon} - (3\mu + 1)}{1 + e^{2\pi\epsilon}}\,\sigma_y^\infty \tag{19}$$

Alternatively, equation (19) also may be obtained from the conditions of continuity of stresses and displacements across the x-axis along which the component σ_x has a jump (see Appendix 3).

The constant A_2 may be related to the rotation at an infinitely remote part of the x, y-plane as follows:

$$A_2 = \frac{\tau_{xy}^\infty}{1 + e^{2\pi\epsilon}} + \frac{2G_1\omega_1^\infty}{1 + \eta_1} = \frac{1}{\mu}\left(\frac{e^{2\pi\epsilon}}{1 + e^{2\pi\epsilon}}\tau_{xy}^\infty + \frac{2G_2\omega_2^\infty}{1 + \eta_2}\right) \tag{20}$$

in which ω_1^∞ and ω_2^∞ denote the rotations at infinity in the upper and lower half-planes, respectively. After some algebraic manipulations, equation (20) gives

$$\omega_2^\infty - \omega_1^\infty = \left(\frac{G_2 - G_1}{2G_1G_2}\right)\tau_{xy}^\infty \tag{21}$$

In contrast to the homogeneous case ($G_1 = G_2$), where the rotation may be assumed to vanish as it does not affect the stresses, ω_1^∞ and ω_2^∞ in the bimaterial problem cannot be set arbitrarily to zero at the same time unless $\tau_{xy}^\infty = 0$.

Hitherto, no consideration has been given to the condition of single-valuedness of displacements. It is necessary and sufficient for the one-valuedness of $u_j + iv_j$ $(j = 1, 2)$ that the integral

$$\int g(z)F(z)dz$$

in equations (14) and (15) be a single-valued function of z. For $|z| > a$, the function $F(z)$ in equation (13) may be represented by a series of the form

$$F(z) = \frac{1}{z} + \frac{2i\epsilon a}{z^2} + \frac{a^2(1 - 4\epsilon^2)}{2z^3} + \ldots \qquad (22)$$

By virtue of equations (17) and (22)

$$\int g(z)F(z)dz = Bz + (C + 2i\epsilon aB)\log z - \left(\frac{1 - 4\epsilon^2}{2}a^2B - 2i\epsilon aC\right)\frac{1}{z} + \ldots$$

For single-valued displacements, i.e., solutions involving no dislocations, the integral should have no logarithmic term. Therefore, the constant C is determined:

$$C = -2i\epsilon aB \qquad (23)$$

where B is given by equation (18). The final result in terms of $g(z)$ may be written as

$$g(z) = \frac{\sigma_y^\infty - i\tau_{xy}^\infty}{1 + e^{2\pi\epsilon}}(z - 2i\epsilon a) \qquad (24)$$

from which the Goursat functions $\Phi_j(z)$ and $\Psi_j(z)$ may be obtained.

Green's Function

The problem of two semi-infinite planes bonded along the x-axis with a crack of length, $2a$, centered at the origin, Fig. 3, and having two equal and opposite forces $R = Q + iP$ applied at $z = b$ is of fundamental interest, since it may be used as a Green's function to form the solution to other problems.

Replacing the constant a in equation (7) by $-b$ and equating the results in terms of stresses with those obtained from equations (11) and (12) in the vicinity of $z = b$, it gives

$$g(z) = \frac{R}{2\pi i}\frac{e^{-\pi\epsilon}}{z - b}(a^2 - b^2)^{1/2}\left(\frac{a - b}{a + b}\right)^{i\epsilon} \qquad (25)$$

where $A = 0$ as the stresses are zero at infinity. The isolated force solution, equation (25), may now be taken as a Green's function to solve problems with any loading desired on the crack surface.

Moreover, by judicious application of the "principle of superposition," the solution of the problem of a crack with surface tractions may be further used to attack all the general problems of an infinite plate with a crack whose surface is free from tractions. First, the stresses $\sigma_y(x, 0)$ and $\tau_{xy}(x, 0)$ on the prospective crack surface with no crack present are computed from the prescribed loading in the original problem. Then, superimposing tractions equal and opposite to those on the prospective crack surface (i.e., to free the crack surface), the result is

$$g(z) = \frac{e^{-\pi\epsilon}}{2\pi}\int_{-a}^{a}[\sigma_y(x, 0) - i\tau_{xy}(x, 0)] \times \left(\frac{a - x}{a + x}\right)^{i\epsilon}\frac{\sqrt{a^2 - x^2}}{z - x}dx \qquad (26)$$

Once $g(z)$ is known, the stresses and displacements are completely determined. Hence, equation (26) provides a direct method of solving any problem involving a crack between two bonded dissimilar materials.

