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The Bending of Plates of Dissimilar Materials With Cracks

This paper considers the problem of bending of a plate composed of two plates of materials having dissimilar elastic properties, bonded together along a straight line which sustains a crack. Both materials are assumed to be isotropic and homogeneous. Upon obtaining stress solutions, it is found that the significant stresses are inversely proportional to the square root of the radial distance from the crack front and have an oscillatory character, which is shown to be confined to the immediate vicinity of the crack tip. A two-parameter set of equations expressing the general form of the stress distribution around the tip of such a crack is provided as it is of primary importance in predicting the strength of cracked plates. Some analogies are also observed between the characteristic equations occurring in the extension and bending of cracked plates composed of dissimilar materials.

A NUMBER of problems involving "two dissimilar media" has appeared in recent publications. Among those of practical interest are the cases of inserts of various shapes in infinite plates [1, 2, 3],[1] two joined half-planes [2, 4], and cracks along the bonded surfaces of half-planes of different materials [5, 6]. The main concern of this investigation will be to determine the stresses associated with "crack-like" imperfections between the surfaces of two joined dissimilar materials owing to bending loads.

One of the early investigations in this field was by Williams [5], who used an eigenfunction approach to determine the singular character of the extensional stress near the tip of a crack at the interface between two materials. He found that the stresses have an oscillatory character with a maximum modulus determined by $r^{-1/2}$, where r is the distance from the crack tip. This behavior was later verified by Erdogan [6] using a complex variable method similar to Muskhelishvili's [1], as formulated by Sherman [7]. Zak and Williams [8] also have investigated the extensional stress field around the root of a crack, which is orientated perpendicularly to the dividing line between two dissimilar media. However, their results are confined to in-plane loads.

More specifically, the problem considered here is that of the bending of two elastic plates bonded along a straight line with a through-the-thickness crack in the line of bonding. For purposes of examining the bending solution in a region near the crack tip, an origin of coordinates is placed at the tip with the positive x-axis lying along the bimaterial interface and the negative x-axis lying along the crack. The elastic constants describing the material in the upper half-plane are different from those in the lower half-plane and accordingly all quantities referred to the regions $y > 0$ and $y < 0$ will be designated by subscripts 1 and 2, respectively. On the basis of the Poisson-Kirchhoff theory of thin plates, the boundary conditions for this problem can be conveniently stated in terms of polar coordinates, r and θ, Fig. 1, as follows:

(a) The crack surfaces at $\theta = \pm\pi$ are assumed to satisfy the free-edge conditions of Kirchhoff given by

$$[(M_\theta)_1]_{\theta=\pi} = 0, \qquad [(M_\theta)_2]_{\theta=-\pi} = 0$$
$$[(V_\theta)_1]_{\theta=\pi} = 0, \qquad [(V_\theta)_2]_{\theta=-\pi} = 0 \tag{1}$$

(b) The bending moments and Kirchhoff shears are continuous across the uncracked portion of interface; i.e.,

$$[(M_\theta)_1]_{\theta=0} = [(M_\theta)_2]_{\theta=0}$$
$$[(V_\theta)_1]_{\theta=0} = [(V_\theta)_2]_{\theta=0} \tag{2}$$

(c) Continuity of the deflection and slope at $\theta = 0$ requires that

$$[w_1]_{\theta=0} = [w_2]_{\theta=0}$$
$$\left[\frac{1}{r}\frac{\partial w_1}{\partial\theta}\right]_{\theta=0} = \left[\frac{1}{r}\frac{\partial w_2}{\partial\theta}\right]_{\theta=0} \tag{3}$$

Unlike the homogeneous case, where the elastic properties are the same throughout the plate, the bimaterial case requires the determination of two deflection functions, w_1 and w_2, each of which must satisfy

$$D_j\nabla^4 w_j(r, \theta) = q(r, \theta), \qquad j = 1, 2 \tag{4}$$

which is the classical fourth-order differential equation governing the deflection of a plate in bending.

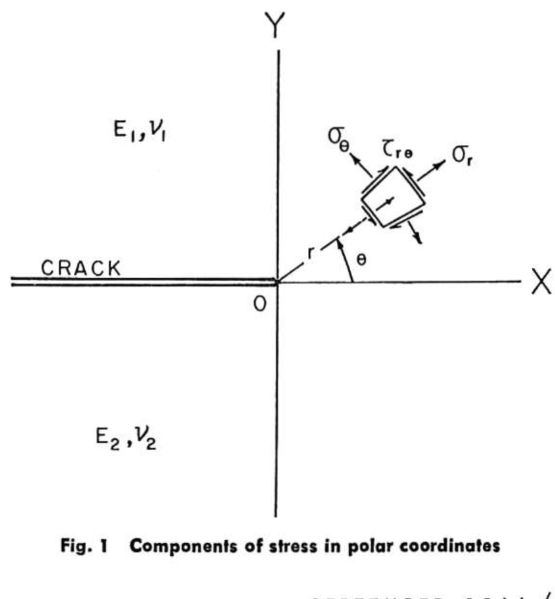

Fig. 1 Components of stress in polar coordinates

[1] Numbers in brackets designate References at end of paper.

