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On the Distribution of Rises and Falls in a Continuous Random Process

This paper is concerned with the statistics of the height of rise and fall for continuous random processes. In particular, approximate methods are given for determining the probability density of the increment in a random continuous function as the function passes from one extremum to the next. Application of the general result is made to the case of processes with a Gaussian distribution. Numerical results are given for four special cases of stationary Gaussian processes. Computed results are found to agree well with available experimental data. The knowledge of such statistical information is of use in studies dealing with fatigue under random loadings.

Statement of Problem

Consider a continuous and twice differentiable random function, $x(t)$. The purpose of this work is to predict the probability density for the height of rise or fall of $x(t)$ as this function passes from one extremum to the next. In what follows it will be convenient to view the independent variable t as the time and $x(t)$ as a random time function.

Assuming that the first extremum is a minimum, the function to be computed is the joint density $P(\alpha, \alpha', \tau)$ where

$P(\alpha, \alpha', \tau)d\alpha\, d\alpha' d\tau$ = probability that, given a minimum of $x(t)$ at $t = 0$, the next maximum will occur in the time interval $\tau < t < \tau + d\tau$, with $\alpha < x(0) < \alpha + d\alpha$ and $\alpha' < x(\tau) < \alpha' + d\alpha'$

This joint density leads to a great deal of information on the statistics of the rise and fall distribution. From it the density function for the height of rise can be computed as follows:

$$P(h) = \int_0^\infty \int_{-\infty}^{+\infty} P(\alpha, \alpha + h, \tau)d\alpha\, d\tau \tag{1}$$

where

$P(h)dh$ = probability that the increment of the function $x(t)$ as this function passes from a minimum at $t = 0$ to the next maximum falls within the range h to $h + dh$

More particular information on the density of the height of rise h for the class of minima occurring at some particular level α may also be obtained. For example, we may write

$$P(h|\alpha) = \frac{1}{Q(\alpha)} \int_0^\infty P(\alpha, \alpha + h, \tau)d\tau \tag{2}$$

where

$Q(\alpha)d\alpha$ = probability that, given a minimum of $x(t)$ at $t = 0$, the value $x(0)$ of this minimum falls within the range α to $\alpha + d\alpha$

and

$P(h|\alpha)dh$ = probability that, given a minimum of $x(t)$ at $t = 0$ with $x(0) = \alpha$, the height of rise in $x(t)$ in going to the next maximum falls within the range h to $h + dh$

Such information is of use in evaluating the fatigue life of structural members subjected to random loadings [1],[1] and in predicting the rate of propagation of a fatigue crack under random loadings [2]. A discussion of the rise and fall problem in somewhat greater detail than to be given here is contained in [5].

Approximate Formulation

Attempts at an exact computation of $P(\alpha, \alpha', \tau)$ by the method of exclusion and inclusion, which will be referred to later, lead to a hopelessly complex expression which can be evaluated only in certain trivial cases. Thus it was found necessary to use the following approximate formulation:

Using the law of conditional probabilities, the probability density $P(\alpha, \alpha', \tau)$ may be expressed as the product

$$P(\alpha, \alpha', \tau) = F_0(\alpha, \alpha'|\tau)F(\tau) \tag{3}$$

where

$F_0(\alpha, \alpha'|\tau)d\alpha d\alpha'$ = probability that, given a minimum of $x(t)$ at $t = 0$ and the next maximum at $t = \tau$, the values of $x(0)$ and $x(\tau)$ fall within the ranges $\alpha < x(0) < \alpha + d\alpha$ and $\alpha' < x(\tau) < \alpha' + d\alpha'$

$F(\tau)d\tau$ = probability that, given a minimum of $x(t)$ at $t = 0$, the next maximum occurs in the interval $\tau < t < \tau + d\tau$

We note that $F(\tau)$ is the probability density for the time between zero crossings of $\dot{x}(t)$. Since no solution is known to the zero-crossing problem, which is valid over the entire range of time between zero crossings, an approximate expression will be developed subsequently for $F(\tau)$. Similarly, we shall approximate the conditional joint density $F_0(\alpha, \alpha'|\tau)$ by writing

$$F_0(\alpha, \alpha'|\tau) \approx f_0(\alpha, \alpha'|\tau) \tag{4}$$

where

$f_0(\alpha, \alpha'|\tau)d\alpha\, d\alpha'$ = probability that, given a minimum of $x(t)$ at $t = 0$ and a maximum at $t = \tau$, the values of $x(0)$ and $x(\tau)$ fall within the ranges $\alpha < x(0) < \alpha + d\alpha$ and $\alpha' < x(\tau) < \alpha' + d\alpha'$

The difference between $F_0(\alpha, \alpha'|\tau)$ and $f_0(\alpha, \alpha'|\tau)$ lies in the fact that in the former case it is assumed that the first maximum after a minimum at $t = 0$ occurs at $t = \tau$, while in the latter case it is assumed only that a maximum (not necessarily the first) after a minimum at $t = 0$ occurs at $t = \tau$. For small values of τ the probability of having a maximum at τ and some other maximum in the interval $0 < t < \tau$ is correspondingly small and the approximation is quite good. For larger values of τ the difference between the two expressions becomes significant but, since $F(\tau)$ approaches zero when τ increases indefinitely because of the high probability of a first maximum having already occurred in the interval $0 < t < \tau$, the discrepancy for large values of τ is not expected to contribute significantly to the error.

