Closure to "Discussion of 'The Bending of Plates of Dissimilar Materials With Cracks'" (1965, ASME J. Appl. Mech., 32, p. 464)
G. C. Sih and J. R. Rice, J. Appl. Mech. **32**, 464 (1965)

The Bending of Plates of Dissimilar Materials With Cracks[1]

M. L. Williams.[2] The authors are to be commended for completing an analysis of the companion bending problem for a cracked bimaterial plate. Inasmuch as they have also found occasion to complete the extensional solution [5],[3] it is convenient to append for ready reference certain other remarks which apply to both solutions because their character is similar.

The major remark relates to a more precise statement of the admissible eigenvalues. In the extensional paper [5], it was pointed out that there were two sets of (complex) solutions corresponding to

$$\lambda_n^{(1)} = (2n+1)/2 + i\lambda_j^{(1)} \text{ and } \lambda_n^{(2)} = n + i\lambda_j^{(2)}$$

The first of these corresponds to those given by the authors in their equation (11) for the bending case. While it was stated in the extensional problem that for $\lambda_j^{(2)}$ not to equal zero it would be the root of

$$\coth \lambda_j \pi = \frac{2k(1-\sigma_2) - 2(1-\sigma_1) - (k-1)}{2k(1-\sigma_2) + 2(1-\sigma_1)}$$

$$k = \frac{\mu_1}{\mu_2},\ \sigma_1 = \frac{\nu_i}{1+\nu_i} \tag{1}$$

it can be shown that the right-hand side is always less than unity, because of physical restrictions in the material properties, and hence this second set of complex eigenvalues cannot exist. However, as may be checked in the basic set of eight homogeneous equations, $\lambda_n^{(2)} = n (n = 0, 1, 2, \ldots)$, i.e., real, does satisfy the matrix in a nontrivial fashion *for the general bimaterial case.* For example, in the extensional case, one quickly finds that

$$a_1 = -\left[\frac{\lambda-1}{\lambda+1}\, k\, \frac{1-\sigma_2}{1-\sigma_1}\right] c_2;\quad a_2 = -\left[\frac{\lambda-1}{\lambda+1}\right] c_2 \tag{2}$$

$$b_1 = -\left[k\,\frac{1-\sigma_2}{1-\sigma_1}\right] d_2;\quad b_2 = -d_2 \tag{3}$$

$$c_1 = \left[k\,\frac{1-\sigma_2}{1-\sigma_1}\right] c_2;\quad d_1 = \left[k\,\frac{1-\sigma_2}{1-\sigma_1}\right] d_2 \tag{4}$$

The point to note is that there remain two independent sets of real constants, $c_2^{(n)}$ and $d_2^{(n)}$, leading to a second set of eigensolutions for $F_n(\theta)$ [or $w_n(\theta)$ in the bending case] in the form

$$r^{\lambda_n+1}F_n(\theta) = [c_2^{(n)}f_n(\theta) + d_2^{(n)}g_n(\theta)]r^{\lambda_n+1} \tag{5}$$

which is the form of solution found for the homogeneous material case [15], and to which (v) reduces in the case of similar properties; i.e., $k = 1$, $\sigma_1 = \sigma_2$. This form of solution, resulting from a real eigenvalue, does *not* produce the same trig-log character of singularity as the complex eigenvalue. The same argument applies to the bending solution for its second, interlaced set of eigenvalues.

There is another interesting point, also common to both problems. The authors show in equation (15) of the paper that seven of the bimaterial constants can be determined in terms of the eighth, taken in their analysis as the complex constant $d_2^{(n)}$. For the homogeneous material, one deduces quickly that $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$, $d_1 = d_2$ as one would expect from requiring continuity. On the other hand, completing the calculation in (15), one finds

$$a = i\left(\frac{\lambda-\mu}{\lambda+1}\right)d;\quad b = -\left(\frac{\lambda+\mu}{\lambda+1}\right)d;\quad c = -id \tag{6}$$

whereas beginning with equation (8) of the paper and making the calculation assuming $\lambda_n^{(1)} = (2n+1)/2$, $(n = 0, 1, 2, \ldots)$, one finds only the requirement

$$a = -\left(\frac{\lambda-\mu}{\lambda+1}\right)c;\quad b = -\left(\frac{\lambda+\mu}{\lambda+1}\right)d \tag{7}$$

with no coupling between c and d. This latter solution produces the form given in (5) of this discussion which corresponds to the limit found in reference [9] of the paper. Whereas at first sight it is not apparent, the form of solution obtained using the one unknown complex loading constant, d, in (6) of this discussion and separating the linearly independent real and imaginary parts is the same as that obtained from (7) because c and d are real.