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Method of Solution

For local investigations near the crack tip, it is sufficient to consider only the homogeneous solution of equation (4). According to Williams [9], an appropriate characteristic solution of this homogeneous equation may be represented by

$$w_j = \sum_{n=1}^{\infty} r^{\lambda_n+1} F_j(\theta;\ \lambda_n), \qquad j = 1, 2 \tag{5}$$

where

$$F_j(\theta;\ \lambda_n) = a_j^{(n)} \sin(\lambda_n + 1)\theta + b_j^{(n)} \cos(\lambda_n + 1)\theta + c_j^{(n)} \sin(\lambda_n - 1)\theta + d_j^{(n)} \cos(\lambda_n - 1)\theta$$

Similar to that of the bimaterial case in plane extension [5], $j = 1, 2$, the eight constants $a_j^{(n)}$, $b_j^{(n)}$, and so on, may be evaluated from the boundary conditions stated by equations (1), (2), and (3). To facilitate calculation, the bending moments per unit length M_r, M_θ, twisting moment per unit length $M_{r\theta}$, and the Kirchhoff shearing forces per unit length V_r, V_θ (see [10] for notation) may be expressed in terms of the eigenfunction, $F_j(\theta;\ \lambda_n)$, as follows:

$$(M_r)_j = -D_j\left[\nu_j \nabla^2 w_j + (1 - \nu_j)\frac{\partial^2 w_j}{\partial r^2}\right] = -D_j \sum_{n=1}^{\infty} r^{\lambda_n - 1}[(\lambda_n + 1)(\lambda_n + \nu_j)F_j + \nu_j F_j'']$$

$$(M_\theta)_j = -D_j\left[\nabla^2 w_j - (1 - \nu_j)\frac{\partial^2 w_j}{\partial r^2}\right] = -D_j \sum_{n=1}^{\infty} r^{\lambda_n - 1}[(\lambda_n + 1)(\nu_j\lambda_n + 1)F_j + F_j'']$$

$$(M_{r\theta})_j = (1 - \nu_j)D_j\left[\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)\right]w_j = (1 - \nu_j)D_j \sum_{n=1}^{\infty} \lambda_n r^{\lambda_n - 1} F_j' \tag{6}$$

$$(V_r)_j = (Q_r)_j - \frac{1}{r}\frac{\partial}{\partial\theta}(M_{r\theta})_j = -\left[D_j \frac{\partial}{\partial r}(\nabla^2 w_j) + \frac{1}{r}\frac{\partial}{\partial\theta}(M_{r\theta})_j\right]$$

$$= -D_j \sum_{n=1}^{\infty} r^{\lambda_n - 2}\{(\lambda_n^2 - 1)(\lambda_n + 1)F_j + [\lambda_n(2 - \nu_j) - 1]F_j''\}$$

$$(V_\theta)_j = (Q_\theta)_j - \frac{\partial}{\partial r}(M_{r\theta})_j = -\left[D_j \frac{1}{r}\frac{\partial}{\partial\theta}(\nabla^2 w_j) + \frac{\partial}{\partial r}(M_{r\theta})_j\right]$$

$$= -D_j \sum_{n=1}^{\infty} r^{\lambda_n - 2}\{[(\lambda_n + 1)^2 + \lambda_n(\lambda_n - 1)(1 - \nu_j)]F_j' + F_j'''\}$$

where $D_j = E_j h_j^3/12(1 - \nu_j^2)$ is the flexural rigidity of the plate; E_j, ν_j, and h_j are, respectively, Young's modulus, Poisson's ratio, and the thickness of the plate. Primes denote differentiation with respect to θ. The eigenparameters, λ_n, in both regions are taken to be the same so that the boundary conditions in equations (2) and (3) will hold for arbitrary values of r. Inserting w_j, $(M_\theta)_j$, and $(V_\theta)_j$ as determined by equations (5) and (6) into equations (1), (2), and (3), the boundary conditions become

$$\begin{aligned}
&(\lambda_n + 1)(\nu_1\lambda_n + 1)F_1(\pi) + F_1''(\pi) = 0\\
&(\lambda_n + 1)(\nu_2\lambda_n + 1)F_2(-\pi) + F_2''(-\pi) = 0\\
&[(\lambda_n + 1)^2 + \lambda_n(\lambda_n - 1)(1 - \nu_1)]F_1'(\pi) + F_1'''(\pi) = 0\\
&[(\lambda_n + 1)^2 + \lambda_n(\lambda_n - 1)(1 - \nu_2)]F_2'(-\pi) + F_2'''(-\pi) = 0\\
&D_1[(\lambda_n + 1)(\nu_1\lambda_n + 1)F_1(0) + F_1''(0)] = D_2[(\lambda_n + 1)(\nu_2\lambda_n + 1)F_2(0) + F_2''(0)]\\
&D_1\{[(\lambda_n + 1)^2 + \lambda_n(\lambda_n - 1)(1 - \nu_1)]F_1'(0) + F_1'''(0)\}\\
&\quad = D_2\{[(\lambda_n + 1)^2 + \lambda_n(\lambda_n - 1)(1 - \nu_2)]F_2'(0) + F_2'''(0)\}\\
&F_1(0) = F_2(0)\\
&F_1'(0) = F_2'(0)
\end{aligned} \tag{7}$$

When the appropriate values of F_1 and F_2 are substituted into equations (7), there results in eight equations in the eight unknown constants $a_j^{(n)}$, $b_j^{(n)}$, $(j = 1, 2)$, and so on