The next step is an approximation for $F(\tau)$. First, an exact ex-

[1] Numbers in brackets designate References at end of paper.

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pression will be developed from which the approximation may be constructed. Instead of restricting the discussion to the case of rise and fall in a random curve, the development will be given in general terms.

Let $G(\tau)d\tau$ be the probability that the first new occurrence of an event which happened at $t = 0$ takes place in the interval $\tau < t < \tau + d\tau$. Let $q(\tau|0 < t < \tau)d\tau$ be the probability that, given that the event occurred at $t = 0$ and given no occurrence of the event in the interval $0 < t < \tau$, the event occurs in the interval $\tau < t < \tau + d\tau$. Then, by the law of conditional probabilities, the probability of a first new occurrence in the interval $\tau < t < \tau + d\tau$ is the probability of an occurrence in that interval given no occurrence for $0 < t < \tau$ multiplied by the probability of no occurrence for $0 < t < \tau$. Thus

$$G(\tau) = q(\tau|0 < t < \tau)\left\{1 - \int_0^\tau G(t)dt\right\} \quad (5)$$

Solving this equation yields the following expression for the probability density of the first new occurrence:

$$G(\tau) = q(\tau|0 < t < \tau) \exp\left\{-\int_0^\tau q(s|0 < t < s)ds\right\} \quad (6)$$

At this point the approximation

$$q(\tau|0 < t < \tau) \approx q(\tau) \quad (7)$$

is made where $q(\tau)\,d\tau$ is the probability of an occurrence in the interval $\tau < t < \tau + d\tau$ given the event occurred at $t = 0$. The corresponding approximation for the probability density of the first new occurrence is

$$G(\tau) \approx q(\tau) \exp\left\{-\int_0^\tau q(t)dt\right\} \quad (8)$$

The foregoing approximation of $q(\tau|0 < t < \tau)$ by $q(\tau)$ results in neglecting the dependence of an occurrence of the event at $t = \tau$ on a prior occurrence of the event in the interval $0 < t < \tau$. However, the dependence of an occurrence of the event at $t = \tau$ on the given occurrence at $t = 0$ is taken into account. The approximate relation (8) may be used for a variety of first new occurrence problems, and the relation becomes exact when the occurrence of an event at $t = \tau$ is independent of the occurrence of the event at some value of t in the interval $0 < t < \tau$.

For very small values of τ, the exponential in (8) is close to unity and the relation reduces to $G(\tau) \approx q(\tau)$; this is a good approximation since the probability of an event occurring in $0 < t < \tau$ is then very small. For very large values of τ, relation (8) may be written in the form

$$G(\tau) \approx \exp\left\{\int_0^\infty [q(\infty) - q(t)]dt\right\} q(\infty) \exp\{-q(\infty)\tau\}$$

which, except for a constant factor, represents the probability density of the first new occurrence of a process consisting of events occurring independently in time.

For values of τ which are neither very large nor very small, little can be said about the closeness of the approximation, but it should be noted that the integral of $G(\tau)$, as approximated in (8), over values of τ from zero to infinity gives unity.

Returning now to $F(\tau)$, the probability density for the first new occurrence of an extremum in a random function $x(t)$, we let

$p(\tau)d\tau$ = probability that, given a minimum of $x(t)$ at $t = 0$, a maximum occurs in the time interval $\tau < t < \tau + d\tau$

Then, under the assumptions leading to the approximation found in (8), we may write

$$F(\tau) \approx p(\tau) \exp\left\{-\int_0^\tau p(t)dt\right\} \quad (9)$$

Recalling the approximation obtained for $F_0(\alpha, \alpha'|\tau)$ in (4) and substituting into (3), we have

$$P(\alpha, \alpha', \tau) \approx f_0(\alpha, \alpha'|\tau)p(\tau)e^{-\int_0^\tau p(t)dt} \quad (10)$$

The product of the first two terms in (10) is simply, by the law of conditional probability, the function $f(\alpha, \alpha', \tau)$ where

$f(\alpha, \alpha', \tau)d\alpha\, d\alpha' d\tau$ = probability that, given a minimum of $x(t)$ at $t = 0$, a maximum will occur in the time interval $\tau < t < \tau + d\tau$, with $\alpha < x(0) < \alpha + d\alpha$ and $\alpha' < x(\tau) < \alpha' + d\alpha'$

Thus the final form of the approximation for the joint density $P(\alpha, \alpha', \tau)$ is

$$P(\alpha, \alpha', \tau) \approx f(\alpha, \alpha', \tau) \exp\left\{-\int_0^\tau p(t)dt\right\} \quad (11)$$

To apply the approximate foregoing formula, the functions $f(\alpha, \alpha', \tau)$ and $p(\tau)$ must be expressed in terms of the joint probability-density functions of the particular continuous random function $x(t)$ under consideration.

To compute $p(\tau)$, let $g(\beta, \gamma)d\beta d\gamma$ = probability that $\beta < \dot{x}(0) < \beta + d\beta$ and $\gamma < \ddot{x}(0) < \gamma + d\gamma$

and

$g(\beta, \gamma;\ \beta', \gamma';\ \tau)d\beta\, d\gamma\, d\beta'\, d\gamma'$ = probability that $\beta < \dot{x}(0) < \beta + d\beta$, $\gamma < \ddot{x}(0) < \gamma + d\gamma$, $\beta' < \dot{x}(\tau) < \beta' + d\beta'$, and $\gamma' < \ddot{x}(\tau) < \gamma' + d\gamma'$.