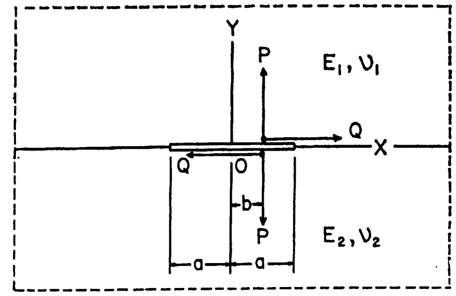

Fig. 3 A finite crack opened by wedge forces

Stress-Intensity Factors in Dissimilar Materials

In a previous paper [2], the stresses in the immediate vicinity of the crack tip were given as a function of r and θ, where r is the distance from the crack front and θ the angle between r and the crack plane. It was found that the singular behavior of the stresses remains proportional to the inverse square root of r (i.e., $r^{-1/2}$) as in the case of homogeneous materials, but the stresses possess a pronounced oscillatory character of the type

$$\sigma \sim r^{-1/2}\begin{pmatrix}\sin\\ \cos\end{pmatrix}(\epsilon \log r) \qquad (27)$$

which was first observed by Williams [1]. Equation (27) shows that the stress-intensity factors $k_j(j = 1, 2)$, used in the Griffith-Irwin theory of fracture, can be evaluated in a manner similar to that of the homogeneous case [7]. However, in the bimaterial problem, k_1 and k_2 can no longer be regarded as the crack tip stress-intensity factors for symmetrical and skew-symmetrical stress distributions. This point will be discussed later.

An examination of equations (43) through (44) in [2] indicates that the parameters k_1 and k_2 in general may be considered as the strength of the stress singularities at crack tips. Quantitatively, $k_j(j = 1, 2)$ depend on the external loads and the crack dimensions. For a given problem, the stress-intensity factors may be computed from the complex potential $\Phi_1(z)$. Take the case of a semi-infinite crack with its tip at the origin, Fig. 1. The stress-intensity factors at $z = 0$ are given by[5]

$$k_1(0) - ik_2(0) = 2\sqrt{2}\, e^{\pi\epsilon} \lim_{z\to 0} z^{\frac{1}{2}+i\epsilon}\Phi_1(z) \qquad (28)$$

As a first example, consider the semi-infinite crack problem stated earlier, Fig. 1. The isolated forces P and Q are located at a distance a away from the origin. Using equations (4) and (8), equation (28) becomes

$$\begin{aligned} k_1(0) &= \frac{1}{\pi}\left(\frac{2}{a}\right)^{1/2}[P\cos(\epsilon\log a) + Q\sin(\epsilon\log a)] \\ k_2(0) &= \frac{1}{\pi}\left(\frac{2}{a}\right)^{1/2}[Q\cos(\epsilon\log a) - P\sin(\epsilon\log a)] \end{aligned} \qquad (29)$$

Contrary to the conclusion in [3], the stress-intensity factors for a single bond do depend on the bielastic constant ϵ. Thus, the dependency of k_j on the material constants is not a simple matter of identifying it with the number of bond lines. As should have been expected, when $\epsilon = 0$, equation (29) reduces to the solution for the homogeneous material.

Similarly, the evaluation of k_j for a finite crack of length $2a$, Fig. 2, may be carried out by redefining equation (28) in the form

$$k_1(a) - ik_2(a) = 2\sqrt{2}\, e^{\pi\epsilon} \lim_{z\to a}(z - a)^{\frac{1}{2}+i\epsilon}\Phi_1(z) \qquad (30)$$

As a second example, consider a straight crack of length $2a$ along the x-axis in an infinite plate with normal and shear stresses at large distances from the crack, Fig. 2. From equations (11),

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[5] Equation (48) in [2].

(24), and (30), the stress-intensity factors at $z = a$ are obtained. They are

$$k_1 = \frac{\left\{\begin{array}{l}\sigma[\cos(\epsilon \log 2a) + 2\epsilon \sin(\epsilon \log 2a)] \\ + \tau[\sin(\epsilon \log 2a) - 2\epsilon \cos(\epsilon \log 2a)]\end{array}\right\}}{\cosh \pi\epsilon} a^{1/2}$$

$$k_2 = \frac{\left\{\begin{array}{l}\tau[\cos(\epsilon \log 2a) + 2\epsilon \sin(\epsilon \log 2a)] \\ - \sigma[\sin(\epsilon \log 2a) - 2\epsilon \cos(\epsilon \log 2a)]\end{array}\right\}}{\cosh \pi\epsilon} a^{1/2} \tag{31}$$