$$\begin{aligned}
&a_1^{(n)}(\lambda_n + 1)\sin\pi\lambda_n + b_1^{(n)}(\lambda_n + 1)\cos\pi\lambda_n + c_1^{(n)}(\lambda_n - \mu_1)\sin\pi\lambda_n + d_1^{(n)}(\lambda_n - \mu_1)\cos\pi\lambda_n = 0\\
&-a_2^{(n)}(\lambda_n + 1)\sin\pi\lambda_n + b_2^{(n)}(\lambda_n + 1)\cos\pi\lambda_n - c_2^{(n)}(\lambda_n - \mu_2)\sin\pi\lambda_n + d_2^{(n)}(\lambda_n - \mu_2)\cos\pi\lambda_n = 0\\
&-a_1^{(n)}(\lambda_n + 1)\cos\pi\lambda_n + b_1^{(n)}(\lambda_n + 1)\sin\pi\lambda_n - c_1^{(n)}(\lambda_n + \mu_1)\cos\pi\lambda_n + d_1^{(n)}(\lambda_n + \mu_1)\sin\pi\lambda_n = 0\\
&a_2^{(n)}(\lambda_n + 1)\cos\pi\lambda_n + b_2^{(n)}(\lambda_n + 1)\sin\pi\lambda_n + c_2^{(n)}(\lambda_n + \mu_2)\cos\pi\lambda_n + d_2^{(n)}(\lambda_n + \mu_2)\sin\pi\lambda_n = 0\\
&b_1^{(n)}\gamma(\lambda_n + 1) - b_2^{(n)}(\lambda_n + 1) + d_1^{(n)}\gamma(\lambda_n - \mu_1) - d_2^{(n)}(\lambda_n - \mu_2) = 0\\
&a_1^{(n)}\gamma(\lambda_n + 1) + c_1^{(n)}\gamma(\lambda_n + \mu_1) - a_2^{(n)}(\lambda_n + 1) - c_2^{(n)}(\lambda_n + \mu_2) = 0\\
&b_1^{(n)} + d_1^{(n)} - b_2^{(n)} - d_2^{(n)} = 0\\
&a_1^{(n)}(\lambda_n + 1) + c_1^{(n)}(\lambda_n - 1) - a_2^{(n)}(\lambda_n + 1) - c_2^{(n)}(\lambda_n - 1) = 0
\end{aligned} \tag{8}$$

where the following contractions have been made

$$\gamma = \frac{D_1}{D_2}\left[\frac{1 - \nu_1}{1 - \nu_2}\right], \qquad \mu_j = \frac{3 + \nu_j}{1 - \nu_j} \tag{9}$$

For a nontrivial solution of the eight homogeneous linear equations, the determinant of the coefficients of the constants must vanish, which leads to a characteristic value equation of the form

$$\cot^2\pi\lambda_n + \left[\frac{\gamma\mu_1(\mu_2 - 1) - \mu_2(\mu_1 - 1)}{\mu_2(\mu_1 + 1) + \gamma\mu_1(\mu_2 + 1)}\right]^2 = 0 \tag{10}$$

In contrast to the homogeneous case, the eigenvalues are observed to be complex. Moreover, there are two admissible sets of complex eigenvalues, which are possible solutions. After some algebraic manipulation, they are found to be

$$\lambda_n = (n - \tfrac{1}{2}) + i\kappa, \qquad \bar{\lambda}_n = (n - \tfrac{1}{2}) - i\kappa, \qquad n = 1, 2, \ldots \tag{11}$$

where κ will be defined as a "bielastic constant" of the form

$$\kappa = -\frac{1}{2\pi}\log\left[\frac{\mu_1}{\mu_2}\left(\frac{\mu_2 + \gamma}{\gamma\mu_1 + 1}\right)\right] \tag{12}$$

Note that n must be restricted to positive integers only, so that the slopes in both regions are finite as $r \rightarrow 0$, the crack tip.

In the particular case where both regions have the same elastic properties and plate thicknesses, i.e., $\mu_1 = \mu_2$, $\gamma = 1$, then $\kappa = 0$ and equation (11) reduces to a single set of real eigenvalues

$$\lambda_n = \bar{\lambda}_n = n - \tfrac{1}{2}, \qquad n = 1, 2, \ldots$$

The minimum eigenvalue, $\lambda_{\min} = \frac{1}{2}$, in this case will give unbounded stresses near the crack tip, i.e., $\sigma \sim r^{-1/2}$, which corresponds to the homogeneous solution obtained by Williams [8]. The occurrence of an imaginary part of the eigenvalue may therefore qualitatively affect the singular behavior of the stresses around the crack front. However, this behavior will be discussed subsequently.

Now, turning to some specific applications of the foregoing results, it is possible to have two plates of identical thickness (e.g., $\nu_1 = \nu_2 = 0.3$ and $h_1 = h_2$) but of different rigidities, say $E_1/E_2 = 3$. From equation (12), $\kappa = -0.236$. In this case, κ may be considered as a measure of the rigidity effects. If $E_1/E_2 = 0$, i.e., an elastic-to-rigid connection, then $\gamma = 0$ and

$$\kappa = -\frac{1}{2\pi}\log\left[\frac{3+\nu_1}{1-\nu_1}\right]$$

Since $0 < \nu_1 < 1$, κ again appears as a negative constant. Subsequently, the ratio of the elastic constants will always be chosen in such a way that κ is negative definite.

Deflection in Polar Coordinates

The transverse deflection, w_j, may be obtained from the sum of two solutions, each of which refers to one of the complex eigenvalues in equation (11), i.e.,

$$w_j = w_j^{(1)}(r, \theta;\ \lambda_n) + w_j^{(2)}(r, \theta;\ \bar{\lambda}_n) \tag{13}$$

where $\bar{\lambda}_n$ is the complex conjugate of λ_n. In addition, since w_j must be real, using equation (5) the solution may be given in the form

$$\begin{aligned} w_j &= \sum_{n=1}^{\infty} [r^{\lambda_n+1}F_j(\theta;\ \lambda_n) + r^{\bar{\lambda}_n+1}F_j(\theta;\ \bar{\lambda}_n)] \\ &= 2\,\mathrm{Re}\left[\sum_{n=1}^{\infty} r^{\lambda_n+1}F_j(\theta;\ \lambda_n)\right] \end{aligned} \tag{14}$$