Then

$$p(\tau) = \frac{\int_0^{-\infty}\int_0^\infty \gamma\gamma' g(0, \gamma;\ 0, \gamma';\ \tau)d\gamma\, d\gamma'}{\int_0^\infty \gamma g(0, \gamma)d\gamma} \quad (12)$$

To compute $f(\alpha, \alpha', \tau)$, let

$g(\alpha, \beta, \gamma;\ \alpha', \beta', \gamma';\ \tau)d\alpha\, d\beta\, d\gamma\, d\alpha'\, d\beta'\, d\gamma'$ = probability that $\alpha < x(0) < \alpha + d\alpha$, $\beta < \dot{x}(0) < \beta + d\beta$, $\gamma < \ddot{x}(0) < \gamma + d\gamma$, $\alpha' < x(\tau) < \alpha' + d\alpha'$, $\beta' < \dot{x}(\tau) < \beta' + d\beta'$, and $\gamma' < \ddot{x}(\tau) < \gamma' + d\gamma'$

Then

$$f(\alpha, \alpha', \tau) = \frac{\int_0^{-\infty}\int_0^\infty \gamma\gamma' g(\alpha, 0, \gamma;\ \alpha', 0, \gamma';\ \tau)d\gamma\, d\gamma'}{\int_0^\infty \gamma g(0, \gamma)d\gamma} \quad (13)$$

Details of the derivations of (12) and (13) have been omitted here. They may easily be filled in by referring to similar derivations given in [3].

Expressions for Error in Approximating

To obtain a better understanding of the error involved in the approximations introduced in the preceding section, the formulas developed there will be compared with exact expressions obtained by the method of inclusion and exclusion. First, consider the approximation

$$F(\tau) \approx p(\tau)e^{-\int_0^\tau p(t)dt} \quad (9)$$

obtained for the probability density of the time τ between successive extrema. The exact value of $F(\tau)$ may be expressed as a series of integrals through a straightforward application of the inclusion and exclusion method as indicated in [3]:

$$F(\tau) = p(\tau) - \frac{1}{1!}\int_0^\tau p(\tau, t)dt_1 + \frac{1}{2!}\int_0^\tau\int_0^\tau p(\tau, t_1, t_2)dt_1\, dt_2 - \ldots \quad (14)$$

where

$p(\tau, t_1, t_2, \ldots, t_k)d\tau\, dt_1\, dt_2 \ldots dt_k$ = probability that, given a minimum at $t = 0$, maxima will occur in the intervals $\tau < t < \tau + d\tau$, $t_1 < t < t_1 + dt_1$, $t_2 < t < t_2 + dt_2$, ..., and $t_k < t < t_k + dt_k$

Expanding the exponential in the right member of (9) in a Taylor series, one obtains

$$p(\tau) \exp\left\{-\int_0^\tau p(t)dt\right\} = p(\tau) - \frac{1}{1!}\int_0^\tau p(\tau)p(t_1)dt_1 + \frac{1}{2!}\int_0^\tau\int_0^\tau p(\tau)p(t_1)p(t_2)dt_1\,dt_2 - \ldots \quad (15)$$

Subtracting (15) from (14) term by term yields the difference between the exact and the approximate expressions for $F(\tau)$:

$$F(\tau) - p(\tau)\exp\left\{-\int_0^\tau p(t)dt\right\} = -\frac{1}{1!}\int_0^\tau \{p(\tau, t_1) - p(\tau)p(t_1)\}dt_1 + \frac{1}{2!}\int_0^\tau\int_0^\tau \{p(\tau, t_1, t_2) - p(\tau)p(t_1)p(t_2)\}dt_1\,dt_2 + \ldots \quad (16)$$

We thus check that the error in the approximation (9) is due to neglecting the dependence of a maximum at some particular time on maxima at other times, though the dependence on the minimum at $t = 0$ is taken into account.

The final approximate expression for $P(\alpha, \alpha', \tau)$, as given in (11), is

$$P(\alpha, \alpha', \tau) \approx f(\alpha, \alpha', \tau)e^{-\int_0^\tau p(t)dt} \quad (11)$$

The exact value of the probability density $P(\alpha, \alpha', \tau)$ may be expressed as a series of integrals by the method of inclusion and exclusion:

$$P(\alpha, \alpha', \tau) = f(\alpha, \alpha', \tau) - \frac{1}{1!}\int_0^\tau f(\alpha, \alpha', \tau, t_1)dt_1 + \frac{1}{2!}\int_0^\tau\int_0^\tau f(\alpha, \alpha', \tau, t_1, t_2)dt_1\,dt_2 - \ldots \quad (17)$$

where

$f(\alpha, \alpha', \tau, t_1, t_2, \ldots, t_k)d\alpha\, d\alpha'\, d\tau\, dt_1\, dt_2 \ldots dt_k$ = probability that, given a minimum at $t = 0$, a maximum will occur in the time interval $\tau < t < \tau + d\tau$ with $\alpha < x(0) < \alpha + d\alpha$ and $\alpha' < x(\tau) < \alpha' + d\alpha'$, and that maxima will occur in the time intervals $t_1 < t < t_1 + dt_1$, $t_2 < t < t_2 + dt_2$, ..., $t_k < t < t_k + dt_k$

