Non-linear Stability Analysis of Steady State Slip in a Single Degree of
Freedom Elastic System with Rate and State Dependent Friction

K. Ranjith and J. R. Rice

Division of Engineering and Applied Sciences and Department of Earth and Planetary Sciences,
Harvard University, Cambridge MA 02138 Fax: 617-495-9837
e-mail: ranjith@esag.harvard.edu, rice@esag.harvard.edu

The stability of quasi-static frictional slip of a rigid block loaded by a linear spring is studied. Motivated by experimental studies of rock and metal friction, the frictional shear stress, \( \tau \), on the block is taken to depend both on the slipping rate, \( V \), and the state of the surface in the form

\[
\tau = \tau_s + A \ln(V/V_s) + B \ln(V_s \theta/L),
\]

where \( A \), \( B \) and \( L \) are constants, \( \tau_s \) and \( V_s \) are reference values of friction stress and sliding velocity, respectively, and \( \theta \) is a state variable. The evolution of the state variable is assumed to follow the Dieterich-Ruina ageing law, given by

\[
\dot{\theta} = 1 - V \theta / L.
\]

An important feature of the ageing law is that friction evolves logarithmically with time during stationary contact (\( V = 0 \)). Another widely used evolution law is the Ruina-Dieterich slip law, given by

\[
\dot{\theta} = -(V \theta / L) \ln(V \theta / L).
\]

Here, in contrast to the Dieterich-Ruina ageing law, friction evolves only with slip.

The steady state of the above system is one wherein the block slides with the velocity of the imposed motion, \( V_s \). Away from steady state,

\[
\dot{\tau} = K(V_s - V),
\]

where \( K \) is the spring stiffness. The non-linear stability of the steady state, with state evolution described by the Ruina-Dieterich slip law, has been studied by Gu, Rice, Ruina and Tse, J. Mech. Phys. Solids, 32, 167-196, 1984. A similar analysis for the Dieterich-Ruina ageing law using analytically determined phase-plane trajectories and Liapunov function techniques is reported in the present work. The stability results are shown to have an extremely simple form:

- With non-zero load-point velocities (\( V_s \neq 0 \)), slip motion is always periodic when

  \[
  K = K_{sr} = (B - A)/L > 0.
  \]

  When \( K > K_{sr} > 0 \), sliding is always stable. In other words, the block always approaches the velocity of the imposed motion. When \( K < K_{sr} \), slip is always unstable with the sliding velocity becoming unbounded. This result has an important implication for slip instabilities in continuum systems. It may be argued that, in a continuum system, a critical nucleation size of coherent slip has to be attained before unstable slip can ensue.

- When the load point is held stationary (\( V_s = 0 \)), the system stably evolves towards the steady state whenever \( K \geq K_{sr} > 0 \). However, when \( 0 < K < K_{sr} \), initial conditions leading to stable and unstable slip motion exist. This shows that self-driven creep modes can exist only in the latter case.

\(^1\)Author for correspondence