Before an explicit expression of w_j can be obtained, it is necessary to determine the constants $a_j^{(n)}$, $b_j^{(n)}$, and so on, by solving equation (8) simultaneously. As a matter of convenience, the results are expressed in terms of $d_2^{(n)}$ as follows:

$$\begin{aligned} a_1^{(n)} &= \frac{i}{\gamma}\left(\frac{\mu_2}{\mu_1}\right)\left[\frac{\lambda_n e^{-2\pi\kappa} - \mu_1}{\lambda_n + 1}\right] d_2^{(n)} \\ b_1^{(n)} &= -\frac{1}{\gamma}\left(\frac{\mu_2}{\mu_1}\right)\left[\frac{\lambda_n e^{-2\pi\kappa} + \mu_1}{\lambda_n + 1}\right] d_2^{(n)} \\ c_1^{(n)} &= -\frac{i}{\gamma}\left(\frac{\mu_2}{\mu_1}\right) e^{-2\pi\kappa} d_2^{(n)} \\ d_1^{(n)} &= \frac{1}{\gamma}\left(\frac{\mu_2}{\mu_1}\right) e^{-2\pi\kappa} d_2^{(n)} \\ a_2^{(n)} &= i\left(\frac{\lambda_n - \mu_2 e^{-2\pi\kappa}}{\lambda_n + 1}\right) d_2^{(n)} \\ b_2^{(n)} &= -\left(\frac{\lambda_n + \mu_2 e^{-2\pi\kappa}}{\lambda_n + 1}\right) d_2^{(n)} \\ c_2^{(n)} &= -i d_2^{(n)} \end{aligned} \tag{15}$$

where $a_1^{(n)}, a_2^{(n)}, \ldots, d_1^{(n)}, d_2^{(n)}$ are all complex constants. Since a knowledge of the stress field near the crack tip is the only requirement for predicting unstable crack extension, it suffices to consider only the minimum eigenvalue; i.e., $n = 1$. Furthermore, the boundary conditions for this problem are independent of r. Therefore, the singular behavior of the stresses must be the same for both materials. As a consequence, it is not necessary to derive both w_1 and w_2, but say, w_1. Accordingly,

$$\begin{aligned} w_1 = 2e^{\kappa\theta}r^{3/2}\Bigg\{ & A_1^{(1)}\Bigg[\cos\left(\frac{\theta}{2} + \kappa\log r\right) \\ & - \frac{3+\nu_1}{1-\nu_1}e^{2\kappa(\pi-\theta)}\cos\left(\frac{3\theta}{2} + \kappa\log r\right) + \sin\theta\sin\left(\frac{\theta}{2} - \kappa\log r\right) \\ & - 2\kappa\sin\theta\cos\left(\frac{\theta}{2} - \kappa\log r\right)\Bigg] + A_2^{(1)}\Bigg[-\sin\left(\frac{\theta}{2} + \kappa\log r\right) \\ & + \frac{3+\nu_1}{1-\nu_1}e^{2\kappa(\pi-\theta)}\sin\left(\frac{3\theta}{2} + \kappa\log r\right) - 2\kappa\sin\theta\sin \\ & \times\left(\frac{\theta}{2} - \kappa\log r\right) - \sin\theta\cos\left(\frac{\theta}{2} - \kappa\log r\right)\Bigg]\Bigg\} + \ldots \end{aligned} \tag{16}$$

where higher order terms in r, the radial distance from the crack front, have been neglected. In equation (16), $A_1^{(1)}$ and $A_2^{(1)}$ are the real and imaginary parts of the complex constants $A^{(1)}$ defined by

$$A^{(1)} = A_1^{(1)} + iA_2^{(1)} = \frac{1}{\gamma}\left(\frac{\mu_2}{\mu_1}\right)\left(\frac{e^{-2\pi\kappa}}{\lambda_1 + 1}\right) d_2^{(1)} \tag{17}$$

In the homogeneous case, $\kappa = 0$, and upon some rearrangement equation (16) reduces to Williams' solution[2]

$$\begin{aligned} w_1 = r^{3/2}\Bigg\{ & \left[3\cos\frac{\theta}{2} - \left(\frac{7+\nu}{1-\nu}\right)\cos\frac{3\theta}{2}\right] A_1^{(1)} \\ & + \left[\left(\frac{5+3\nu}{1-\nu}\right)\sin\frac{3\theta}{2} - 3\sin\frac{\theta}{2}\right] A_2^{(1)}\Bigg\} + \ldots \end{aligned} \tag{18}$$

when

$$A_1^{(1)} = \frac{1-\nu}{7+\nu}\, b_1^{(1)}, \qquad A_2^{(1)} = \frac{1-\nu}{5+3\nu}\, b_2^{(1)} \tag{19}$$

in which $b_1^{(1)}$ and $b_2^{(1)}$ are the constants used in [8].

In the usual manner, equation (16) may be used to derive the significant stress field around the tip of a crack. According to the classical theory of thin plates, the bending stresses are distributed linearly through the thickness of the plate, i.e.,

$$\begin{bmatrix} (\sigma_r)_j \\ (\sigma_\theta)_j \\ (\sigma_{r\theta})_j \end{bmatrix} = \frac{12\delta}{h_j^3}\begin{bmatrix} (M_r)_j \\ (M_\theta)_j \\ (M_{r\theta})_j \end{bmatrix} \tag{20}$$

where δ is the thickness coordinate measured from the middle plane of the plate. The transverse shear stresses, which satisfy the conditions that $\tau_{rz} = \tau_{\theta z} = 0$ for $\delta = \pm(h/2)$, can be obtained from the equations of equilibrium and are

$$\begin{bmatrix} (\tau_{rz})_j \\ (\tau_{\theta z})_j \end{bmatrix} = \frac{3(h_j^2 - 4\delta^2)}{2h_j^3}\begin{bmatrix} (Q_r)_j \\ (Q_\theta)_j \end{bmatrix} \tag{21}$$

Stress Distribution Near Crack Tip

The determination of the distribution of stress in the vicinity of a crack plays an important role in the "Griffith-Irwin" theory of fracture, since it may be used subsequently to analyze the stability of a crack. This is more easily accomplished from the Goursat functions [1] than from the transverse-deflection expression (16).