Expanding the exponential in the right member of (10) in a Taylor series, one obtains

$$f(\alpha, \alpha', \tau)e^{-\int_0^\tau p(t)dt} = f(\alpha, \alpha', \tau) - \frac{1}{1!}\int_0^\tau f(\alpha, \alpha', \tau)p(t_1)dt_1 + \frac{1}{2!}\int_0^\tau\int_0^\tau f(\alpha, \alpha', \tau)p(t_1)p(t_2)dt_1\,dt_2 - \ldots \quad (18)$$

Subtracting (18) from (17) term by term yields the difference between the exact and the approximate expressions for $P(\alpha, \alpha', \tau)$:

$$P(\alpha, \alpha', \tau) - f(\alpha, \alpha', \tau)e^{-\int_0^\tau p(t)dt} = \frac{-1}{1!}\int_0^\tau \{f(\alpha, \alpha', \tau, t_1) - f(\alpha, \alpha', \tau)p(t_1)\}dt_1 + \frac{1}{2!}\int_0^\tau\int_0^\tau \{f(\alpha, \alpha', \tau, t_1, t_2) - f(\alpha, \alpha', \tau)p(t_1)p(t_2)\}dt_1dt_2 - \ldots \quad (19)$$

Again we note that an error is introduced by neglecting the dependence of a maximum at some particular time on maxima at other times, although the dependence on the minimum at $t = 0$ is taken into account. An additional error is introduced, however, by neglecting the dependence between the occurrence of a maximum at some value of t in the interval $0 < t < \tau$ and the values α and α' of the function at the end points of the interval, although the dependence between a maximum at $t = \tau$ and the values α and α' is taken into account.

As a closing note to this section, it should be indicated that, since the function $P(\alpha, \alpha', \tau)$ is integrated over all τ-values in relations (1) and (2), one should expect the error on the probability densities $P(h)$ or $P(h|\alpha)$ to be smaller than the error on $P(\alpha, \alpha', \tau)$.

Average Height of Rise and Fall

For stationary processes, the average height of rise and fall may be very simply computed, although the determination of the distribution of rises and falls remains an extremely difficult problem. The method consists of finding the average rise and fall per unit time and dividing the value obtained by the expected number of extrema per unit time. Now, the average rise and fall per unit time is the expected value $\langle|\dot{x}|\rangle$ of the absolute value of the derivative of the process $x(t)$. Thus, if N_e is the expected number of extrema per unit time,

$$\bar{h} = \frac{\langle|\dot{x}|\rangle}{N_e} \quad (20)$$

It is shown in [3] that

$$N_e = \int_{-\infty}^{+\infty} |\gamma| g_{\dot{x}\ddot{x}}(0, \gamma)d\gamma \quad (21)$$

where $g_{\dot{x}\ddot{x}}(\beta, \gamma)$ is the joint probability density of the first and second derivatives. Denoting by $g_{\ddot{x}|\dot{x}}(\gamma|\beta)$ and $g_{\dot{x}}(\beta)$, respectively, the conditional density of $\ddot{x}$ given $\dot{x}$, and the density of $\dot{x}$, one obtains by the law of conditional probability

$$N_e = g_{\dot{x}}(0)\int_{-\infty}^{+\infty} |\gamma| g_{\ddot{x}|\dot{x}}(\gamma|0)d\gamma$$
$$N_e = g_{\dot{x}}(0)\langle|\ddot{x}|\,|\dot{x} = 0\rangle \quad (22)$$

Thus, the average rise and fall may be expressed as

$$\bar{h} = \frac{\langle|\dot{x}|\rangle}{g_{\dot{x}}(0)\langle|\ddot{x}|\,|\dot{x} = 0\rangle} \quad (23)$$

In certain cases the average height of rise may be expressed in terms of the ratio of the expected number of mean level crossings per unit time to the expected number of extrema per unit time. To illustrate, suppose that the process $x(t)$ has zero mean. Then the expected number of mean level crossings (or zero crossings) per unit time is [3]

$$N_0 = \int_{-\infty}^{+\infty} |\beta| g_{x\dot{x}}(0, \beta)d\beta \quad (24)$$

A reasoning similar to the one used to derive (22) yields

$$N_0 = g_x(0)\langle|\dot{x}|\,|x = 0\rangle \quad (25)$$

where $g_x(\alpha)$ denotes the probability density of x. Now suppose the process $x(t)$ is such that

$$\langle|\dot{x}|\,|x = 0\rangle = \langle|\dot{x}|\rangle \quad (26)$$

Then, by equations (20) and (25), one obtains in that case

$$\bar{h} = \frac{1}{g_x(0)}\frac{N_0}{N_e} \quad (27)$$

It is noted in passing that the condition expressed by equation (26) is satisfied in at least one type of stationary process, namely, a process in which the joint density of x and $\dot{x}$ is two-dimensional Gaussian. In any stationary process, the function x and its derivative $\dot{x}$ are uncorrelated random variables, but this condition alone does not ensure that (26) will hold. A somewhat stronger condition is required; namely, that the conditional density of $\dot{x}$, given that x attains its mean value, is identical to the density of $\dot{x}$, or