An interesting feature of equation (31) is that both the symmetric and skew-symmetric loadings, σ_y^∞ and τ_{xy}^∞, are intermixed in the expressions for k_1 and k_2. As a result, the $k_j(j = 1, 2)$ do not have the simple physical interpretation as in the homogeneous case where the symmetric and skew-symmetric loads are separately contained in $k_1 = \sigma_y^\infty a^{1/2}$ and $k_2 = \tau_{xy}^\infty a^{1/2}$ for $\epsilon = 0$. When $\epsilon \neq 0$, even if the external loads were symmetric, say $\tau_{xy}^\infty = 0$ in equation (31), more than one stress-intensity factor is involved. Hence, in the application of the Griffith-Irwin theory of fracture, it is necessary to assume that a function of k_1, k_2 will cause the crack to grow upon reaching some critical value. The criterion may be written as

$$f(k_1, k_2) = f_{cr} \tag{32}$$

The specific form of equation (32) must be determined experimentally. Such studies will be left for future investigations.

Conclusions

A simple method for determining the Goursat functions for two dissimilar (or similar) materials bonded along straight-line segments is developed. The unbonded portion of the interface may be regarded as cracklike imperfections. The derivation combines an eigenfunction-expansion method with the complex-function theory of Muskhelishvili. The problem of isolated forces on the crack surface is solved with the aid of Boussinesq's solution.

In general, the results in this paper can be used in any one of the current fracture-mechanics theories. In particular, it is shown that the concept of stress-intensity factor in the Griffith-Irwin theory of fracture may be extended to cracks in dissimilar materials.

The Goursat functions for out-of-plane bending of cracks along the interface of two joined materials may be obtained in the same way. These results will be reported at a later date.

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APPENDIX 1

Degenerate Case of Goursat Functions

When the Goursat functions $\Phi_j(z)$ and $\Psi_j(z)$ degenerate to the constants A_j and B_j, respectively, equations (1) and (2) may be written as

$$\begin{aligned}(\sigma_x)_j + (\sigma_y)_j &= 2(A_j + \bar{A}_j) \\ (\sigma_y)_j - (\sigma_x)_j + 2i(\tau_{xy})_j &= 2B_j \\ 2G_j(u_j + iv_j) &= (\eta_j A_j - \bar{A}_j - \bar{B}_j)z\end{aligned} \tag{33}$$

where $j = 1, 2$. Now, consider a uniaxial state of stress parallel to the crack surface which is not affected by the presence of the crack. This is given by

$$(\sigma_y)_1 + i(\tau_{xy})_1 = (\sigma_y)_2 + i(\tau_{xy})_2 = 0$$

Hence, equation (33) yields

$$-B_j = A_j + \bar{A}_j, \quad j = 1, 2 \tag{34}$$

From equation (34) and the continuity of displacements along the bond line, i.e.,

$$u_1 + iv_1 = u_2 + iv_2, \quad \text{at} \quad y = 0$$

it is found that

$$G_2(\eta_1 + 1)A_1 = G_1(\eta_2 + 1)A_2 \tag{35}$$

To simplify the notation, let $A_1 = A$. Thus, the Goursat functions for the two materials become

$$\Phi_1 = A, \quad \Psi_1 = -(A + \bar{A}), \quad \text{for} \quad y > 0$$

and

$$\begin{aligned}\Phi_2 &= \frac{G_2}{G_1}\left(\frac{\eta_1 + 1}{\eta_2 + 1}\right)A, \\ \Psi_2 &= -\frac{G_2}{G_1}\left(\frac{\eta_1 + 1}{\eta_2 + 1}\right)(A + \bar{A}), \quad \text{for} \quad y < 0\end{aligned} \tag{36}$$

APPENDIX 2

An Infinite Row of Collinear Cracks

The problem of an infinite series of equal cracks of length $2a$ along the bond line of two dissimilar materials and spaced at constant intervals $b(>2a)$ may be solved by the method described earlier. Referring to Fig. 4 for notation and the external loads at infinity, analogously to the expressions of (11), (13), and (17),