Since w_1 is biharmonic, it may be represented by two complex functions $\phi_1(z)$ and $\chi_1(z)$ of the variable, $z = x + iy$, i.e.,

$$w_1 = \mathrm{Re}[\bar{z}\phi_1(z) + \chi_1(z)] \tag{22}$$

By means of equations (11) and (14), the Goursat functions for

[2] See equation (8) in reference [8].

this problem may be constructed and written in the form[3]

$$\phi_1(z) = 2z^{-\frac{1}{2}-i\kappa} \sum_{n=1}^{\infty} [(n + \tfrac{1}{2}) - i\kappa]\bar{A}^{(n)}z^n$$

$$\chi_1(z) = -2\mu_1 e^{2\pi\kappa} z^{\frac{1}{2}+i\kappa} \sum_{n=1}^{\infty} A^{(n)}z^n - 2z^{\frac{1}{2}-i\kappa} \sum_{n=1}^{\infty} [(n - \tfrac{1}{2}) - i\kappa]\bar{A}^{(n)}z^n \qquad (23)$$

The relationships for determining the moments and shears from the functions $\phi_1(z)$ and $\chi_1(z)$ in classical plate theory may be rewritten, using equation (22), to read

$$(M_r)_1 + (M_\theta)_1 = -4D_1(1 + \nu_1)\,\mathrm{Re}[\phi_1'(z)]$$

$$(M_\theta)_1 - (M_r)_1 + 2i(H_{r\theta})_1 = 2D_1(1 - \nu_1)e^{2i\theta}[\bar{z}\phi_1''(z) + \chi_1''(z)] \qquad (24)$$

$$(Q_r)_1 - i(Q_\theta)_1 = -4D_1e^{i\theta}\phi_1''(z)$$

where $(H_{r\theta})_j = -(M_{r\theta})_j$. The local bending-stress components corresponding to $n = 1$ are found upon combining equations (20), (23), and (24). The results are

$$\frac{(\sigma_r)_1}{G_1} = \frac{2\delta K_1}{(1 - \nu_1)r^{1/2}} \left\langle e^{-\kappa(\pi-\theta)}\left\{-(2 + 3\nu_1)\cos\left(\frac{\theta}{2} + \kappa\log r\right) + \nu_1 \sin\theta \sin\left(\frac{\theta}{2} - \kappa\log r\right) + [\cos\theta + 2\kappa(1 - \nu_1)\sin\theta] \times \cos\left(\frac{\theta}{2} - \kappa\log r\right)\right\} + (3 + \nu_1)e^{\kappa(\pi-\theta)}\cos\left(\frac{3\theta}{2} + \kappa\log r\right)\right\rangle$$
$$+ \frac{2\delta K_2}{(1 - \nu_1)r^{1/2}} \left\langle e^{-\kappa(\pi-\theta)}\left\{-(2 + 3\nu_1)\sin\left(\frac{\theta}{2} + \kappa\log r\right) + \nu_1\sin\theta\cos\left(\frac{\theta}{2} - \kappa\log r\right) - [\cos\theta + 2\kappa(1 - \nu_1)\sin\theta]\sin\left(\frac{\theta}{2} - \kappa\log r\right)\right\} + (3 + \nu_1)e^{\kappa(\pi-\theta)}\sin\left(\frac{3\theta}{2} + \kappa\log r\right)\right\rangle + \ldots \qquad (25)$$

$$-\frac{(\sigma_\theta)_1}{G_1} = \frac{2\delta K_1}{(1 - \nu_1)r^{1/2}} \left\langle e^{-\kappa(\pi-\theta)}\left\{(2 + \nu_1)\cos\left(\frac{\theta}{2} + \kappa\log r\right) + \nu_1\sin\theta\sin\left(\frac{\theta}{2} - \kappa\log r\right) + [\cos\theta + 2\kappa(1 - \nu_1)\sin\theta] \times \cos\left(\frac{\theta}{2} - \kappa\log r\right)\right\} + (3 + \nu_1)e^{\kappa(\pi-\theta)}\cos\left(\frac{3\theta}{2} + \kappa\log r\right)\right\rangle$$
$$+ \frac{2\delta K_2}{(1 - \nu_1)r^{1/2}} \left\langle e^{-\kappa(\pi-\theta)}\left\{(2 + \nu_1)\sin\left(\frac{\theta}{2} + \kappa\log r\right) + \nu_1\sin\theta\cos\left(\frac{\theta}{2} - \kappa\log r\right) - [\cos\theta + 2\kappa(1 - \nu_1)\sin\theta]\sin\left(\frac{\theta}{2} - \kappa\log r\right)\right\} + (3 + \nu_1)e^{\kappa(\pi-\theta)}\sin\left(\frac{3\theta}{2} + \kappa\log r\right)\right\rangle + \ldots \qquad (26)$$

$$\frac{(\tau_{r\theta})_1}{G_1} = \frac{2\delta K_1}{(1 - \nu_1)r^{1/2}} \left\langle e^{-\kappa(\pi-\theta)}\left\{\nu_1\sin\left(\frac{\theta}{2} + \kappa\log r\right) - \nu_1\sin\theta \times \cos\left(\frac{\theta}{2} - \kappa\log r\right)\right\} + [\cos\theta + 2\kappa(1 - \nu_1)\sin\theta] \times \sin\left(\frac{\theta}{2} - \kappa\log r\right)\right\} + (3 + \nu_1)e^{\kappa(\pi-\theta)}\sin\left(\frac{3\theta}{2} + \kappa\log r\right)\right\rangle$$
$$+ \frac{2\delta K_2}{(1 - \nu_1)r^{1/2}} \left\langle e^{-\kappa(\pi-\theta)}\left\{-\nu_1\sin\left(\frac{\theta}{2} + \kappa\log r\right) + \nu_1\sin\theta\sin\left(\frac{\theta}{2} - \kappa\log r\right) + [\cos\theta + 2\kappa(1 - \nu_1)\sin\theta]\cos\left(\frac{\theta}{2} - \kappa\log r\right)\right\} - (3 + \nu_1)e^{\kappa(\pi-\theta)}\cos\left(\frac{3\theta}{2} + \kappa\log r\right)\right\rangle + \ldots \qquad (27)$$