In conclusion, it should be emphasized that the character of the lowest mathematically admissible eigensolution is as reported by the authors; the amplification is related to the identification of the additional set of higher interlaced eigenfunctions frequently needed for a mathematically complete solution, and some reconsideration of the loading constants for the limiting case of a homogeneous material.

Authors' Closure

Professor Williams' comments are well applied to this paper and are greatly appreciated by the authors. The additional set of eigenvalues consisting of $\lambda_n^{(2)} = n$ $(n = 0, 1, 2, \ldots)$ is indeed necessary for mathematical completeness, particularly for the formulation of crack problems involving finite regions. It will be shown that this infinite number of real eigenvalues leads to a set of complex functions, holomorphic in the plane $z = x + iy$, with the possible exception of the point at infinity. For discussion purposes, let these functions be denoted by $\phi_j^{(2)}(z)$ and $\chi_j^{(2)}(z)$. The behavior of $\phi_j^{(2)}(z)$ and $\chi_j^{(2)}(z)$ is determined by the stresses and/or displacements prescribed on the boundary of the region under consideration. In the case of an infinite region with a uniform state of stress at infinity, $\phi_j^{(2)}(z)$ may be at most a linear function of z and $\chi_j^{(2)}(z)$ a quadratic function of z, as terms of higher order result in unbounded stresses as $|z| \to \infty$. The coefficients of z in these functions are determined so that $\phi_j(z) = \phi_j^{(1)}(z) + \phi_j^{(2)}(z)$ and $\chi_j(z) = \chi_j^{(1)}(z) + \chi_j^{(2)}(z)$ (where $\phi_j^{(1)}$, $\chi_j^{(1)}(z)$ are the Goursat functions given in the paper) satisfy the imposed uniform stress state at infinity. Now, if the infinite region contains a semi-infinite crack such as the configuration in Fig. 1 of the paper, it is necessary that $\phi_j^{(2)}(z)$ and $\chi_j^{(2)}(z)$ result at most in a uniform normal stress state acting parallel to the crack plane which gives no singularity at the crack tip or branch cut along the crack line. Otherwise, no equilibrium may result, as the problem of an infinite elastic body with a semi-infinite crack subjected to any other uniform stress state has no stress solution that is bounded at finite distances from the crack tip. In this respect, the second set of eigenvalues does not contribute to the solution of the present problem. However, the authors agree with Professor Williams that, in general, the additional eigenvalues $\lambda_n^{(2)}$ should be included in the analysis for a mathematically complete solution.

[1] By G. C. Sih and J. R. Rice, published in the September, 1964, issue of the JOURNAL OF APPLIED MECHANICS, vol. 31, TRANS. ASME, vol. 86, Series E, pp. 477–482.

[2] Professor of Aeronautics, Firestone Flight Sciences Laboratory, Graduate Aeronautical Laboratories, California Institute of Technology, Pasadena, Calif.

[3] Numbers in brackets designate References in original paper.

In order to be more specific, the Goursat functions corresponding to $\lambda_n^{(2)}$ in the case of plane extension may be obtained:

$$\begin{aligned}\phi_1^{(2)}(z) &= \sum_{n=1}^{\infty} D^{(n)} z^n \\ \chi_1^{(2)}(z) &= \sum_{n=1}^{\infty} \left[\frac{D^{(n)} - \bar{D}^{(n)}}{n+1} - D^{(n)}\right] z^{n+1}\end{aligned} \quad (1)$$

and

$$\begin{aligned}\phi_2^{(2)}(z) &= \frac{G_2}{G_1}\left(\frac{\eta_1+1}{\eta_2+1}\right) \sum_{n=1}^{\infty} D^{(n)} z^n \\ \chi_2^{(2)}(z) &= \frac{G_2}{G_1}\left(\frac{\eta_1+1}{\eta_2+1}\right) \sum_{n=1}^{\infty} \left[\frac{D^{(n)} - \bar{D}^{(n)}}{n+1} - D^{(n)}\right] z^{n+1}\end{aligned} \quad (2)$$

in which $D^{(n)}$ is a complex constant given by

$$D^{(n)} = 2\,\frac{G_1}{G_2}\left(\frac{\eta_2+1}{\eta_1+1}\right)[d_2^{(n)} - ic_2^{(n)}]$$

The subscripts 1 and 2 refer, respectively, to the materials on the upper and lower sides of the cracked bond line. In equation (2), G_j is the shear modulus, $\eta_j = (3 - \nu_j)/(1 + \nu_j)$ for plane stress and $3 - 4\nu_j$ for plane strain, and ν_j is the Poisson's ratio, where $j = 1, 2$. It should be pointed out that the Goursat functions in equations (1) and (2) correspond to a stress field free from singularities at the crack site. It may be checked that the stresses are in fact zero on the crack surface as well as along the bond line, i.e., the entire x-axis.