$$g_{\dot{x}|x}(\beta|0) = g_{\dot{x}}(\beta) \tag{28}$$

For processes in which (26) holds, equation (27) is valid, and this leads to a convenient experimental method for determining $\bar{h}$ which clearly does not involve the actual measurement of rise and fall heights. Equation (27) also leads to an upper bound for $\bar{h}$. Since the expected number of extrema per unit time must always be greater than or equal to the expected number of zero crossings per unit time, one must have

$$\bar{h} \leqslant \frac{1}{g_x(0)} \tag{29}$$

Application to Stationary Gaussian Processes

In this section it is assumed that $x(t)$, $\dot{x}(t)$, and $\ddot{x}(t)$ form a multidimensional Gaussian distribution and have zero mean values. For the stationary case, such a process is statistically completely defined by the correlation function

$$R(\tau) = \langle x(t)x(t+\tau)\rangle \tag{30}$$

Expressions for $p(\tau)$, equation (12), and $f(\alpha, \alpha', \tau)$, equation (13), will be given in terms of this function and its derivatives. Since a function almost identical to $p(\tau)$ has been used in [3], only the resulting expression will be given here and details of the derivation will be omitted:

$$p(\tau) = \frac{1}{2\pi}\left[\frac{-R''(0)}{R^{\text{iv}}(0)}\right]^{1/2}[J^2 - K^2]^{1/2} \times[\{R''(0)\}^2 - \{R''(\tau)\}^2]^{-3/2}[1 + H\cot^{-1}(-H)] \tag{31}$$

where

$$J = R^{\text{iv}}(0)[\{R''(0)\}^2 - \{R''(\tau)\}^2] + R''(0)\{R'''(\tau)\}^2$$

$$K = -R^{\text{iv}}(\tau)[\{R''(0)\}^2 - \{R''(\tau)\}^2] - R''(\tau)\{R'''(\tau)\}^2$$

$$H = K[J^2 - K^2]^{-1/2}$$

and

$$0 \leqslant \cot^{-1}(-H) \leqslant \pi \tag{32}$$

The expression for $f(\alpha, \alpha', \tau)$ cannot be integrated in closed form, but it can be reduced to a single integral. First consider the integral in the denominator in (13). It may be written, as shown in [4],

$$\int_0^\infty |\gamma| g(0, \gamma) d\gamma = \frac{1}{2\pi}\left[\frac{R^{\text{iv}}(0)}{-R''(0)}\right]^{1/2} \tag{33}$$

and represents the expected number of maxima per unit time (or the reciprocal of the mean time between successive maxima).

The integrand in the numerator of (13) may be evaluated from the multidimensional Gaussian distribution as shown in [4]. Replacing α' by $\alpha + h$ and writing out the Gaussian expression for $g(\alpha, 0, \gamma;\ \alpha + h, 0, \gamma';\ \tau)$, one obtains after reductions

$$f(\alpha, \alpha + h, \tau) = \left(\frac{1}{2\pi}\right)^2\left[\frac{-R''(0)}{R^{\text{iv}}(0)}\right]^{1/2} \times \frac{1}{\sqrt{|M|}}\exp\left[-(s_{11} + s_{14})(\alpha h + \alpha^2) - \tfrac{1}{2}s_{11}h^2\right]$$
$$\times \int_0^{-\infty}\int_0^{\infty}\gamma\gamma' \exp\left[-\tfrac{1}{2}s_{33}(\gamma^2 + \gamma'^2) - s_{36}\gamma\gamma' - s_{13}h\gamma' - s_{16}h\gamma - (s_{13} + s_{16})\alpha(\gamma + \gamma')\right]d\gamma\, d\gamma' \tag{34}$$

Here $|M|$ is the determinant of the correlations matrix $[M]$, and s_{ij} is the element in the ith row and jth column of the inverse of the correlation matrix $[M]$, where

$$[M] = \begin{bmatrix} R(0) & 0 & R''(0) & R(\tau) & R'(\tau) & R''(\tau) \\ 0 & -R''(0) & 0 & -R'(\tau) & -R''(\tau) & -R'''(\tau) \\ R''(0) & 0 & R^{\text{iv}}(0) & R''(\tau) & R'''(\tau) & R^{\text{iv}}(\tau) \\ R(\tau) & -R'(\tau) & R''(\tau) & R(0) & 0 & R''(0) \\ R'(\tau) & -R''(\tau) & R'''(\tau) & 0 & -R''(0) & 0 \\ R''(\tau) & -R'''(\tau) & R^{\text{iv}}(\tau) & R''(0) & 0 & R^{\text{iv}}(0) \end{bmatrix} \tag{35}$$

Note that $|M|$ and the elements s_{ij} are functions of τ.