[3] The complex constant $A^{(n)}$ in equation (23) may be determined in the usual way [1] from the prescribed boundary conditions of a particular problem.

where

$$K_1 + iK_2 = 2e^{\kappa\pi}(\tfrac{1}{2} - i\kappa)(\tfrac{3}{2} - i\kappa)(A_1^{(1)} - iA_2^{(2)}) \qquad (28)$$

In equations (25)-(27), $G_1 = E_1/2(1 + \nu_1)$ is the shear modulus of elasticity for the material in region $y > 0$. As is customary in plate-bending theory, the stress, $(\sigma_z)_j$, is assumed to be small compared to the other stress components and it is neglected in the stress-strain relations. Making use of equations (19) and (28) and setting $\kappa = 0$ for the homogeneous case, the parts multiplied by the constant K_1 in equations (25), (26), and (27) reduce, respectively, to Williams' solution [9], equations (19), (20), and (21) for symmetrical bending with respect to the crack line. In a similar fashion, the parts containing the constant K_2 in equations (25), (26), and (27) reduce, respectively, to equations (26), (27), and (28) in [9] for the skew-symmetrical local stress distributions.

The transverse shear stresses, associated with the condition of vanishing tractions parallel to the surfaces of the plate, are obtained from equations (21), (23), and the remaining expression in equation (24). They are given by

$$\frac{(\tau_{rz})_1}{G_1} = \frac{e^{-\kappa(\pi-\theta)}(h_1^2 - 4\delta^2)}{2(1 - \nu_1)r^{3/2}}\left[K_1\cos\left(\frac{\theta}{2} + \kappa\log r\right) + K_2\sin\left(\frac{\theta}{2} + \kappa\log r\right)\right] + \ldots \qquad (29)$$

$$\frac{(\tau_{\theta z})_1}{G_1} = \frac{e^{-\kappa(\pi-\theta)}(h_1^2 - 4\delta^2)}{2(1 - \nu_1)r^{3/2}}\left[K_1\sin\left(\frac{\theta}{2} + \kappa\log r\right) - K_2\cos\left(\frac{\theta}{2} + \kappa\log r\right)\right] + \ldots \qquad (30)$$

Differing from the in-plane bending and shear stresses shown in equations (25)-(27), equations (29) and (30) suggest a stronger stress singularity, of the order of $r^{-3/2}$, for the transverse shear components. However, this point will be reserved for further discussions with reference to results based on the Reissner plate theory for the homogeneous case in the section to follow.

Discussion of Results

In order to study the local behavior of the bending stresses, it is convenient to rearrange them into the form

$$\sigma \sim r^{-1/2}[J(\theta)\sin(\kappa\log r) + H(\theta)\cos(\kappa\log r)] \qquad (31)$$

where J and H are functions of θ only. It is evident from equation (31) that the stresses will undergo a rapid reversal of sign as the origin is approached; i.e., $r \to 0$. This highly oscillatory character of the stress will be shown to be confined to a very small region surrounding the end of the crack. For the purpose of illustrating this fact, consider the extreme case of elastic-to-rigid connection, i.e., $E_2 \to \infty$, and assume $\nu_1 = 0.3$. Using equation (12), the bielastic constant, κ, is -0.244. For the sake of definiteness, let the radial distance, r, be compared to some planar dimension, c, of the initially flat plate in the form of the ratio, r/c. For instance, in the case of a finite crack in an infinite plate loaded uniformly at infinity, c would be the crack length.

In the case of a semi-infinite crack in an infinite plate loaded with a concentrated couple on the crack surface, c would be the distance of the load, say from the crack point. Since κ is negative, the bending stresses in equation (31) will remain unchanged in sign if the range, $0 < \kappa \log (r/c) < \pi/2$, is observed. Thus, for values of r/c smaller than $\exp(\pi/2\kappa) = 1.58 \times 10^{-3}$, the bending stresses will begin to oscillate between positive and negative values. Now, bear in mind that equation (31) is only valid for those stresses in a small region surrounding the crack tip; i.e., for values of the ratio r/c small relative to unity. As $r/c = 1.58 \times 10^{-3}$ is well within that order of magnitude, the region in which rapid oscillation occurs is indeed in close proximity to the end of the crack. Owing to these conditions, the stresses in the immediate neighborhood of the crack front are seen to be of an oscillating nature with a singularity strength determined by $r^{-1/2}$. In the same way, the transverse shear forces, Q_r and Q_θ, can also be shown to oscillate and become infinite in the order of $r^{-3/2}$, as $r \rightarrow 0$. This singular behavior of Q_r and Q_θ is true only in the Kirchhoff sense (i.e., the three free-edge conditions prescribing M_θ, $M_{r\theta}$, and Q_θ have been contracted into two conditions).