For a complete solution, equation (1) should be added to equation (41) in the paper. The same applies to equation (2). A complete solution to the problem of two bonded dissimilar planes with a finite crack on the dividing line is further elaborated by the authors in another paper.[3] There only the terms corresponding to $n = 1$ appear as the boundedness of the stresses, for large $|z|$ requires that $D^{(n)} = \bar{D}^{(n)} = 0$ for $n \geq 2$. The constant $D^{(1)}$ is related to the applied stresses at infinity.

In the same fashion, a second set of eigenvalues consisting of positive integers also arises in the plate bending problem of two dissimilar materials. Solving equation (8) in the paper for the constants $a_j^{(n)}$, $b_j^{(n)}$, etc., in terms of the two independent real constants $c_2^{(n)}$ and $d_2^{(n)}$, it is found that

$$a_1^{(n)} = -\left(\frac{1+\mu_2}{1+\mu_1}\right)\left(\frac{n+\mu_1}{n+1}\right)c_2^{(n)};\quad d_1^{(n)} = \left(\frac{1+\mu_2}{1+\mu_1}\right)d_2^{(n)}$$

$$b_1^{(n)} = -\left(\frac{1+\mu_2}{1+\mu_1}\right)\left(\frac{n-\mu_1}{n+1}\right)d_2^{(n)};$$

$$a_2^{(n)} = -\left(\frac{n+\mu_2}{n+1}\right)c_2^{(n)} \quad (3)$$

$$c_1^{(n)} = \left(\frac{1+\mu_2}{1+\mu_1}\right)c_2^{(n)};\quad b_2^{(n)} = -\left(\frac{n-\mu_2}{n+1}\right)d_2^{(n)}$$

where $n = 0, 1, 2, \ldots$. Making use of equation (3), the Goursat functions which should be added to equation (23) of the paper are:

$$\begin{aligned}\phi_1^{(2)}(z) &= \sum_{n=1}^{\infty} C^{(n)} z^n \\ \chi_1^{(2)}(z) &= \sum_{n=1}^{\infty} \left[\frac{C^{(n)} + \mu_1 \bar{C}^{(n)}}{n+1} - C^{(n)}\right] z^{n+1}\end{aligned} \quad (4)$$

for the region $y > 0$. For the region $y < 0$, the Goursat functions are of the form

$$\begin{aligned}\phi_2^{(2)}(z) &= \left(\frac{1+\mu_1}{1+\mu_2}\right) \sum_{n=1}^{\infty} C^{(n)} z^n \\ \chi_2^{(2)}(z) &= \left(\frac{1+\mu_1}{1+\mu_2}\right) \sum_{n=1}^{\infty} \left[\frac{C^{(n)} + \mu_2 \bar{C}^{(n)}}{n+1} - C^{(n)}\right] z^{n+1}\end{aligned} \quad (5)$$

The complex constant $C^{(n)}$ is defined as

$$C^{(n)} = 2\left(\frac{1+\mu_1}{1+\mu_2}\right)[d_2^{(n)} - ic_2^{(n)}]$$

and $\mu_j (j = 1, 2)$ is given by equation (9) of the paper. Since equations (4) and (5) are power series of the same degree as those given by equations (1) and (2), the role of this second set of eigenvalues in a complete solution for the plate bending and the plane extensional problems is basically the same.

As mentioned before, equations (4) and (5) are inconsequential to the physical problem described in the paper. To cite an example, consider the problem of a semi-infinite crack between two bonded dissimilar materials. Concentrated couples M and N are applied to the crack surface at $z = -a$ as it is shown in Fig. 1. The condition of vanishing stresses at infinity demands that $C^{(n)} = 0$ for all values of n. Hence, the complete solution to this problem is given by equation (23) of the paper. Moreover, by letting

$$f(z) = 2 \sum_{n=9}^{\infty} [(n - \tfrac{1}{2}) - i\kappa][(n + \tfrac{1}{2}) - i\kappa]\bar{A}^{(n)} z^{n-1} \quad (6)$$