Since finding the rise and fall distribution entails an integration on α, let

$$f(h, \tau) = \int_{-\infty}^{+\infty} f(\alpha, \alpha + h, \tau) d\alpha \tag{36}$$

The expression obtained after integration and a change of variables is

$$f(h, \tau) = A\exp[-Bh^2]\int_0^\infty\int_0^\infty uv \exp[-(u^2 + 2cuv + v^2) - 2Dh(u + v)]du\, dv \tag{37}$$

where

$$A = \frac{1}{\pi[\pi|M|(s_{11} + s_{14})]^{1/2}\left[s_{33} - \dfrac{(s_{13} + s_{16})^2}{2(s_{11} + s_{14})}\right]^2}\left[\frac{-R''(0)}{R^{\text{iv}}(0)}\right]^{1/2} \tag{38}$$

$$B = \tfrac{1}{4}(s_{11} - s_{14}) \tag{39}$$

$$C = \frac{s_{36} - \dfrac{(s_{13} + s_{16})^2}{2(s_{11} + s_{14})}}{s_{33} - \dfrac{(s_{13} + s_{16})^2}{2(s_{11} + s_{14})}} \tag{40}$$

$$D = \frac{s_{13} - s_{16}}{2\sqrt{2}\left[s_{33} - \dfrac{(s_{13} + s_{16})^2}{2(s_{11} + s_{14})}\right]^{1/2}} \tag{41}$$

All attempts at evaluating the double integral in (37) in closed form have proved futile. However, through the transformation of coordinates $u = r\cos\theta$ and $v = r\sin\theta$, the double integral may be reduced to a single integral in θ; and the infinite upper limit, inconvenient for numerical evaluation, may be eliminated. The result, after integration in r and noting the symmetry about $\theta = \pi/4$, is

$$f(h, \tau) = \tfrac{1}{2}A\exp[-Bh^2]\int_0^{\pi/4}\frac{\sin 2\theta}{(1 + C\sin 2\theta)^2} [\sqrt{\pi}z(\tfrac{3}{2} + z^2)\exp(z^2)\{1 + \text{erf}(z)\} + (1 + z^2)]d\theta \tag{42}$$

where

$$z = z(h, \tau, \theta) = \frac{Dh(\sin\theta + \cos\theta)}{(1 + C\sin 2\theta)^{1/2}} \tag{43}$$

and here erf (x) is defined by

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \tag{44}$$

Some observations will now be made on the behavior of $p(\tau)$ and $f(h, \tau)$ for values of τ close to zero. Since numerical integrations must be used to evaluate $f(h, \tau)$ and finally to carry out the integration over τ from 0 to ∞, the behavior near $\tau = 0$ is quite important. The reason is that, because of the structure of the equations, slight numerical inaccuracies are particularly critical for small values of τ. Fortunately, both $p(\tau)$ and $f(h, \tau)$ are equal to zero for $\tau = 0$. In fact, it has been shown in reference [3] that the form of $p(\tau)$ for τ near zero is

$$p(\tau) \approx \frac{\tau}{8}\left[\frac{\{R^{(4)}(0)\}^2 - R''(0)R^{(6)}(0)}{R''(0)R^{(4)}(0)}\right] \tag{45}$$

Since by its definition

$$p(\tau) = \int_{-\infty}^{+\infty} f(h, \tau)dh \tag{46}$$

and since $f(h, \tau) > 0$ for all values of h, it follows that $f(h, \tau)$ also varies linearly with τ when τ is near zero.

The average height of rise and fall for stationary Gaussian processes is readily computed from equation (20). The expected absolute value of $\dot{x}$ is

$$\langle|\dot{x}|\rangle = \int_{-\infty}^{+\infty} |\beta| \frac{1}{[-2\pi R''(0)]^{1/2}} \exp\left[\frac{1}{2}\frac{\beta^2}{R''(0)}\right] d\beta = \left[-\frac{2}{\pi}R''(0)\right]^{1/2} \tag{47}$$

The expected number of extrema per unit time is, from reference [3]

$$N_e = \frac{1}{2\pi}\left[\frac{R^{\text{iv}}(0)}{-R''(0)}\right]^{1/2} \tag{48}$$

Thus, the average height of rise and fall is

$$\bar{h} = -R''(0)\left[\frac{2\pi}{R^{\text{iv}}(0)}\right]^{1/2} \tag{49}$$

Since, for a stationary Gaussian process $x(t)$, the function and its derivative are independent random variables, equations (27) and (29) are valid. But, for a Gaussian process with zero mean

$$g_x(0) = \left[\frac{1}{2\pi R(0)}\right]^{1/2} \tag{50}$$

Thus, equations (27) and (29) yield, respectively

$$\bar{h} = [2\pi R(0)]^{1/2}\frac{N_0}{N_e} \tag{51}$$

$$\bar{h} \leqslant [2\pi R(0)]^{1/2} \tag{52}$$

Equation (51) leads to a particularly simple method for the experimental determination of $\bar{h}$. Instead of actually measuring rise and fall heights, one need only determine the variance $R(0)$ of the process and make a count, over a sufficiently large time interval, of the number of mean crossings and extrema. As far as equation (52) is concerned, it will be seen later that the upper bound is approached as the spectral "bandwidth" of the process is narrowed.

Numerical Examples for Stationary Gaussian Processes

Some numerical examples are given here for stationary Gaussian random functions with power spectra constant over a certain frequency domain and zero elsewhere. Letting σ^2 represent the variance of $x(t)$, the power spectrum $F(\omega)$ may be expressed as

$$F(\omega) = \begin{cases} \dfrac{\sigma^2}{(1-\beta)\omega_c} & \text{for } \beta\omega_c < \omega < \omega_c \\ 0 & \text{otherwise} \end{cases} \tag{53}$$

where $0 \leqslant \beta < 1$, ω_c is an upper cut-off frequency, and $\beta\omega_c$ is a lower cut-off frequency. The case $\beta = 0$ corresponds to the ideal low-pass filter, while cases for which β is a substantial fraction of one correspond to narrow-band filters.