In contrast to this result, Knowles and Wang [11] have used a more refined theory due to Reissner for the homogeneous case, where all three conditions are satisfied individually, and found that the shear forces, Q_r and Q_θ, are actually finite at the tip of a crack, while the inverse square-root of r characteristic for the inplane stresses is the same as that of the Kirchhoff small-deflection theory. Consequently, it is possible to define "bending stress-intensity factors" for each of the two theories, i.e., the classical and Reissner, in such a way that the results are identical. In fact, for the symmetric and homogeneous case, Williams [12] has already pointed out the difference to be a factor of $(1 + \nu)/(3 + \nu)$. Therefore, the small-deflection theory does preserve the character of the bending-stress radial decay around the crack tip. This alone is sufficient for the purpose of merely establishing a fracture criterion.

Referring to the general forms of equations (25), (26), and (27), the distribution of the bending stresses is seen to have the same functional form in r and θ near the singular point of a crack between two dissimilar materials. They will, however, differ quantitatively from one problem to the next through the constants K_1 and K_2, which are dependent upon the loads and a characteristic length, such as crack length or length of the bond line.

According to the "Griffith-Irwin" theory of fracture, K_1 and K_2 may be considered as stress-intensity factors that cause unstable crack extension upon reaching some critical values or combinations. In the usual manner, they may be evaluated from the Goursat function [13] as follows:

$$K_1 + iK_2 = e^{\kappa\pi} \lim_{z\rightarrow 0} z^{1/2+i\kappa}\phi_1'(z) \qquad (32)$$

Finally, it is interesting to observe an analogy between Williams' solution[4] for the extensional case

$$\cot^2 \pi\lambda_n + \frac{1}{4}\left[\frac{\frac{E_1}{E_2}(1-\nu_2) - (1-\nu_1)}{1 + \frac{E_1}{E_2}}\right]^2 = 0 \qquad (33)$$

and equation (10) for the bending case

$$\cot^2 \pi\lambda_n + \frac{1}{4}\left[\frac{\frac{E_1}{E_2}\left(\frac{1+\nu_2}{1+\nu_1}\right)\left(\frac{3+\nu_1}{3+\nu_2}\right)(1+\nu_2) - (1+\nu_1)}{1 + \frac{E_1}{E_2}\left(\frac{1+\nu_2}{1+\nu_1}\right)\left(\frac{3+\nu_1}{3+\nu_2}\right)}\right]^2 = 0 \qquad (34)$$

[4] Equation (16) in reference [5].

Both equations (33) and (34) have been rearranged to facilitate comparison. Since both problems are geometrically identical, the characteristic equations can also be made into a single generalization upon observing that

$$E_j \text{ corresponds with } \quad \text{const}\left(\frac{3+\nu_j}{1+\nu_j}\right)E_j \qquad (35)$$

$$\nu_j \text{ corresponds with } \quad -\nu_j \qquad (36)$$

Though the last result was already observed by Southwell [14], to the best of the authors' knowledge, the correspondence of Young's modulus (determined within a constant) has not appeared elsewhere.

Note on the Extensional Solution

In [5], Williams has shown that an oscillatory type of stress singularity exists at the tip of a crack along the interface between two dissimilar materials, which is subject to in-plane loads. With the aid of Kolosov-Muskhelishvili's stress combinations, his work may be extended to derive the crack-tip stress field. A short derivation is included in the Appendix.

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APPENDIX

Derivation of Extensional Stresses Near a Crack Tip

Consider the extension of a bimaterial plate with a crack along the bond line as shown in Fig. 1. The Airy stress function for this problem is given by [5]

$$U_1 = 2\,\mathrm{Re} \sum_{n=1}^{\infty} r^{\lambda_n+1}\,[a_1^{(n)} \sin(\lambda_n + 1)\theta + b_1^{(n)} \cos(\lambda_n + 1)\theta + c_1^{(n)} \sin(\lambda_n - 1)\theta + d_1^{(n)} \cos(\lambda_n - 1)\theta] \quad (37)$$

where

$$\lambda_n = (n - \tfrac{1}{2}) + i\epsilon$$

and

$$\begin{aligned} a_1^{(n)} &= i\left(\frac{1 + \lambda_n e^{-2\pi\epsilon}}{\lambda_n + 1}\right) d_2^{(n)} \\ b_1^{(n)} &= \left(\frac{1 - \lambda_n e^{-2\pi\epsilon}}{\lambda_n + 1}\right) d_2^{(n)} \\ c_1^{(n)} &= -ie^{-2\pi\epsilon}\, d_2^{(n)} \\ d_1^{(n)} &= e^{-2\pi\epsilon} d_2^{(n)} \end{aligned} \quad (38)$$

are all complex constants. The bielastic constant for the plane stress case is

$$\epsilon = \frac{1}{2\pi} \log \left[\frac{\left(\frac{3 - \nu_1}{1 + \nu_1}\right)\left(\frac{1}{G_1}\right) + \frac{1}{G_2}}{\left(\frac{3 - \nu_2}{1 + \nu_2}\right)\left(\frac{1}{G_2}\right) + \frac{1}{G_1}}\right] \quad (39)$$

where G_1 and G_2 are the shear moduli of elasticity for materials 1 and 2, respectively. In the case of plane strain, replace ν_j by $\nu_j/(1 - \nu_j)$, $j = 1, 2$.