and

$$\phi_j'(z) = \Phi_j(z), \qquad \chi_j''(z) = \Psi_j(z), \qquad j = 1, 2$$

The complex functions $\Phi_j(z)$ and $\Psi_j(z)$ may be written in terms of $f(z)$ alone, i.e.,

$$\begin{aligned}\Phi_1(z) &= z^{-\frac{1}{2}-i\kappa} f(z) \\ \Psi_1(z) &= -\mu_1 e^{2\pi\kappa} z^{-\frac{1}{2}+i\kappa} \bar{f}(z) - [(\tfrac{1}{2} - i\kappa) f(z) + z f'(z)] z^{-\frac{1}{2}-i\kappa}\end{aligned} \quad (7)$$

for the upper half plane and

$$\begin{aligned}\Phi_2(z) &= \gamma\left(\frac{\mu_1}{\mu_2}\right) e^{2\pi\kappa} z^{-\frac{1}{2}-i\kappa} f(z) \\ \Psi_2(z) &= -\gamma\mu_1 z^{-\frac{1}{2}+i\kappa} \bar{f}(z) \\ &\quad - \gamma\left(\frac{\mu_1}{\mu_2}\right) e^{2\pi\kappa}[(\tfrac{1}{2} - i\kappa) f(z) + z f'(z)] z^{-\frac{1}{2}-i\kappa}\end{aligned} \quad (8)$$

for the lower half plane. The physical constants γ and κ are given by equations (9) and (12) in the paper, respectively. The unknown function $f(z)$ may be evaluated from the solution of the problem of a semi-infinite plate subjected to concentrated couples M and N on its edge. The plate occupies the upper half plane with material properties G_1 and ν_1. This is analogous to the Boussinesq solution in plane elasticity. In plate bending, the solution is

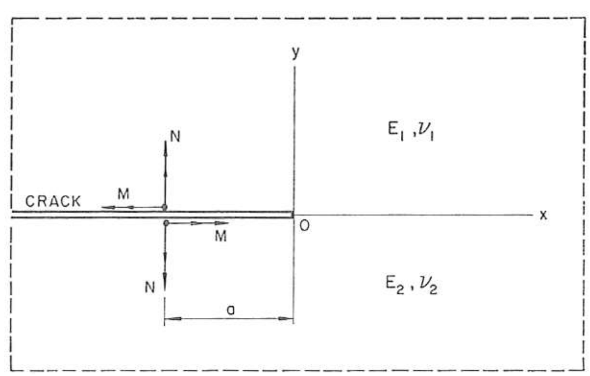

Fig. 1 Concentrated couples on crack surface between two dissimilar materials

[3] J. R. Rice and G. C. Sih, "Plane Problems of Cracks in Dissimilar Media," ASME Paper No. 65—APM-4; see this issue of the Journal, pp. 418–423.

$$\Phi_1(z) = \frac{M - iN}{2\pi i D_1(3 + \nu_1)} \frac{1}{z + a}$$

$$\Psi_1(z) = -\frac{M + iN}{2\pi i D_1(1 - \nu_1)} \frac{1}{z + a} - \frac{M - iN}{2\pi i D_1(3 + \nu_1)} \frac{a}{(z + a)^2} \quad (9)$$

Knowing that in the vicinity of the point $z = -a$ the stress distribution derived from equation (7) must be of the same form as that obtained by equation (9), $f(z)$ is found to be

$$f(z) = \frac{M - iN}{2\pi D_1(3 + \nu_1)} \frac{e^{-\pi\kappa}}{z + a} a^{\frac{1}{2}+i\kappa} \quad (10)$$

Substituting equation (10) into equations (7) and (8) yields the Goursat functions from which the stresses and displacements throughout the two dissimilar materials can be computed.

Briefly, the authors wish to conclude that the inclusion or exclusion of the second set of eigenvalues $\lambda_n^{(2)} = n$ $(n = 0, 1, 2, \ldots)$ in the final solution of a given problem depends mainly upon the geometry and the external loading. The manner in which the eigenvalues $\lambda_n^{(2)}$ affect the solution in terms of the constants $C^{(n)}$ (in bending) and $D^{(n)}$ (in extension) for two different groups of boundary-value problems can be outlined as follows:

(A) *Finite Regions*

The complex constants

$C^{(n)}, n = 1, 2, \ldots$ (Bending)

and

$D^{(n)}, n = 1, 2, \ldots$ (Extension)

can be determined from the boundary values of the stresses and/or displacements.