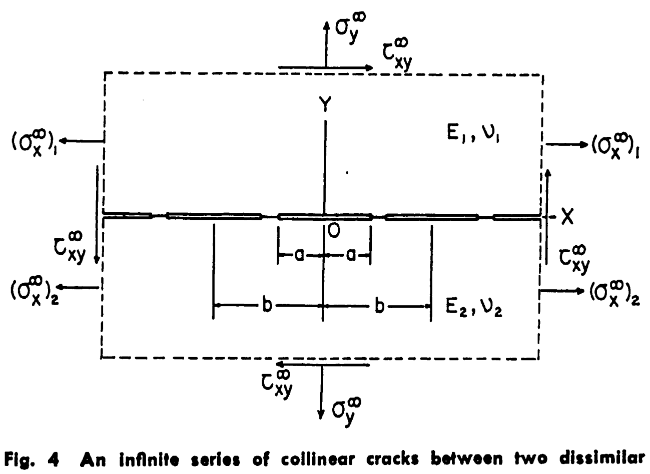

Fig. 4 An infinite series of collinear cracks between two dissimilar materials

$\Phi_1(z)$ is represented in the periodic form, giving stress-free crack surface and uniform stress at infinity,

$$\Phi_1(z) = B \prod_{n=-\infty}^{+\infty} (z - nb - d)F(z - nb) + A$$

$$= B\left(\frac{z-d}{b}\right)\left(\frac{z-a}{b}\right)^{-\frac{1}{2}-i\epsilon}\left(\frac{z+a}{b}\right)^{-\frac{1}{2}+i\epsilon}$$

$$\times \prod_{n=1}^{\infty}\left\{\left[1-\frac{1}{n^2}\left(\frac{z-d}{b}\right)^2\right]\left[1-\frac{1}{n^2}\left(\frac{z-a}{b}\right)^2\right]^{-\frac{1}{2}-i\epsilon}\right.$$

$$\left.\times\left[1-\frac{1}{n^2}\left(\frac{z+a}{b}\right)^2\right]^{-\frac{1}{2}+i\epsilon}\right\} + A$$

One may show that for single-valued displacements, $d = 2i\epsilon a$; and that boundary conditions at infinity are satisfied by expressing $A = A_1 + iA_2$ and $B = B_1 + iB_2$ in terms of the applied stresses and rotation through equations identical to (18) and (20).

Noting that

$$\sin \pi t = \pi t \prod_{n=1}^{\infty} (1 - t^2/n^2),$$

$$\Phi_1(z) = B \sin\frac{\pi(z - 2i\epsilon a)}{b}\left[\sin\frac{\pi(z-a)}{b}\right]^{-\frac{1}{2}-i\epsilon} \times \left[\sin\frac{\pi(z+a)}{b}\right]^{-\frac{1}{2}+i\epsilon} + A \quad (37)$$

and the remaining Goursat functions are given by

$$\Psi_1(z) = e^{2\pi\epsilon}[\bar{\Phi}_1(z) - \bar{A}] - [\Phi_1(z) - A + z\Phi_1'(z)] - (A + \bar{A})$$

$$\Phi_2(z) = e^{2\pi\epsilon}[\Phi_1(z) - A] + \mu A$$

$$\Psi_2(z) = [\bar{\Phi}_1(z) - \bar{A}] - e^{2\pi\epsilon}[\Phi_1(z) - A + z\Phi_1'(z)] - \mu(A + \bar{A})$$

In the special case when $\epsilon = 0$, the foregoing solution reduces to that obtained by Koiter [8] for two similar materials.

APPENDIX 3

Stress Jump Across Interface

In general, equation (19) may be derived by considering the equilibrium of an element occupying both the region $y > 0$ and $y < 0$, $y = 0$ being the bond line. The stress component σ_x is taken to be discontinuous across the line $y = 0$ and the strain component ϵ_x to be continuous along such a line, i.e.,

$$(\epsilon_x)_1 = (\epsilon_x)_2$$

It follows from the strain-stress relations that

$$(\sigma_x)_2 = \frac{E_2}{E_1}(\sigma_x)_1 + \left[\nu_2 - \frac{E_2}{E_1}\nu_1\right]\sigma_y \quad (38)$$

for plane stress and

$$(\sigma_x)_2 = \frac{E_2}{E_1}\left(\frac{1-\nu_1^2}{1-\nu_2^2}\right)(\sigma_x)_1 + \left[\frac{\nu_2}{1-\nu_2} - \frac{E_2}{E_1}\frac{\nu_1(1+\nu_1)}{1-\nu_2^2}\right]\sigma_y \quad (39)$$

for plane strain. Equations (38) and (39) may be made into a single generalization upon defining $\eta_j (j = 1, 2)$ such that $\eta_j = 3 - 4\nu_j$ for plane strain and $\eta_j = (3 - \nu_j)/(1 + \nu_j)$ for plane stress. Moreover, using equation (6), the final result may be put into the form of equation (19).