Since U_1 is biharmonic, equation (1) may be expressed in terms of two stress functions of the complex variable, $z = x + iy$; i.e.,

$$U_1 = \mathrm{Re}[\bar{z}\phi_1(z) + \chi_1(z)] \quad (40)$$

It follows from equation (37) that the complex functions are

$$\phi_1(z) = 2z^{-1/2-i\epsilon} \sum_{n=1}^{\infty} [(n + \tfrac{1}{2}) - i\epsilon]\, \overline{B}^{(n)} z^n \quad (41)$$

$$\chi_1(z) = 2e^{2\pi\epsilon} z^{1/2+i\epsilon} \sum_{n=1}^{\infty} B^{(n)} z^n - 2z^{1/2-i\epsilon} \sum_{n=1}^{\infty} [(n - \tfrac{1}{2}) - i\epsilon]\overline{B}^{(n)} z^n$$

in which

$$(\lambda_n + 1)B^{(n)} = e^{-2\pi\epsilon} d_2^{(n)}$$

Using the Kolosov-Muskhelishvili formulas

$$\begin{aligned} (\sigma_r)_1 + (\sigma_\theta)_1 &= 4\,\mathrm{Re}[\phi_1'(z)] \\ (\sigma_\theta)_1 - (\sigma_r)_1 + 2i(\tau_{r\theta})_1 &= 2e^{2i\theta}[\bar{z}\phi_1''(z) + \chi_1''(z)] \end{aligned} \quad (42)$$

the stresses near the crack tip are obtained by setting $n = 1$.

$$\begin{aligned} (\sigma_r)_1 = \frac{k_1}{2(2r)^{1/2}} \Big\{ & e^{-\epsilon(\pi-\theta)}\Big[3\cos\left(\frac{\theta}{2} + \epsilon\log r\right) + 2\epsilon\sin\theta\cos\left(\frac{\theta}{2} - \epsilon\log r\right) - \sin\theta\sin\left(\frac{\theta}{2} - \epsilon\log r\right)\Big] \\ & - e^{\epsilon(\pi-\theta)}\cos\left(\frac{3\theta}{2} + \epsilon\log r\right)\Big\} \\ - \frac{k_2}{2(2r)^{1/2}} \Big\{ & e^{-\epsilon(\pi-\theta)}\Big[3\sin\left(\frac{\theta}{2} + \epsilon\log r\right) - \sin\theta\cos\left(\frac{\theta}{2} - \epsilon\log r\right) - 2\epsilon\sin\theta\sin\left(\frac{\theta}{2} - \epsilon\log r\right)\Big] \\ & - e^{\epsilon(\pi-\theta)}\sin\left(\frac{3\theta}{2} + \epsilon\log r\right)\Big\} + \ldots \end{aligned} \quad (43)$$

$$\begin{aligned} (\sigma_\theta)_1 = \frac{k_1}{2(2r)^{1/2}} \Big\{ & e^{-\epsilon(\pi-\theta)}\Big[\cos\left(\frac{\theta}{2} + \epsilon\log r\right) - 2\epsilon\sin\theta\cos\left(\frac{\theta}{2} - \epsilon\log r\right) + \sin\theta\sin\left(\frac{\theta}{2} - \epsilon\log r\right)\Big] \\ & + e^{\epsilon(\pi-\theta)}\cos\left(\frac{3\theta}{2} + \epsilon\log r\right)\Big\} \\ - \frac{k_2}{2(2r)^{1/2}} \Big\{ & e^{-\epsilon(\pi-\theta)}\Big[\sin\left(\frac{\theta}{2} + \epsilon\log r\right) + \sin\theta\cos\left(\frac{\theta}{2} - \epsilon\log r\right) + 2\epsilon\sin\theta\sin\left(\frac{\theta}{2} - \epsilon\log r\right)\Big] \\ & + e^{\epsilon(\pi-\theta)}\sin\left(\frac{3\theta}{2} + \epsilon\log r\right)\Big\} + \ldots \end{aligned} \quad (44)$$

$$\begin{aligned} (\tau_{r\theta})_1 = \frac{k_1}{2(2r)^{1/2}} \Big\{ & e^{-\epsilon(\pi-\theta)}\Big[\sin\left(\frac{\theta}{2} + \epsilon\log r\right) - \sin\theta\cos\left(\frac{\theta}{2} - \epsilon\log r\right) - 2\epsilon\sin\theta\sin\left(\frac{\theta}{2} - \epsilon\log r\right)\Big] \\ & + e^{\epsilon(\pi-\theta)}\sin\left(\frac{3\theta}{2} + \epsilon\log r\right)\Big\} \\ - \frac{k_2}{2(2r)^{1/2}} \Big\{ & e^{-\epsilon(\pi-\theta)}\Big[-\cos\left(\frac{\theta}{2} + \epsilon\log r\right) - 2\epsilon\sin\theta\cos\left(\frac{\theta}{2} - \epsilon\log r\right) + \sin\theta\sin\left(\frac{\theta}{2} - \epsilon\log r\right)\Big] \\ & - e^{\epsilon(\pi-\theta)}\cos\left(\frac{3\theta}{2} + \epsilon\log r\right)\Big\} + \ldots \end{aligned} \quad (45)$$

where

$$k = k_1 - ik_2 = 4\sqrt{2}e^{\epsilon\pi}(\tfrac{1}{2} - i\epsilon)(\tfrac{3}{2} - i\epsilon)\overline{B}^{(1)} \quad (46)$$

When equations (43)–(45) are expressed in Cartesian coordinates, they are identical with equations (49)–(51) in [6], respectively, if $\epsilon = -\gamma$, $k_1 - ik_2 = 2\sqrt{2}\,(A_1 + iA_2)$. For $\epsilon = 0$, equations (43)–(45) reduce to the homogeneous solution in [15].

The stress-intensity factors k_1 and k_2 are of special interest in fracture mechanics, since they govern the onset of rapid crack extension. As in the immediate vicinity of the crack tip

$$\phi_1'(z) = \frac{e^{-\epsilon\pi}}{2\sqrt{2}}\,(k_1 - ik_2)z^{-1/2-i\epsilon} \quad (47)$$

they may be evaluated from the complex function $\phi_1'(z)$ alone; i.e.,

$$k_1 - ik_2 = 2\sqrt{2}\,e^{\epsilon\pi} \lim_{z\to 0} z^{1/2+i\epsilon}\,\phi_1'(z) \quad (48)$$