(B) *Infinite Regions*

It is necessary that

$C^{(n)} = \bar{C}^{(n)} = 0, \quad n \geq 2$ (Bending)

and

$D^{(n)} = \bar{D}^{(n)} = 0, \quad n \geq 2$ (Extension)

The nonzero constants $C^{(1)}$ and $D^{(1)}$ are to be evaluated from the uniform stresses at infinity. For finite cracks, any arbitrary uniform stress state may be specified. In the case of semi-infinite cracks, only a uniform state of normal stress acting parallel to the crack plane may be imposed for reasons stated earlier.

Unsymmetric Buckling of Thin Shallow Spherical Shells[1]

R. R. PARMERTER.[2] The author is to be congratulated for this apparently correct solution to the unsymmetrical buckling problem. It may be of interest to note that the Galerkin solution by Professor Y. C. Fung and myself (reference [9] of the paper) was subsequently refined by including higher-order asymmetric modes and produced results in essential agreement with those of Huang for $N = 1, 2, 3, 4$. Because of the disagreement between these results and the results of Weinitschke (reference [10], Huang), we conducted a series of experiments with electroformed shells and carefully controlled boundary conditions.

The Galerkin solution and the experimental results were pre-

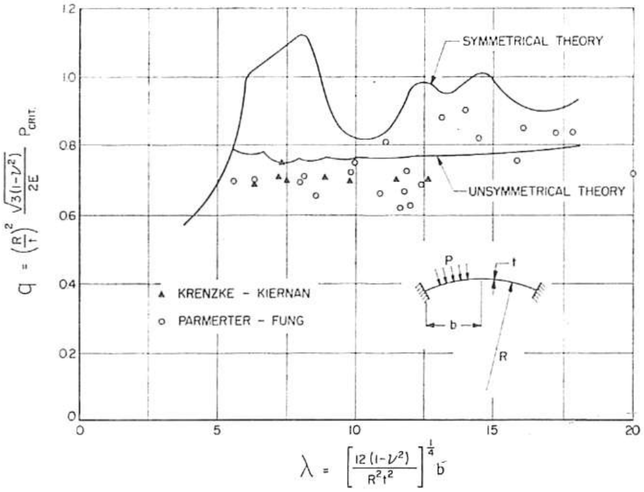

Fig. 1

sented in the writer's PhD thesis at the California Institute of Technology, and later in reference [1].[3] The experimental results are compared to Huang's theory in Fig. 1. Also included are the experimental results of Krenzke [2].

The latter tests are of particular interest. The shells tested in [1] all satisfied the shallowness criterion rise/radius < 1/8. In addition, the shells had ratios of spherical radius/thickness between 750 and 4200. This specification is of considerable importance as the membrane stress at buckling is inversely proportional to this ratio. The membrane stress at buckling must be kept well below the yield stress of the material or the combined bending and membrane stress may invalidate the assumption of elastic behavior up to the initiation of buckling. The four shells tested in [2] for $\lambda > 8$ (Fig. 1) all violate the shallowness assumption, with segment angles up to 60 deg. In addition, the spherical radius/thickness ratio of the shells ranged from 75 to 210. Thus the membrane stresses at buckling are at least an order of magnitude greater than the stresses in [1], leaving some question that the shells were elastic up to the buckling load.

It is extremely significant, therefore, that the buckling pressures observed in [2] are adequately predicted by elastic, shallow shell theory. It appears that, at least with regard to buckling load, the shallowness criterion places an overly conservative limitation on the theory.

It is also of interest that our experimental buckling loads exceed those predicted by Huang's theory for $\lambda > 13$, although they are below the symmetrical buckling loads given by Budiansky. It is not known whether this is due to a consistent error in the experiments or a failure of the unsymmetrical theory. The question is presently under investigation.

References

1 R. R. Parmerter, "The Buckling of Clamped Shallow Spherical Shells Under Uniform Pressure," AF OSR 5362, Graduate Aeronautical Laboratories, California Institute of Technology, Pasadena, Calif., November, 1963.

2 M. A. Krenzke and T. J. Kiernan, "Elastic Stability of Near Perfect Shallow Spherical Shells," *AIAA Journal*, December, 1963, p. 2855.

Author's Closure

The author appreciates very much the discussion by Prof. R. R. Parmerter to show the recent experimental results which support the author's theory for certain range of values of λ.

The discrepancy between numerical results due to Weinitschke and author is also clarified. In reference [3], Weinitschke points out that a hidden error is found in his machine program and

[1] By Nai-Chien Huang, published in the September, 1964, issue of the JOURNAL OF APPLIED MECHANICS, vol. 31, TRANS. ASME, vol. 86, Series E, pp. 447–457.

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[3] Numbers in brackets designate References at end of Discussion and Closure.