In performing numerical computations, it is convenient to use dimensionless quantities. To this end, we define the random process with unit variance and zero mean $y(t)$ as

$$y(t) = \frac{x(t) - \langle x(t)\rangle}{\sigma} \tag{54}$$

and introduce the dimensionless time $\varphi = \omega_c t$. We shall compute the rise and fall density for $y(t)$, which is equivalent to computing the density of h/σ where h is a rise or fall height of $x(t)$. The correlation function of $y(t)$, expressed in terms of φ, is given by

$$R(\varphi) = \frac{1}{\sigma^2}\int_0^\infty F(\omega)\cos\left(\frac{\omega}{\omega_c}\varphi\right)d\omega = \frac{1}{(1-\beta)\varphi}[\sin\varphi - \sin\beta\varphi] \tag{55}$$

In the present notation, and in the light of the approximation developed in equation (11), the expression (1) for the rise and fall density becomes

$$P(h/\sigma) = \int_0^\infty f(h/\sigma, \varphi)\exp\left[-\int_0^\varphi p(\lambda)d\lambda\right]d\varphi \tag{56}$$

where $f(h/\sigma, \varphi)$ and $p(\varphi)$ are defined for stationary Gaussian processes by equations (42) and (31). The evaluation of these functions requires a knowledge of $R(\varphi)$ and its first four derivatives for all φ.

The average height of rise and fall, evaluated from equation (49), is

$$\bar{h}/\sigma = \left[\frac{10\pi}{(1-\beta)(1-\beta^5)}\right]^{1/2}\frac{1-\beta^3}{3} \tag{57}$$

It is easily verified that $\bar{h}/\sigma$ approaches its upper bound of $(2\pi)^{1/2}$ when β approaches 1.

The actual evaluation of $P(h/\sigma)$ required extensive digital computing. Results were obtained for 19 values of h/σ ranging from 0 to 7.2 with spacings of 0.4. The integrations in φ and λ indicated in (56) were carried out by computing the integrand for particular values of φ spaced at intervals of 0.5, which is about 1/8 of the expected distance between successive extrema. The integration from 0 to ∞ in equation (56) was actually carried out from 0 to 50. An additional integration in θ indicated in equation (42) was carried out by computing the integrand at intervals of $\pi/32$. Aside from the time-consuming numerical integrations, the primary difficulty was in the accurate determination of the inverse members of the correlation matrix of (35). The inversion had to be carried out with an accuracy of 16 digits in order to yield meaningful results, and even then the inverse members could not be obtained for φ less than 2.

The rise and fall density was computed for four values of β; namely $\beta = 0$, 0.25, 0.50, and 0.75. Results are shown in Figs. 1 through 4.

The dashed line shown in Fig. 1 is a plot of experimental data obtained by Leybold [1] from a digitally generated random function with an ideal low-pass-filter power spectrum. It represents a sample of approximately 53,000 rises and falls. Agreement is seen to be good. The dashed line in Fig. 4 is a plot of a Rayleigh distributed rise and fall distribution which would occur in an extremely narrow-band process representable as a sine wave with an amplitude varying negligibly from peak to peak. The comparison seems appropriate since the case for which $\beta = 0.75$ has an average rise and fall height differing by only a few percent from the upper bound of $(2\pi)^{1/2}$. The computed results in this case for small h/σ are not shown since they were wildly fluctuating and attained a value at $h/\sigma = 0.4$ which was nearly three times the maximum value shown in Fig. 4.

Figs. 2 and 3 for $\beta = 0.25$ and 0.50 show an unexpected hump for small h/σ. Owing to the extreme complexity of the numerical computations, we are unable to say whether this inconsistency for small h/σ is due to numerical inaccuracies or is inherent in the approximation developed. The agreement with experimental data on the one hand and the limiting case of a Rayleigh distribution on the other indicates that the approximation developed gives a reasonable prediction of the rise and fall density for the larger values of h/σ. Since in applications [2] one is frequently interested in only the higher moments of the rise and fall density, the results are sufficient.

In preparing Figs. 1 through 3, the computed results have been corrected slightly for numerical inaccuracies. A check of the derivation for the rise and fall density will show that the integral from $-\infty$ to $+\infty$ of $P(h/\sigma)$ should give unity. Numerical computations for negative h/σ gave a computed result for the area which varied from about 0.95 to 0.97. Taking this to be a measure of computer error due to the various numerical integrations, we have divided all results by the computed area.

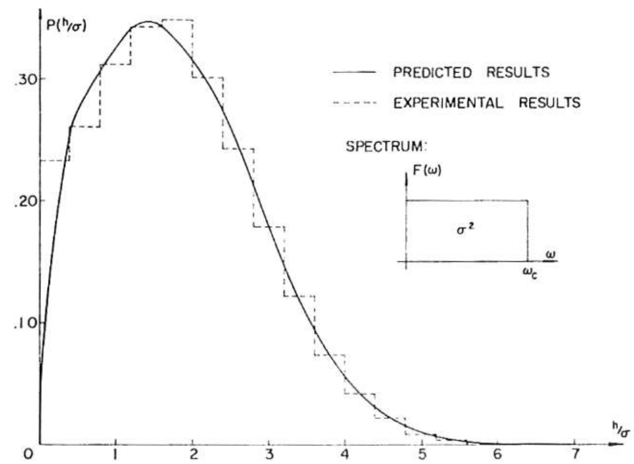

Fig. 1 Rise and fall density, $\beta = 0$ (ideal low-pass filter)

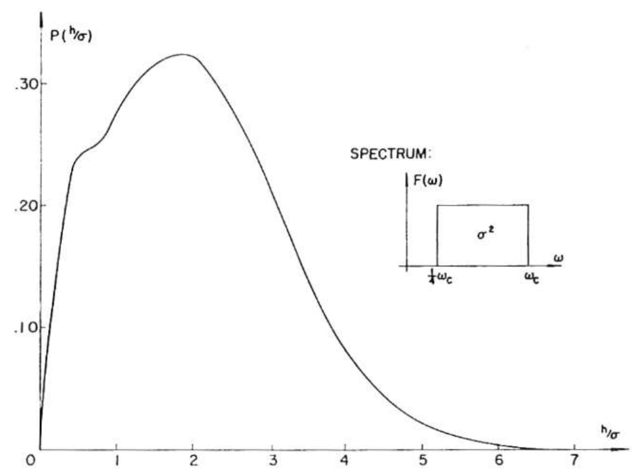

Fig. 2 Rise and fall density, $\beta = 1/4$

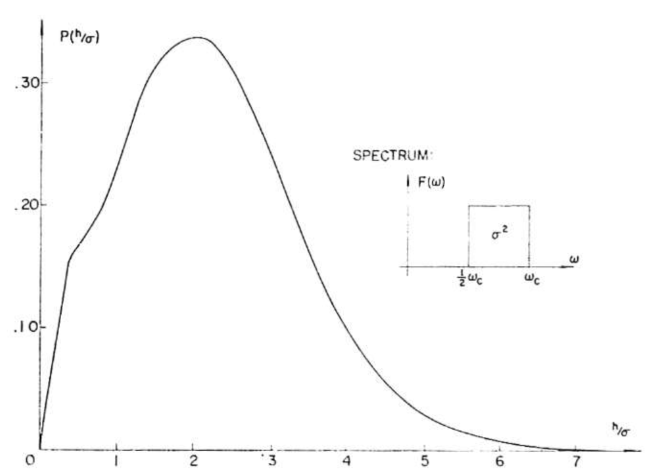

Fig. 3 Rise and fall density, $\beta = 1/2$

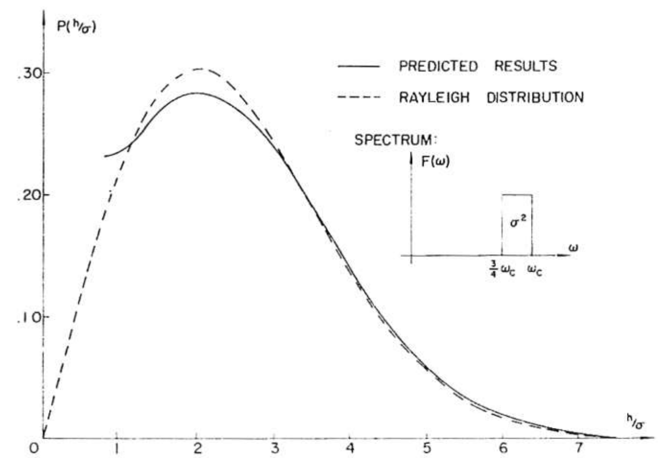

Fig. 4 Rise and fall density, $\beta = 3/4$ (narrow-band filter)

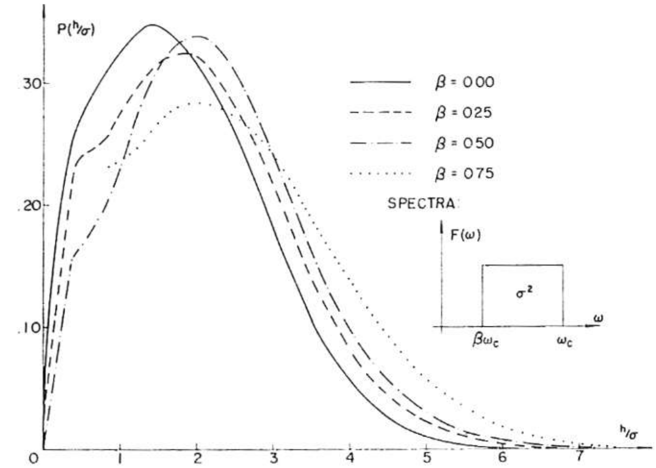

Fig. 5 Comparison of computed rise and fall densities

Table 1

Moment	$\beta = 0$	$\beta = 0.25$	$\beta = 0.50$	$\beta = 0.75$
$(\bar{h}/\sigma)$ exact	1.868	2.111	2.351	2.478
$(\bar{h}/\sigma)$ computed	1.810	2.026	2.244	...
$(\bar{h}^2/\sigma^2)$ computed	4.469	5.473	6.415	...
$(\bar{h}^3/\sigma^3)$ computed	13.112	17.484	21.491	...
$(\bar{h}^4/\sigma^4)$ computed	43.573	63.110	81.297	...

Fig. 5 is a combined plot of all the computed curves in Figs. 1 through 4. Table 1 gives a summary of the first four moments of h/σ for the various cases considered.

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